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INFRA-SOLVMANIFOLDS OF $\mathrm{Sol_1}^4$

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Abstract. The purpose of this paper is to classify all compact manifolds modeled on the 4-dimensional solvable Lie group $\mathrm{Sol_1}^4$, and more generally, the crystallographic groups of $\mathrm{Sol_1}^4$. The maximal compact subgroup of $\mathrm{Isom}(\mathrm{Sol_1}^4)$ is $D_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$. We shall exhibit an infra-solvmanifold of $\mathrm{Sol_1}^4$ whose holonomy is D_4. This implies that all possible holonomy groups do occur; the trivial group, $\mathbb{Z}_2$ (5 families), $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$ (5 families), and $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ (2 families).

The 4-dimensional Lie group $\mathrm{Sol_1}^4$ is the subgroup of $\mathrm{GL}(3,\mathbb{R})$ defined as

$$\mathrm{Sol_1}^4 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, x, y, z, u \in \mathbb{R} \right\}.$$

The nilradical of $\mathrm{Sol_1}^4$ is the 3-dimensional Heisenberg group Nil (the elements of $\mathrm{Sol_1}^4$ with $u = 0$). It has 1-dimensional center (the elements of $\mathrm{Sol_1}^4$ with $x = y = u = 0$), and the quotient of $\mathrm{Sol_1}^4$ by the center is isomorphic to Sol^3. Recall that both Nil and Sol^3 are model spaces for 3-dimensional geometry. Let C be a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol_1}^4)$. A cocompact discrete subgroup

$$\Pi \subset \mathrm{Sol_1}^4 \rtimes C$$

is a *crystallographic group* of $\mathrm{Sol_1}^4$. The motivation for this arises from the crystallographic groups of Euclidean space $\mathbb{R}^n$, that is, the cocompact discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n,\mathbb{R})$. In general, the classification of crystallographic groups of nilpotent Lie groups, or certain well-behaved solvable Lie groups (such as $\mathrm{Sol_1}^4$), is an important question. For example, crystallographic groups of $\mathbb{R}^n$ are classified for $n \leq 4$. See [1] for a classification. Dekimpe provides a classification of crystallographic groups of 4-dimensional nilpotent Lie groups in [5]. A classification of crystallographic groups of Sol^3 is given by K. Y. Ha and J. B. Lee in [7].

Since the Bieberbach theorems generalize to $\mathrm{Sol_1}^4$ [6], the translation subgroup of Π, $\Pi \cap \mathrm{Sol_1}^4$, is of finite index in Π, and is a cocompact discrete

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subgroup (that is, a lattice) of Sol_1^4. Fortunately for us, the maximal compact subgroup C is very small. It is D_4, the dihedral group of 8 elements. Therefore, all crystallographic groups of Sol_1^4 are extensions of a lattice by a subgroup Φ of the finite group D_4. On the other hand, there are many non-isomorphic lattices, which makes things quite complicated. We shall classify the crystallographic groups of Sol_1^4 (this will include the classification of crystallographic groups of Sol^3).

A crystallographic group $\Pi \subset \mathrm{Sol}_1^4 \rtimes C$ acts naturally on Sol_1^4; that is, for $(a,\alpha) \in \Pi$, $x \in \mathrm{Sol}_1^4$, $(a,\alpha)\cdot x = a\alpha(x)$. The orbit space of Sol_1^4 by the action of a torsion free crystallographic group Π, $\Pi\backslash\mathrm{Sol}_1^4$, is an *infra-solvmanifold* of Sol_1^4. By the generalized Bieberbach theorems, two infra-solvmanifolds of Sol_1^4, say $\Pi\backslash\mathrm{Sol}_1^4$ and $\Pi'\backslash\mathrm{Sol}_1^4$, are (affinely) diffeomorphic precisely when Π and Π' are isomorphic. We shall exhibit an infra-solvmanifold of Sol_1^4 with maximal holonomy D_4, the largest possible. This implies that all possible holonomy groups do occur; the trivial group, $\mathbb{Z}_2$ (5 families), $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$ (5 families), and $\mathbb{Z}_4\rtimes\mathbb{Z}_2$ (2 families).

This paper is organized as follows. In Section 1, we determine $\mathrm{Aut}(\mathrm{Sol}_1^4)$, and show the dihedral group D_4 of order 8 is the maximal compact subgroup.

In Section 2, we recall the classification of lattices of Sol^3: all are isomorphic to $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$, for some $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{tr}(\mathcal{S}) > 2$.

In Section 3, we recall the result of [7] that any crystallographic group Q of Sol^3 can be viewed as an extension

$$1 \to \mathbb{Z}^2 \to Q \to \mathbb{Z}_\Phi \to 1,$$

where $\mathbb{Z}_\Phi$ itself is an extension $1 \to \mathbb{Z} \to \mathbb{Z}_\Phi \to \Phi \to 1$ for $\Phi \subset D_4$. Using the results of [7], Theorem 3.3 classifies all possible abstract kernels $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$.

In Section 4, we study the classification of Sol^3-crystallographic groups, in a similar fashion to that in [7]. We show an isomorphism between $H^2_\varphi(\mathbb{Z}_\Phi, \mathbb{Z}^2)$ and $H^1(\Phi, \mathrm{Coker}(I-\mathcal{S}))$, which greatly simplifies the calculations in [7]. The list is deferred until Section 6.

In Section 5, the classification of Sol_1^4-lattices as lifts of Sol^3-lattices is given.

In Section 6, the main classification theorem of crystallographic groups of Sol_1^4, Theorem 6.13, is proved. We find 8 categories; some are never torsion free, some are always torsion free, and some contain mixed cases. We determine this by examining the action of a crystallographic group on Sol_1^4. This theorem also serves as a classification of Sol^3-crystallographic groups, by considering the groups modulo the center of Sol_1^4.

In Section 7, we first show that Sol_1^4 admits an affine structure. It is much easier to represent crystallographic groups using this affine structure. We exhibit two examples of infra-Sol_1^4 manifolds. The first one is where the lattice is "non-standard". The second one is a space with the maximal holonomy group D_4. Both yield non-orientable manifolds.

All calculations were done by the program Mathematica [17], and were hand-checked.

1. The automorphism groups of $\mathbf{Sol^3}$ and $\mathbf{Sol_1^4}$

The group $\mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R}$ has group operation

$$(\mathbf{x}, u)(\mathbf{y}, v) = (\mathbf{x} + E^u\mathbf{y}, u + v), \text{ where } E^u = \begin{bmatrix} e^{-u} & 0 \\ 0 & e^u \end{bmatrix}.$$

Let α be an automorphism of Sol^3. Since $\mathbb{R}^2$ is the nilradical (maximal normal nilpotent subgroup) of Sol^3, α induces an automorphism A of $\mathbb{R}^2$, and hence, also an automorphism $\bar{A}$ of the quotient $\mathbb{R}$. Thus, there is a homomorphism

$$\begin{array}{ccc} \mathrm{Aut}(\mathrm{Sol}^3) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Aut}(\mathbb{R}) \\ \alpha & \longrightarrow & (A, \bar{A}). \end{array}$$

The following is known.

Proposition 1.1 ([7, p. 2]). *We have* $\mathrm{Aut}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$*, where* D_4 *is the dihedral group with 8 elements. Under this isomorphism,* Sol^3 *acts as inner automorphisms, and* $(\mathbb{R}^+ \times D_4)$ *is identified with the group of matrices* $\mathbb{R}^+ \times D_4 = \left\langle k \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], k \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\rangle$*,* $k > 0$*, (*$k = 1$ *yields* D_4*),* $A \in \mathbb{R}^+ \times D_4$ *acts on* Sol^3 *as*

$$A : \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \left(A \begin{bmatrix} x \\ y \end{bmatrix}, \bar{A}u \right).$$

($\bar{A} = +1$ *if* A *is diagonal,* $\bar{A} = -1$ *otherwise.)*

We now turn our attention to $\mathrm{Sol_1}^4$, embedded in $\mathrm{GL}(3, \mathbb{R})$ as

$$\mathrm{Sol_1}^4 = \left\{ s(x, y, z, u) := \begin{bmatrix} 1 & x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z, u \in \mathbb{R} \right\}.$$

By writing every element as a product

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & 1 \end{bmatrix} := \mathbf{x}\mathbf{e}^u,$$

we see that $\mathrm{Sol_1}^4$ is the semi-direct product $\mathrm{Nil} \rtimes \mathbb{R}$, where

$$(\mathbf{x}, u) \cdot (\mathbf{y}, v) = (\mathbf{x} \cdot \mathbf{e}^u \mathbf{y} \mathbf{e}^{-u}, u + v).$$

Nil is the nil-radical of $\mathrm{Sol_1}^4$, and the center of Nil, $\mathbb{R} = \{s(0, 0, z, 0) \mid z \in \mathbb{R}\}$, is also the center of $\mathrm{Sol_1}^4$. Evidently, $\mathrm{Sol_1}^4/\mathbb{R} \cong \mathrm{Sol}^3$. Thus we have a commuting diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{R} = \mathscr{Z}(\mathrm{Nil}) & = & \mathbb{R} = \mathscr{Z}({\mathrm{Sol}_1}^4) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
(1.1)\quad 1 & \longrightarrow & \mathrm{Nil} & \longrightarrow & {\mathrm{Sol}_1}^4 = \mathrm{Nil} \rtimes \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

The rows split, but the columns do not.

An automorphism $\hat{\alpha}$ of ${\mathrm{Sol}_1}^4$ induces automorphisms of the center $\mathbb{R}$ and the quotient Sol^3:

$$
\begin{array}{ccccc}
\mathrm{Aut}({\mathrm{Sol}_1}^4) & \longrightarrow & \mathrm{Aut}(\mathscr{Z}({\mathrm{Sol}_1}^4)) \times \mathrm{Aut}(\mathrm{Sol}^3) & \longrightarrow & \mathrm{Aut}(\mathbb{R}) \times \mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Aut}(\mathbb{R}) \\
\hat{\alpha} & \longrightarrow & (\hat{A}, \alpha) & \longrightarrow & (\hat{A}, A, \bar{A}).
\end{array}
$$

Similar to the case of Nil, $\hat{A}$ is multiplication by $\det(A)$. Conversely, every automorphism of Sol^3 induces an automorphism of ${\mathrm{Sol}_1}^4$, and $\mathrm{Aut}(\mathrm{Sol}^3)$ lifts to a subgroup of $\mathrm{Aut}({\mathrm{Sol}_1}^4)$. More specifically, we have:

Proposition 1.2.

$$
\begin{aligned}
\mathrm{Aut}({\mathrm{Sol}_1}^4) \cong \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3) &\cong \mathbb{R} \rtimes (\mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)) \\
&\cong (\mathbb{R} \times \mathrm{Sol}^3) \rtimes (\mathbb{R}^+ \times D_4),
\end{aligned}
$$

where $\mathrm{Sol}^3 \cong \mathrm{Inn}({\mathrm{Sol}_1}^4)$. *The group* $\mathbb{R}$ *is the kernel of the homomorphism*

$$\mathrm{Aut}({\mathrm{Sol}_1}^4) \to \mathrm{Aut}(\mathrm{Sol}^3).$$

The automorphism $\hat{k}$, $k \in \mathbb{R}$, *is given by*

$$
\hat{k} : \begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^u x & z + ku \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.
$$

This commutes with the inner automorphisms of ${\mathrm{Sol}_1}^4$, *and* $A \in \mathbb{R}^+ \times D_4$ *acts on this* $\mathbb{R}$ *by* ${}^A\hat{k} = (\hat{A} \cdot \bar{A}) \cdot \hat{k}$.

Proof. We have seen that the image of $\mathrm{Aut}({\mathrm{Sol}_1}^4)$ under

$$\mathrm{Aut}({\mathrm{Sol}_1}^4) \to \mathrm{Aut}(\mathbb{R}) \times \mathrm{Aut}(\mathrm{Sol}^3) \to \mathrm{Aut}(\mathbb{R}) \times \mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Aut}(\mathbb{R})$$

is determined by its image in $\mathrm{Aut}(\mathbb{R}^2)$. On the other hand, $\mathrm{Aut}(\mathrm{Sol}^3)$ lifts back to $\mathrm{Aut}({\mathrm{Sol}_1}^4)$. Recall the isomorphism $\mathrm{Aut}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$ given in

Proposition 1.1. First, $\mathrm{Sol}^3 \subset \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$, corresponding to the inner automorphisms of Sol^3 lifts to the inner automorphisms of $\mathrm{Aut}(\mathrm{Sol_1}^4)$. Note that $\mathrm{Inn}(\mathrm{Sol_1}^4) = \mathrm{Inn}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3$.

For the subgroup $\mathbb{R}^+ \times D_4$ of $\mathrm{Aut}(\mathrm{Sol}^3)$, we have that a diagonal or off-diagonal matrix $A \in \mathrm{GL}(2,\mathbb{R})$ can be lifted to an automorphism of $\mathrm{Sol_1}^4$:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^{\bar{A}u}(ax+by) & \frac{1}{2}(abx^2+2bcxy+cdy^2+2(ad-bc)z) \\ 0 & e^{\bar{A}u} & (cx+dy) \\ 0 & 0 & 1 \end{bmatrix}.$$

This formula is valid only for the cases when either $a = d = 0$ ($\bar{A} = -1$) or $b = c = 0$ ($\bar{A} = +1$).

The kernel of $\mathrm{Aut}(\mathrm{Sol_1}^4) \to \mathrm{Aut}(\mathrm{Sol}^3)$ is the group of crossed homomorphisms $Z^1(\mathrm{Sol}^3,\mathbb{R})$. Since Sol^3 acts trivially on the center $\mathbb{R}$, the crossed homomorphisms become genuine homomorphisms, and

$$Z^1(\mathrm{Sol}^3,\mathbb{R}) = \hom(\mathrm{Sol}^3,\mathbb{R}) = \hom(\mathbb{R},\mathbb{R}) = \mathbb{R}.$$

Thus we have a splitting $\mathrm{Aut}(\mathrm{Sol_1}^4) \cong \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$. □

Proposition 1.3. *The dihedral group* $D_4 = \left\langle \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\rangle$ *is the maximal compact subgroup of both* $\mathrm{Aut}(\mathrm{Sol}^3)$ *and* $\mathrm{Aut}(\mathrm{Sol_1}^4)$. *Furthermore, it is unique up to conjugation.*

Proof. The statement on uniqueness follows from [14, Theorem 3.1]. □

Remark 1.4. Up to the $\mathbb{R} = Z^1(\mathrm{Sol}^3,\mathbb{R})$-factor, $\mathrm{Aut}(\mathrm{Sol_1}^4) = \mathrm{Aut}(\mathrm{Sol}^3)$, and we may denote an automorphism in $D_4 \subset \mathrm{Aut}(\mathrm{Sol_1}^4)$ by a 2×2 matrix A only (suppressing even $\bar{A}$ and $\hat{A}$) when there is no confusion likely.

Remark 1.5. Both Sol^3 and $\mathrm{Sol_1}^4$ admit a left-invariant metric so that

$$\mathrm{Isom}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes D_4 \quad \text{and} \quad \mathrm{Isom}(\mathrm{Sol_1}^4) = \mathrm{Sol_1}^4 \rtimes D_4.$$

2. The Lattices of Sol

Let $\mathcal{S} = \left[\begin{smallmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$. Such a matrix has two positive eigenvalues satisfying $\lambda + \frac{1}{\lambda} > 0$. Then we can find a diagonalizing matrix $P \in \mathrm{GL}(2,\mathbb{R})$, with $\det(P) = 1$, diagonalizing $\mathcal{S}$: $P\mathcal{S}P^{-1} = \Delta$.

Notation 2.1. For uniformity of statements, we always take

$$\Delta = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{bmatrix} \text{ with } \tfrac{1}{\lambda} < 1 < \lambda.$$

With such P and Δ for $\mathcal{S}$, the relation $P\mathcal{S}P^{-1} = \Delta$ allows us to embed the semidirect product $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ as a lattice of Sol^3,

$$\begin{aligned} \phi : \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} &\longrightarrow \qquad \mathrm{Sol}^3 \\ \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) &\longmapsto \left(P\begin{bmatrix} x \\ y \end{bmatrix}, u\ln(\lambda) \right). \end{aligned} \tag{2.1}$$

It maps the generators as follows:

$$\begin{aligned} \mathbf{e}_1 &= \left(\begin{bmatrix}1\\0\end{bmatrix}, 0\right) \longmapsto \mathbf{t}_1 = P\mathbf{e}_1, \\ \mathbf{e}_2 &= \left(\begin{bmatrix}0\\1\end{bmatrix}, 0\right) \longmapsto \mathbf{t}_2 = P\mathbf{e}_2, \\ \mathbf{e}_3 &= \left(\begin{bmatrix}0\\0\end{bmatrix}, 1\right) \longmapsto \mathbf{t}_3 = (\mathbf{0}, \ln(\lambda)). \end{aligned} \tag{2.2}$$

We denote image of $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ by $\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle \subset \mathrm{Sol}^3$, which has relations

$$[\mathbf{t}_1, \mathbf{t}_2] = 1, \quad \mathbf{t}_3 \cdot \mathbf{t}_1 \cdot \mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{11}} \cdot \mathbf{t}_2^{\sigma_{21}}, \quad \mathbf{t}_3 \cdot \mathbf{t}_2 \cdot \mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{12}} \cdot \mathbf{t}_2^{\sigma_{22}}. \tag{2.3}$$

Notation 2.2. We shall refer to a lattice of Sol^3 generated by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ of the form in assignment (2.2) as a *standard lattice* of Sol^3.

Conversely, any lattice of Sol^3 is isomorphic to such a $\Gamma_{\mathcal{S}}$ as the following proposition shows. We say $\mathcal{S}, \mathcal{S}' \in \mathrm{SL}(2,\mathbb{Z})$ are *weakly conjugate* if and only if $\mathcal{S}'$ is conjugate, via an element of $\mathrm{GL}(2,\mathbb{Z})$, to $\mathcal{S}$ or $\mathcal{S}^{-1}$.

Proposition 2.3 ([7, Theorem 3.4])**.** *There is a one-one correspondence between the isomorphism classes of* Sol^3*-lattices and the weak-conjugacy classes of* $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$*. Therefore, any lattice of* Sol^3 *is conjugate to* $\Gamma_{\mathcal{S}}$*, for some* $\mathcal{S}$*, by an element of* $\mathrm{Aff}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$*.*

Proof. The isomorphism statement follows from Theorem 3.4 in [7]. The conjugacy statement follows from Theorem 3.1 below. This can also be seen by direct computation, as we do for Sol_1^4 lattices in Proposition 6.1. □

3. Compatibility of $\mathcal{S}$ with automorphisms

Both Sol^3 and Sol_1^4 are type (R) Lie groups that admit generalizations of Bieberbach's theorems for crystallographic groups of $\mathbb{R}^n$ [6, 11].

Theorem 3.1 ([11, Theorem 8.3.4 and Theorem 8.4.3])**.** *Let G denote either* Sol^3 *or* Sol_1^4*, and C denote a maximal compact subgroup of* $\mathrm{Aut}(G)$*.*

(1) *For a crystallographic group $\mathit{\Pi} \subset G \rtimes C$ of G, the translation subgroup $\mathit{\Pi} \cap G$ is a lattice of G, with $\Phi := \mathit{\Pi}/(\mathit{\Pi} \cap G) \subset C$ finite, the holonomy group.*

(2) *Any isomorphism between two crystallographic groups of G is conjugation by an element of* $\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$*.*

When G is either Sol^3 or Sol_1^4, C is conjugate in $\mathrm{Aut}(G)$ to D_4 (Proposition 1.3). Therefore, we can assume that $C = D_4$ in either case. We will see that every Sol_1^4-crystallographic group $\mathit{\Pi} \subset \mathrm{Sol}_1^4 \rtimes D_4$ projects to some Sol^3-crystallographic group $Q \subset \mathrm{Sol}^3 \rtimes D_4$ under the natural projection $\mathrm{Sol}_1^4 \rtimes D_4 \to \mathrm{Sol}^3 \rtimes D_4$. Therefore, we first recall the classification of Sol^3-crystallographic groups in [7]. We use different notation that is more amenable to lifting to the Sol_1^4 case.

Proposition 3.2. *Any crystallographic group Q' of Sol^3 can be conjugated in $\mathrm{Aff}(\mathrm{Sol}^3)$ to $Q \subset \mathrm{Sol}^3 \rtimes D_4$ so that $Q \cap \mathrm{Sol}^3 = \Gamma_{\mathcal{S}}$. That is, the translation subgroup of Q is a standard lattice of Sol^3, generated by $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ as in (2.2). Thus, Q is generated by $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle$, and at most two isometries of the form $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in \mathrm{Sol}^3 \rtimes D_4$, where a_i are rational numbers.*

Proof. This follows from Theorem 3.1. and Proposition 2.3. □

We will assume our Sol^3–crystallographic group Q is embedded in $\mathrm{Sol}^3 \rtimes D_4$ as in Proposition 3.2. Note that $Q \cap \mathbb{R}^2 = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \cong \mathbb{Z}^2$ is a lattice of $\mathbb{R}^2$. Denote $Q/\langle \mathbf{t}_1, \mathbf{t}_2 \rangle$ by $\mathbb{Z}_\Phi$ so that we have the commuting diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^2 & = & \mathbb{Z}^2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle /\mathbb{Z}^2} & & \downarrow{\scriptstyle /\mathbb{Z}^2} & & \| & & \\
1 & \longrightarrow & \mathbb{Z} = \langle \mathbf{t}_3 \rangle & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{3.1}
$$

To classify Q as extensions of $\mathbb{Z}^2$ by $\mathbb{Z}_\Phi$ as in (3.1), we need all possible *abstract kernels*

$$\varphi : \mathbb{Z}_\Phi \longrightarrow \mathrm{GL}(2, \mathbb{Z}).$$

Now $\mathbb{Z}_\Phi$ is generated by $\mathbf{t}_3$ together with $\bar{\alpha} = (\mathbf{t}_3^{a_3}, A)$ (with possibly an additional generator $\bar{\beta} = (\mathbf{t}_3^{b_3}, B)$):

$$\mathbb{Z}_\Phi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{a_3}, A), \bar{\beta} = (\mathbf{t}_3^{b_3}, B) \rangle.$$

Note we only need to consider $\Phi \subset D_4$ up to conjugacy. By definition, $\varphi(\mathbf{t}_3) = \mathcal{S}$. Since $\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle$ is embedded in Sol^3 as in (2.1), as an automorphism of $\mathbb{Z}^2 = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle$, $\bar{\alpha} = (\mathbf{t}_3^{a_3}, A)$ should map by φ to $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3}\widetilde{A}$, where

$$\mathcal{S}^{a_3} = P^{-1}\Delta^{a_3}P,$$
$$\widetilde{A} = P^{-1}AP.$$

The action of $A \in D_4$ on $\mathbb{Z} = \langle \mathbf{t}_3 \rangle$ in $\mathbb{Z}_\Phi$ is the induced action of A, $\bar{A}$, on the quotient $\mathbb{R} = \mathrm{Sol}^3/\mathbb{R}^2$. Thus, if $A \in D_4$ is a diagonal matrix, then $\bar{A} = +1$. Otherwise $\bar{A} = -1$, see Proposition 1.1. So, if $\bar{A} = +1$, $\varphi(\bar{\alpha})$ must commute with $\mathcal{S}$. Otherwise, $\varphi(\bar{\alpha})$ conjugates $\mathcal{S}$ to its inverse.

Theorem 3.3 below follows from Theorem 8.2 of [7]. In the proof we explain differences in notation.

Theorem 3.3 ([7, cf. Theorem 8.2]). *The following is a complete list of* $\mathbb{Z}_\Phi$ *and homomorphisms* $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *with* $\varphi(\mathbf{t}_3) = \mathcal{S}$ *and*

$$\varphi(\mathbf{t}_3^{a_3}, A) = \mathcal{S}^{a_3}\widetilde{A},$$
$$\varphi(\mathbf{t}_3^{b_3}, B) = \mathcal{S}^{b_3}\widetilde{B},$$

up to conjugation in $\mathrm{GL}(2,\mathbb{Z})$*, that is, change of generators for* $\langle \mathbf{t}_1, \mathbf{t}_2\rangle \cong \mathbb{Z}^2$.

(0) Φ *is trivial,*
$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3\rangle$.

(1) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
• $\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, *and* $\mathcal{S} = nK + I$.

(2a) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
• $\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$.

(2b) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
• $\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$, *and* $\mathcal{S} = nK - I$.

(3) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
• $\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{12} = -\sigma_{21}$.

(3*i*) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
• $\varphi(\bar{\alpha}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{11} = \sigma_{22}$.

(4) $\Phi = \mathbb{Z}_4 : A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_4 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
• $\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and symmetric.*

(5) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$.
• $\varphi(\bar{\alpha}) = -K$, $\varphi(\bar{\beta}) = B$ (1)+(2a)
• $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, *and* $\mathrm{tr}(K) = n > 0$.

(6a) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$.
• $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = B$ (3)+(2a)
• $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{12} = -\sigma_{21}$.

(6a*i*) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$.
• $\varphi(\bar{\alpha}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $\varphi(\bar{\beta}) = B$ (3*i*)+(2a)
• $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{11} = \sigma_{22}$.

(6b) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac12}, B)\rangle$.

• $\varphi(\bar\alpha) = A$, $\varphi(\bar\beta) = -K$ (3)+(2b)

• $\mathcal{S} = nK - I$, *where* $K \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K) = n > 2$; $k_{12} = -k_{21}$.

(6bi) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac12}, B)\rangle$.

• $\varphi(\bar\alpha) = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, $\varphi(\bar\beta) = -K$ (3i)+(2b)

• $\mathcal{S} = nK - I$, *where* $K \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K) = n > 2$; $k_{11} = k_{22}$.

(7) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 : A = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], B = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac12}, B)\rangle$.

• $\varphi(\bar\alpha) = A$, $\varphi(\bar\beta) = -K$ (*includes* (6a)) (3)+(1)

• $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$; $k_{12} = -k_{21}$.

(7i) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 : A = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], B = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac12}, B)\rangle$.

• $\varphi(\bar\alpha) = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, $\varphi(\bar\beta) = -K$ (*includes* (6ai)) (3i)+(1)

• $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, $k_{11} = k_{22}$.

Proof. The 9 families of Sol^3-crystallographic groups in Theorem 8.2 of [7] are labeled E_0, E_1, $E_2^\pm$, E_3, E_5, E_8, E_9, E_{10}, and E_{11}. The table below shows our notation convention:

E_0	E_1	E_2^+	E_2^-	E_3	E_5	E_8	E_9	E_{10}	E_{11}
(0)	(2a)	(2b)	(1)	(3), (3i)	(5)	(6a), (6ai)	(6b), (6bi)	(4)	(7), (7i)

From Theorem 8.2 of [7], $\varphi(\bar\alpha) = \varphi(\mathbf{t}_3^0, A)$ where $A = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] \in D_4$ can act on $\langle \mathbf{t}_1, \mathbf{t}_2\rangle \cong \mathbb{Z}^2$ in two different ways: either $P^{-1}AP = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ or $P^{-1}AP = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$.

In Theorem 8.2 of [7], cases E_3, E_8, E_9, and E_{11} contain such a holonomy element, and therefore we split each into two cases, depending on how $\varphi(\bar\alpha)$ acts on $\langle \mathbf{t}_1, \mathbf{t}_2\rangle \cong \mathbb{Z}^2$. We will see that one case always lifts to crystallographic groups of Sol_1^4 with torsion, whereas the other can lift to torsion free crystallographic groups.

When $\bar\alpha = (\mathbf{t}_3^{\frac12}, A)$, A is necessarily diagonal of order 2, and

$$\varphi(\bar\alpha) = P^{-1}\Delta^{\frac12}AP = -K,$$

where $(-K)^2 = K^2 = \mathcal{S}$. Letting $n = \mathrm{tr}(K)$, it follows that $\mathcal{S} = nK + I$ when $\det(K) = -1$, and $\mathcal{S} = nK - I$ when $\det(K) = 1$. This applies to the cyclic holonomy cases (1), (2b).

When $\bar\alpha = (\mathbf{t}_3^0, A)$, $\varphi(\bar\alpha) = P^{-1}AP$. If $A = -I$, $\varphi(\bar\alpha) = P(-I)P^{-1} = -I$ (regardless of $\mathcal{S}$ and P). For other choices of A, we have:

(1) $P^{-1}\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]P = \left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$ if and only if $P = \pm\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$.

$\mathcal{S}$ is diagonalized by such a P if and only if $\sigma_{12} = \sigma_{21}$.

(2) $P^{-1}\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]P = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ if and only if $P = \pm\begin{bmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{bmatrix}$.

$\mathcal{S}$ is diagonalized by such a P if and only if $\sigma_{12} = -\sigma_{21}$.

(3) $P^{-1}\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]P=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$ if and only if $P=\pm\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}t&-\frac{1}{t}\\t&\frac{1}{t}\end{smallmatrix}\right]$, $t\neq 0$.
$\mathcal{S}$ is diagonalized by such a P if and only if $\sigma_{11}=\sigma_{22}$.

This applies to the cyclic holonomy cases (3), (3*i*), and (4), and forces the stated conditions on $\mathcal{S}$. The two generator cases follow from the cyclic cases. □

4. Crystallographic groups of Sol³

With a fixed abstract kernel $\varphi:\mathbb{Z}_\Phi\to \mathrm{GL}(2,\mathbb{Z})$ from Theorem 3.1, the set of all equivalence classes of extensions Q in (3.1) is in one-one correspondence with the group $H^2_\varphi(\mathbb{Z}_\Phi,\mathbb{Z}^2)$. The following theorem greatly simplifies the computations in [7].

Theorem 4.1. *For each homomorphism* $\varphi:\mathbb{Z}_\Phi\to\mathrm{GL}(2,\mathbb{Z})$, *in* Theorem 3.3, *we have an isomorphism*

$$H^2_\varphi(\mathbb{Z}_\Phi;\mathbb{Z}^2)\cong H^1(\Phi;\mathrm{Coker}(I-\mathcal{S})),$$

where $\mathrm{Coker}(I-\mathcal{S})\cong(I-\mathcal{S})^{-1}\mathbb{Z}^2/\mathbb{Z}^2\subset T^2$ *is a finite abelian group. So, the set of equivalence classes of extensions* Q,

$$1\longrightarrow\mathbb{Z}^2\longrightarrow Q\longrightarrow\mathbb{Z}_\Phi\longrightarrow 1,$$

is in one-one correspondence with $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$.

Proof. Since $\det(I-\mathcal{S})\neq 0$, $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ is finite, as $\mathrm{Coker}(I-\mathcal{S})$ is finite. First, we verify that $\varphi(\mathbb{Z}_\Phi)\subset\mathrm{GL}(2,\mathbb{Z})=\mathrm{Aut}(\mathbb{Z}^2)$ leaves the group $(I-\mathcal{S})^{-1}\mathbb{Z}^2\subset\mathbb{R}^2$ containing $\mathbb{Z}^2$ invariant. Suppose there exists $\mathbf{a}\in\mathbb{R}^2$ such that $(I-\mathcal{S})\mathbf{a}=\mathbf{z}\in\mathbb{Z}^2$. Then,

$$(I-\mathcal{S})\big(\varphi(\mathbf{t}_3)\mathbf{a}\big)=(I-\mathcal{S})\big(\mathcal{S}\mathbf{a}\big)=\mathcal{S}\big((I-\mathcal{S})(\mathbf{a})\big)=\mathcal{S}(\mathbf{z})\in\mathbb{Z}^2.$$

Now for $\varphi(\bar\alpha)$, if $\bar A=+1$,

$$(I-\mathcal{S})\big(\varphi(\bar\alpha)\mathbf{a}\big)=\varphi(\bar\alpha)(I-\mathcal{S})\mathbf{a}=\varphi(\bar\alpha)\mathbf{z}\in\mathbb{Z}^2;$$

and if $\bar A=-1$, then $\varphi(\alpha)$ conjugates $\mathcal{S}$ to $\mathcal{S}^{-1}$, and so,

$$(I-\mathcal{S})\big(\varphi(\bar\alpha)\mathbf{a}\big)=\varphi(\bar\alpha)(-\mathcal{S}^{-1})(I-\mathcal{S})\mathbf{a}=\varphi(\bar\alpha)(-\mathcal{S}^{-1})\mathbf{z}\in\mathbb{Z}^2.$$

This shows that, if $\mathbf{a}\in(I-\mathcal{S})^{-1}\mathbb{Z}^2$, then so are $\varphi(\mathbf{t}_3)\mathbf{a}$ and $\varphi(\bar\alpha)\mathbf{a}$. Consequently, $(I-\mathcal{S})^{-1}\mathbb{Z}^2$ is $\varphi(\mathbb{Z}_\Phi)$-invariant. Since $\mathbf{a}-\varphi(\mathbf{t}_3)\mathbf{a}=(I-\mathcal{S})\mathbf{a}\in\mathbb{Z}^2$, $\mathbf{t}_3$ acts as the identity on $\mathrm{Coker}(I-\mathcal{S})$. We obtain an induced action of $\mathbb{Z}_\Phi/\langle\mathbf{t}_3\rangle\cong\Phi$ on $\mathrm{Coker}(I-\mathcal{S})$, and so $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ is defined.

Suppose we have a class in $H^2(\mathbb{Z}_\Phi;\mathbb{Z}^2)$ defining an extension Q. Since $\mathbb{Z}^2\subset\mathbb{R}^2$ has the unique automorphism extension property, there exists a push-out $\widetilde{Q}$ [11, (5.3.4)] fitting the commuting diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & Q & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & 1\\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
1 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \widetilde{Q} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & 1
\end{array}$$

Note that $H^2(\mathbb{Z}_\Phi;\mathbb{R}^2)$ is annihilated by the (finite) index of $\mathbb{Z}=\langle \mathbf{t}_3\rangle$ in $\mathbb{Z}_\Phi$ [2, Proposition 10.1]. Therefore, $H^2(\mathbb{Z}_\Phi;\mathbb{R}^2)$ vanishes, and $\widetilde{Q}$ is the split extension $\mathbb{R}^2\rtimes\mathbb{Z}_\Phi$. Since $\mathbb{Z}\subset\mathbb{Z}_\Phi$ lifts back to $\Gamma_\mathcal{S}$, it lifts back to $\widetilde{Q}$ so that $\widetilde{Q}$ contains $(\mathbf{0},\mathbf{t}_3)\in\mathbb{R}^2\rtimes\mathbb{Z}_\Phi$. For each element $\mathbf{t}_3^n\bar\alpha\in\mathbb{Z}_\Phi$, pick a preimage $\alpha=(a,\mathbf{t}_3^n\bar\alpha)\in\mathbb{R}^2\rtimes\mathbb{Z}_\Phi$, taking care that $a=\mathbf{0}$ if $\bar\alpha=\mathrm{id}$. Then $\mathbf{t}_3^n\bar\alpha\mapsto a$ defines a map $\eta:\mathbb{Z}_\Phi\to\mathbb{R}^2/\mathbb{Z}^2=T^2$, and in fact, η maps into $\mathrm{Coker}(I-\mathcal{S})\subset T^2$. Thus we have

$$\eta:\Phi\to\mathrm{Coker}(I-\mathcal{S}).$$

We claim that η is a crossed homomorphism. Let $\bar\alpha,\bar\beta\in\Phi$, and $\eta(\bar\alpha)=\mathbf{a}$, $\eta(\bar\beta)=\mathbf{b}$. For preimages $(\mathbf{a},\mathbf{t}_3^m\bar\alpha)$ and $(\mathbf{b},\mathbf{t}_3^n\bar\beta)$ in $\widetilde{Q}$,

$$\begin{aligned}(\mathbf{a},\mathbf{t}_3^m\bar\alpha)(\mathbf{b},\mathbf{t}_3^n\bar\beta)&=(\mathbf{a}+\varphi(\mathbf{t}_3^m\bar\alpha)(\mathbf{b}),\mathbf{t}_3^m\bar\alpha\mathbf{t}_3^n\bar\beta)\\&=(\mathbf{a}+\varphi(\mathbf{t}_3^m)(\varphi(\bar\alpha)(\mathbf{b})),\mathbf{t}_3^m(\bar\alpha\mathbf{t}_3^n\bar\alpha^{-1})\bar\alpha\bar\beta).\end{aligned}$$

Since $\bar\alpha\mathbf{t}_3^n\bar\alpha^{-1}=\mathbf{t}_3^\ell$ for some $\ell\in\mathbb{Z}$,

$$\begin{aligned}\eta(\bar\alpha\bar\beta)&=\mathbf{a}+\varphi(\mathbf{t}_3^m)(\varphi(\bar\alpha)(\mathbf{b}))\\&=\eta(\bar\alpha)+\mathcal{S}^m(\varphi(\bar\alpha)(\eta(\bar\beta)))\\&=\eta(\bar\alpha)+\varphi(\bar\alpha)(\eta(\bar\beta)),\end{aligned}$$

where the last equality holds because $\varphi(\bar\alpha)(\eta(\bar\beta))\in\mathrm{Coker}(I-\mathcal{S})$, and the action of $\mathcal{S}$ on $\mathrm{Coker}(I-\mathcal{S})$ is trivial (if $\mathbf{a}\in\mathrm{Coker}(I-\mathcal{S})$, then $(I-\mathcal{S})\mathbf{a}\in\mathbb{Z}^2$, and hence $\mathbf{a}=\mathcal{S}\mathbf{a}$ modulo $\mathbb{Z}^2$). Thus η is a crossed homomorphism. Conversely, such a crossed homomorphism η clearly gives rise to an extension Q. Thus, we obtain a surjective map

$$Z^1(\Phi;\mathrm{Coker}(I-S))\to H^2(\mathbb{Z}_\Phi;\mathbb{Z}^2),$$

which we claim is a homomorphism. To see this, given

$$\eta:\Phi\to\mathrm{Coker}(I-\mathcal{S}),$$

we find a 2-cocycle $f:\mathbb{Z}_\Phi\times\mathbb{Z}_\Phi\to\mathbb{Z}^2$ representing the extension Q corresponding to η. Fix a lift $\widetilde\eta:\Phi\to(I-S)^{-1}(\mathbb{Z}^2)$ (not a homomorphism in general) of η. Then we can write any element of Q as

$$(\mathbf{n}+\widetilde\eta(\bar\alpha),\mathbf{t}_3^m\bar\alpha),$$

where $\mathbf{n}\in\mathbb{Z}^2$, $m\in\mathbb{Z}$. Now, for $(\mathbf{n}_1+\widetilde\eta(\bar\alpha),\mathbf{t}_3^{m_1}\bar\alpha)$ and $(\mathbf{n}_2+\widetilde\eta(\bar\beta),\mathbf{t}_3^{m_2}\bar\beta)\in Q$,

$$\begin{aligned}&(\mathbf{n}_1+\widetilde\eta(\bar\alpha),\mathbf{t}_3^{m_1}\bar\alpha)(\mathbf{n}_2+\widetilde\eta(\bar\beta),\mathbf{t}_3^{m_2}\bar\beta)=\\&(\mathbf{n}_1+\mathcal{S}^{m_1}\varphi(\bar\alpha)(\mathbf{n}_2)+\widetilde\eta(\bar\alpha)+\mathcal{S}^{m_1}\varphi(\bar\alpha)(\widetilde\eta(\bar\beta)),\mathbf{t}_3^{m_1}\bar\alpha\mathbf{t}_3^{m_2}\bar\beta).\end{aligned}$$

Therefore, Q is represented by the 2-cocycle $f:\mathbb{Z}_\Phi\times\mathbb{Z}_\Phi\to\mathbb{Z}^2$ defined by

$$f(\mathbf{t}_3^{m_1}\bar\alpha,\mathbf{t}_3^{m_2}\bar\beta)=\widetilde\eta(\bar\alpha)+\mathcal{S}^{m_1}\varphi(\bar\alpha)(\widetilde\eta(\bar\beta))-\widetilde\eta(\bar\alpha\bar\beta).$$

It is now clear that addition of crossed homomorphisms in $Z^1(\Phi;\mathrm{Coker}(I-S))$ corresponds to addition of 2-cocycles in $Z^2(\mathbb{Z}_\Phi;\mathbb{Z}^2)$.

We shall prove that Q splits if and only if the corresponding η is a coboundary, i.e., $\eta \in B^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$. Note that this will imply that $Z^1(\Phi; \mathrm{Coker}(I -S)) \to H^2(\mathbb{Z}_\Phi; \mathbb{Z}^2)$ induces an isomorphism

$$H^1(\Phi; \mathrm{Coker}(I-S)) \cong H^2(\mathbb{Z}_\Phi; \mathbb{Z}^2).$$

A splitting $\mathbb{Z}_\Phi \to Q$ induces a homomorphism

$$s : \mathbb{Z}_\Phi \to \widetilde{Q}.$$

Suppose $s(\mathbf{t}_3) = (z, \mathbf{t}_3)$ with $z \in \mathbb{Z}^2$. Even in this case, our definition of η shows that, we will pick $(\mathbf{0}, \mathbf{t}_3)$ as our preimage of $\mathbf{t}_3$ so that $\eta(\mathbf{t}_3) = \mathbf{0}$, and $\eta(\bar{\alpha}) = \mathbf{a}$ if $s(\bar{\alpha}) = (\mathbf{a}, \bar{\alpha})$ for others.

Let $y = -(I-\mathcal{S})^{-1}z$. Then

$$\begin{aligned}(y, I)(z, \mathbf{t}_3)(-y, I) &= (y + z - \varphi(\mathbf{t}_3)(y), \mathbf{t}_3) = (z + (I-\mathcal{S})(y), \mathbf{t}_3)\\ &= (\mathbf{0}, \mathbf{t}_3)\end{aligned}$$

and

$$\begin{aligned}(y, I)(\mathbf{a}, \bar{\alpha})(-y, I) &= (y + \mathbf{a} - \varphi(\bar{\alpha})y, \bar{\alpha}) = (\mathbf{a} + (I-\varphi(\bar{\alpha}))y, \bar{\alpha})\\ &= (\mathbf{v}, \bar{\alpha}),\ \text{by setting } \mathbf{a} + (I-\varphi(\bar{\alpha}))y = \mathbf{v}.\end{aligned}$$

Now,

$$\begin{aligned}(\mathbf{v}, \bar{\alpha})(\mathbf{0}, \mathbf{t}_3)(\mathbf{v}, \bar{\alpha})^{-1} &= (\mathbf{v} - (\bar{\alpha}\mathbf{t}_3\bar{\alpha}^{-1})\mathbf{v}, \bar{\alpha}\mathbf{t}_3\bar{\alpha}^{-1}) = (\mathbf{v} - \mathbf{t}_3^{\bar{A}}\mathbf{v}, \mathbf{t}_3^{\bar{A}})\\ &= ((I-\mathcal{S}^{\bar{A}})\mathbf{v}, \mathbf{t}_3^{\bar{A}}).\end{aligned}$$

Since $\mathbb{Z}$ is normal in $\mathbb{Z}_\Phi$, for s to be a homomorphism, we must have $(I-\mathcal{S}^{\bar{A}})\mathbf{v} = \mathbf{0}$. This happens if and only if $\mathbf{v} = 0$ since $(I-\mathcal{S}^{\bar{A}})$ is invertible, which holds if and only if

$$\eta(\bar{\alpha}) = \mathbf{a} = (\varphi(\bar{\alpha}) - I)(-y) = (\delta y)(\bar{\alpha}),$$

so that η is a coboundary. □

An alternate argument for Theorem 4.1 is provided by the long exact sequence

$$\cdots \to H^1_\varphi(\mathbb{Z}_\Phi; \mathbb{R}^2) \to H^1_\varphi(\mathbb{Z}_\Phi; T^2) \to H^2_\varphi(\mathbb{Z}_\Phi; \mathbb{Z}^2) \to H^2_\varphi(\mathbb{Z}_\Phi; \mathbb{R}^2) \to \cdots,$$

induced by the short exact sequence of coefficients $0 \to \mathbb{Z}^2 \to \mathbb{R}^2 \to T^2 \to 0$.

Since both $H^1_\varphi(\mathbb{Z}_\Phi; \mathbb{R}^2)$ and $H^2_\varphi(\mathbb{Z}_\Phi; \mathbb{R}^2)$ vanish, we obtain an isomorphism

$$H^1_\varphi(\mathbb{Z}_\Phi; T^2) \cong H^2_\varphi(\mathbb{Z}_\Phi; \mathbb{Z}^2).$$

To establish that $H^1_\varphi(\mathbb{Z}_\Phi; T^2) \cong H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$, note that any class in $H^1_\varphi(\mathbb{Z}_\Phi; T^2)$ is represented by a crossed homomorphism, mapping $\mathbf{t}_3$ to the identity of T^2, and such a crossed homomorphism $\widetilde{\eta} : \mathbb{Z}_\Phi \to T^2$ induces $\eta : \Phi \to T^2$. The image of η must lie in $(I-S)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$, and so η defines an element of $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$. It is straightforward to check that this is an isomorphism.

On the other hand, our proof of Theorem 4.1 establishes the precise one-one correspondence between $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ and the set of all equivalence classes of extensions Q,

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow Q \longrightarrow \mathbb{Z}_\Phi \longrightarrow 1.$$

Remark 4.2. For each subgroup Φ of D_4, we describe both $Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ and $B^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$, where the action of Φ on $\mathrm{Coker}(I-\mathcal{S})$ is induced from a $\varphi:\mathbb{Z}_\Phi\to \mathrm{GL}(2,\mathbb{Z})$ in Theorem 3.3. For $\Phi\cong\mathbb{Z}_2\times\mathbb{Z}_2$, we need to check that the commutator of $(\mathbf{a},\bar\alpha)$ and $(\mathbf{b},\bar\beta)$ is in $\mathbb{Z}^2$. For $\mathbb{Z}_4$, there is no cocycle condition to check (since $I+\varphi(\bar\alpha)+\varphi(\bar\alpha)^2+\varphi(\bar\alpha)^3=0$). Likewise for $\mathbb{Z}_4\rtimes\mathbb{Z}_2$, there is no cocycle condition for the order 4 element.

(1) $\Phi=\mathbb{Z}_2=\langle\bar\alpha\rangle$,

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\in\mathrm{Coker}(I-\mathcal{S})\mid\ (I+\varphi(\bar\alpha))\mathbf{a}\equiv\mathbf{0}\},$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(I-\varphi(\bar\alpha))\mathbf{v}\mid\ \mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\}.$$

(2) $\Phi=\mathbb{Z}_4=\langle\bar\alpha\rangle$,

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\in\mathrm{Coker}(I-\mathcal{S})\},$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(I-\varphi(\bar\alpha))\mathbf{v}\mid\ \mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\}.$$

(3) $\Phi=\mathbb{Z}_2\times\mathbb{Z}_2=\langle\bar\alpha,\bar\beta\rangle$,

$$\begin{aligned}Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(\mathbf{a},\mathbf{b})\mid \mathbf{a},\mathbf{b}\in\mathrm{Coker}(I-\mathcal{S}),\\ (I+\varphi(\bar\alpha))\mathbf{a}\equiv(I+\varphi(\bar\beta))\mathbf{b}\equiv\mathbf{0},\\ (I-\varphi(\bar\alpha))\mathbf{b}\equiv(I-\varphi(\bar\beta))\mathbf{a}\},\end{aligned}$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{((I-\varphi(\bar\alpha))\mathbf{v},(I-\varphi(\bar\beta))\mathbf{v})\mid \mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\}.$$

(4) $\Phi=\mathbb{Z}_4\rtimes\mathbb{Z}_2=\langle\bar\alpha,\bar\beta|\ \bar\alpha^2,\bar\beta^2,(\bar\beta\bar\alpha)^4\rangle$,

$$\begin{aligned}Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(\mathbf{a},\mathbf{b})\mid \mathbf{a},\mathbf{b}\in\mathrm{Coker}(I-\mathcal{S}),\\ (I+\varphi(\bar\alpha))\mathbf{a}\equiv(I+\varphi(\bar\beta))\mathbf{b}\equiv\mathbf{0}\},\end{aligned}$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{((I-\varphi(\bar\alpha))\mathbf{v},(I-\varphi(\bar\beta))\mathbf{v})\mid \mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\}.$$

Suppose we have an extension Q; that is, $\eta\in H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ with $\eta(\bar\alpha)=\mathbf{a}=\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right]$. Then

$$Q=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3},A)\rangle\subset\mathrm{Sol}^3\rtimes D_4$$

has the following presentation

$$\begin{aligned}
&\mathbf{t}_3(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\mathbf{t}_3^{-1}=\mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2},\ \text{where}\begin{bmatrix}m_1\\m_2\end{bmatrix}=\mathcal{S}\begin{bmatrix}n_1\\n_2\end{bmatrix},\\
&\alpha(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\alpha^{-1}=m_1'm_2',\ \text{where}\begin{bmatrix}m_1'\\m_2'\end{bmatrix}=\varphi(\bar\alpha)\begin{bmatrix}n_1\\n_2\end{bmatrix},\\
&\alpha\mathbf{t}_3\alpha^{-1}=\mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar A},\ \text{where}\begin{bmatrix}w_1\\w_2\end{bmatrix}=\left(I-\mathcal{S}^{\bar A}\right)\begin{bmatrix}a_1\\a_2\end{bmatrix},
\end{aligned}$$

$$\alpha^2 = \mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}\mathbf{t}_3^{(1+\bar{A})a_3}, \text{ where } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (I+\varphi(\bar{\alpha}))\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \text{ if } A^2 = I,$$
$$\alpha^4 = \mathrm{id}, \text{ if } \operatorname{ord}(A) = 4.$$

Corollary 4.3. *Let* $Q = \langle \Gamma_{\mathcal{S}}, (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)\rangle$ *be a* Sol^3*-crystallographic group with standard lattice* $\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\rangle$*. Suppose* $\varphi(\bar{\alpha}) = -K$ *and* $\mathcal{S} = nK \pm I$*. Recall that by Theorem 3.3, A has order 2,* $\bar{A} = 1$*, and* $a_3 = \frac{1}{2}$*. Then*

$$H^1(\Phi; \operatorname{Coker}(I-\mathcal{S})) = 0.$$

In fact, there exists $\mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}$ *which conjugates* $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A)$ *to* $(\mathbf{t}_3^{\frac{1}{2}}, A)$ *and leaves* $\Gamma_{\mathcal{S}}$ *invariant.*

Proof. We have $\det(I-\varphi(\bar{\alpha})) = \det(I+K) = 1+\det(K)+\operatorname{tr}(K)$. By Theorem 3.3, when $\det(K) = -1$, $\operatorname{tr}(K) > 0$; and when $\det(K) = 1$, $\operatorname{tr}(K) > 2$.

Consequently, $I - \varphi(\bar{\alpha})$ is always non-singular and we may take $\mathbf{v} = (I - \varphi(\bar{\alpha}))^{-1}\mathbf{a}$. Then $(\mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}, I) \in \mathrm{Sol}^3 \rtimes D_4$ conjugates $(\mathbf{t}_3^{\frac{1}{2}}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A)$. It remains to show $\mathbf{v} \in (I-\mathcal{S})^{-1}\mathbb{Z}^2$. This condition guarantees conjugation by $(\mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}, I)$ leaves $\Gamma_{\mathcal{S}}$ invariant. Since $\varphi(\bar{\alpha}) = -K$ is a square root of $\mathcal{S}$ and $\mathbf{v} = (I-\varphi(\bar{\alpha}))^{-1}\mathbf{a}$,

$$(I-\mathcal{S})\mathbf{v} = (I+\varphi(\bar{\alpha}))(I-\varphi(\bar{\alpha}))\mathbf{v} = (I+\varphi(\bar{\alpha}))\mathbf{a} \in \mathbb{Z}^2,$$

where the last inclusion holds by the cocycle conditions in Remark 4.2. Therefore $\Phi \ni A \mapsto \mathbf{a} \in \operatorname{Coker}(I-\mathcal{S})$ is a coboundary, and $H^1(\Phi; \operatorname{Coker}(I-\mathcal{S}))$ vanishes. □

Corollary 4.3 greatly simplifies the computation of $H^1(\Phi; \operatorname{Coker}(I-\mathcal{S}))$. For example, in cases (1), (2b), and (5) of Theorem 3.3, we can take $\mathbf{a} = \mathbf{0}$, whereas in cases (6b), (6bi), (7), and (7i), we can take $\mathbf{b} = \mathbf{0}$.

The complete list of crystallographic groups for Sol^3 will follow from our classification of crystallographic groups of Sol_1^4. However, we will need to analyze how a type (3i) or (6i) crystallographic group of Sol^3 acts on Sol^3. This will be critical to determining when a crystallographic group of Sol_1^4 has torsion.

Lemma 4.4. *Let Q be a crystallographic group of* Sol^3 *of type* (3i) *or* (6bi).

When Q is of type (3i)*,*

$$Q = \left\langle \Gamma_{\mathcal{S}}, \alpha = \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right)\right\rangle, \text{ and}$$

$Q\backslash\mathrm{Sol}^3$ *can be described as* $T^2 \times I$ *with* $T^2 \times \{0\}$ *identified to itself by the affine involution* $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right)$*, and* $T^2 \times \{1\}$ *identified to itself by the affine involution* $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} \sigma_{11} & -\sigma_{12} \\ \sigma_{21} & -\sigma_{11} \end{smallmatrix}\right]\right)$*. Here* T^2 *is the 2-dimensional torus.*

If $\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ *is used instead of* $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$*, then* $Q\backslash\mathrm{Sol}^3$ *can be described as* $T^2 \times I$ *with* $T^2 \times \{0\}$ *identified to itself by the affine involution* $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)$*, and* $T^2 \times \{1\}$ *identified to itself by the affine involution* $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} -\sigma_{11} & \sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{smallmatrix}\right]\right)$.

When Q is of type (6b*i*),

$$Q = \left\langle \Gamma_{\mathcal{S}}, \alpha = \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right), \beta = \left(\mathbf{t}_3^{\frac{1}{2}}, \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right)\right\rangle, \text{ and}$$

$Q\backslash\mathrm{Sol}^3$ can be described as $T^2 \times I$ with $T^2 \times \{0\}$ identified to itself by the affine involution $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right)$, and $T^2 \times \{1\}$ identified to itself by the affine involution $\left(\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} -k_{11} & k_{12} \\ -k_{21} & k_{11} \end{smallmatrix}\right]\right)$.

Proof. The action of $\Gamma_{\mathcal{S}}$ on Sol^3 is equivalent to the action of $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ on $\mathbb{R}^2 \rtimes_{\mathcal{S}} \mathbb{R}$. A fundamental domain for this action is given by the unit cube I^3, and evidently $Q\backslash\mathrm{Sol}^3$ is given by $T^2 \times I$ with $T^2 \times \{0\}$ identified to $T^2 \times \{1\}$ via $\mathcal{S}$, which we view as a self-diffeomorphism of T^2. Note that $\mathbb{R}^2 \to \mathbb{R}^2 \rtimes_{\mathcal{S}} \mathbb{R} \to \mathbb{R}$ induces the fiber bundle with infinite cyclic structure group generated by $\mathcal{S}$:

$$T^2 \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3 \to S^1.$$

Now suppose Q is of type (3*i*). Then $Q\backslash\mathrm{Sol}^3$ is the quotient of $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$ by the involution defined by $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$. Here α acts as a reflection on the base S^1. A fundamental domain for this action is given by $T^2 \times \left[0, \frac{1}{2}\right]$. Now α identifies $T^2 \times \{0\}$ to itself and $T^2 \times \{\frac{1}{2}\}$ to itself.

Indeed, $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A) \cdot \mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2} = \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})$ shows that α acts on $T^2 \times \{0\}$ as the affine transformation $(\mathbf{a}, \varphi(\bar{\alpha}))$. For $\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{2}} \in T^2 \times \{\frac{1}{2}\}$,

$$\begin{aligned} \mathbf{t}_3(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A) \cdot \mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{2}} &= \mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})A(\mathbf{t}_3^{\frac{1}{2}}) = \mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{2}} \\ &= \left(\mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-1}\right)\left(\mathbf{t}_3 A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-1}\right)\mathbf{t}_3^{\frac{1}{2}} \in T^2 \times \{\tfrac{1}{2}\}. \end{aligned}$$

Since conjugation by $\mathbf{t}_3$ is the action of $\mathcal{S}$, we see that α acts on T^2 as the affine transformation $(\mathcal{S}\mathbf{a}, \mathcal{S}\varphi(\bar{\alpha}))$. But since $\mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})$, this simplifies to $(\mathbf{a}, \mathcal{S}\varphi(\bar{\alpha}))$. Note that the condition that $\sigma_{11} = \sigma_{22}$ ensures that $\mathcal{S}\varphi(\bar{\alpha})$ has order 2.

The argument in case (6b*i*) is nearly identical. In this case, note that Q contains a group of type (2b), say Q', as an index 2 subgroup,

$$Q' = \left\langle \Gamma_{\mathcal{S}}, \beta = \left(\mathbf{t}_3^{\frac{1}{2}}, \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right)\right\rangle.$$

Therefore, $Q\backslash\mathrm{Sol}^3$ is the quotient of $Q'\backslash\mathrm{Sol}^3$ by $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$. Now $Q'\backslash\mathrm{Sol}^3$ is the quotient of $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$ by the involution defined by β. On the base of $T^2 \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3 \to S^1$, β acts as a translation. Thus a fundamental domain for the action of β is given by $T^2 \times \left[0, \frac{1}{2}\right]$. Note that β identifies $T^2 \times \{0\}$ with $T^2 \times \{\frac{1}{2}\}$ via $\varphi(\bar{\beta}) = -K$, which is a square root of $\mathcal{S}$, and $Q'\backslash\mathrm{Sol}^3$ is the mapping torus of $\varphi(\bar{\beta})$. Now because $Q'\backslash\mathrm{Sol}^3$ admits the structure of a T^2 bundle over S^1, the construction in (3*i*) applies. A fundamental domain for the action of α on $Q'\backslash\mathrm{Sol}^3$ is given by $T^2 \times \{\frac{1}{4}\}$. As in case (3*i*), α acts on $T^2 \times \{0\}$ affinely as $(\mathbf{a}, \varphi(\bar{\alpha}))$. For $\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{4}} \in T^2 \times \{\frac{1}{4}\}$,

$$(\mathbf{t}_3^{\frac{1}{2}}, B)(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A) \cdot \mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{4}} = \mathbf{t}_3^{\frac{1}{2}} B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}) BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}) BA(\mathbf{t}_3^{\frac{1}{4}})$$

$$\begin{aligned}
&= \mathbf{t}_3^{\frac{1}{2}} B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})\mathbf{t}_3^{-\frac{1}{2}} \mathbf{t}_3^{\frac{1}{2}} BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{4}} \\
&= \left(\mathbf{t}_3^{\frac{1}{2}} B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})\mathbf{t}_3^{-\frac{1}{2}}\right) \left(\mathbf{t}_3^{\frac{1}{2}} BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{2}}\right) \mathbf{t}_3^{\frac{1}{4}} \\
&\in T^2 \times \{\tfrac{1}{4}\}.
\end{aligned}$$

Now conjugation by $(\mathbf{t}_3^{\frac{1}{2}}, B)$ is the action of $\varphi(\bar{\beta}) = -K$ on T^2. Hence α acts affinely on $T^2 \times \{\frac{1}{4}\}$ as $(\varphi(\bar{\beta})\mathbf{a}, \varphi(\bar{\beta})\varphi(\bar{\alpha}))$. The commutator cocycle conditions for $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ in Remark 4.2, with $\mathbf{b} = \mathbf{0}$ implies $(I - \varphi(\bar{\beta}))\mathbf{a} = (I+K)\mathbf{a} \in \mathbb{Z}^2$, so this simplifies to $(\mathbf{a}, \varphi(\bar{\beta})\varphi(\bar{\alpha})) = (\mathbf{a}, (-K)\varphi(\bar{\alpha}))$. ☐

5. Lattices of Sol_1^4

In this section we classify the lattices of Sol_1^4. Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of Sol_1^4, $\widetilde{\Gamma}_{\mathcal{S}} \cap \mathcal{Z}(\mathrm{Sol}_1^4) \cong \mathbb{Z}$ is a lattice of $\mathcal{Z}(\mathrm{Sol}_1^4) \cong \mathbb{R}$, and the projection map,

$$G \to G/\mathcal{Z}(G) \cong \mathrm{Sol}^3,$$

carries $\widetilde{\Gamma}_{\mathcal{S}}$ to a lattice of Sol^3, isomorphic to $\Gamma_{\mathcal{S}}$, for some $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{trace}(\mathcal{S}) > 2$. Thus, $\widetilde{\Gamma}_{\mathcal{S}}$ is the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}_{\mathcal{S}} \longrightarrow \Gamma_{\mathcal{S}} \longrightarrow 1.$$

As is well known, such central extensions of $\mathbb{Z}$ by $\Gamma_{\mathcal{S}}$ are classified by the second cohomology group $H^2(\Gamma_{\mathcal{S}};\mathbb{Z})$.

Theorem 5.1. *Let $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{trace}(\mathcal{S}) > 2$. There is a one-one correspondence between the equivalence classes of all central extensions*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma_{\mathcal{S}} \longrightarrow 1$$

and the group $\mathbb{Z} \oplus$ Coker$(\mathcal{S} - I)$. Note $\mathrm{Coker}(\mathcal{S} - I)$ *is finite.*

Proof. Recall $\Gamma_{\mathcal{S}} = \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$. Then

$$\begin{aligned}
H^2(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z}) &= \mathrm{Free}\left(H_2(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z})\right) \oplus \mathrm{Torsion}\left(H_1(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z})\right) \\
&= \mathbb{Z} \oplus (\mathbb{Z}^2/(\mathcal{S} - I)\mathbb{Z}^2) = \mathbb{Z} \oplus \mathrm{Coker}(\mathcal{S} - I).
\end{aligned}$$
☐

For $\{q, (m_1, m_2)\} \in \mathbb{Z} \oplus \mathrm{Coker}(\mathcal{S} - I)$, denote the corresponding extension $\widetilde{\Gamma}$ by $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ whose presentation is given in Lemma 5.2. We show that $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ with $q \neq 0$ embeds as a lattice in Sol_1^4 (when $q = 0$, $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ embeds into $\mathrm{Sol}^3 \times \mathbb{R}$). An $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$ produces P and Δ, where $P \in \mathrm{SL}(2,\mathbb{R})$ diagonalizes $\mathcal{S}$, $P\mathcal{S}P^{-1} = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}$, $\frac{1}{\lambda} < 1 < \lambda$. We had the embedding of $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ into Sol^3 in (2.1):

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, u\right) \longmapsto \left(P\begin{bmatrix} x \\ y \end{bmatrix}, u\ln(\lambda)\right).$$

The quotient of Sol_1^4 by its center is isomorphic to Sol^3 by the projection

$$\begin{bmatrix}1 & e^u x & z\\ 0 & e^u & y\\ 0 & 0 & 1\end{bmatrix} \longmapsto \left(\begin{bmatrix}x\\ y\end{bmatrix}, u\right).$$

Under this projection, we will find all lattices of Sol_1^4 projecting to $\Gamma_{\mathcal{S}}$. Let

$$(5.1)\quad \begin{aligned} \mathbf{e}_1 &= \left(\begin{bmatrix}1\\ 0\end{bmatrix}, 0\right) \longmapsto (P\mathbf{e}_1, 0) \longmapsto \mathbf{t}_1 = \begin{bmatrix}1 & p_{11} & c_1\\ 0 & 1 & p_{21}\\ 0 & 0 & 1\end{bmatrix},\\ \mathbf{e}_2 &= \left(\begin{bmatrix}0\\ 1\end{bmatrix}, 0\right) \longmapsto (P\mathbf{e}_2, 0) \longmapsto \mathbf{t}_2 = \begin{bmatrix}1 & p_{12} & c_2\\ 0 & 1 & p_{22}\\ 0 & 0 & 1\end{bmatrix},\\ \mathbf{e}_3 &= \left(\begin{bmatrix}0\\ 0\end{bmatrix}, 1\right) \longmapsto (0, \ln(\lambda)) \longmapsto \mathbf{t}_3 = \begin{bmatrix}1 & 0 & c_3\\ 0 & \lambda & 0\\ 0 & 0 & 1\end{bmatrix},\\ \mathbf{t}_4 &= \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \end{aligned}$$

where c_i's are to be determined. Then $[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4$ (regardless of the c_i's).

Lemma 5.2. *For any integers q, m_1, m_2, there exist unique c_1, c_2 for which $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}\}$ forms a group $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ with the presentation*

$$\begin{aligned}\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \mid [\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4,\ &\mathbf{t}_4 \text{ is central},\\ &\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{11}}\mathbf{t}_2^{\sigma_{21}}\mathbf{t}_4^{\frac{m_1}{q}},\\ &\mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{12}}\mathbf{t}_2^{\sigma_{22}}\mathbf{t}_4^{\frac{m_2}{q}}\rangle.\end{aligned}$$

Consequently, $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is solvable and contains $\Gamma_q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4^{\frac{1}{q}}\rangle$ as its discrete nil-radical, where Γ_q is a lattice of Nil.

Proof. We only need to verify the last two equalities. But they become a system of equations on c_i's

$$(5.2)\quad \begin{aligned}(1-\sigma_{11})c_1 - \sigma_{21}c_2 &= \frac{m_1}{q} - \frac{\sigma_{21}(\sigma_{12} + 1 - \sigma_{11} + \sigma_{11}\sqrt{T^2-4})}{2\sqrt{T^2-4}},\\ -\sigma_{12}c_1 + (1-\sigma_{22})c_2 &= \frac{m_2}{q} + \frac{\sigma_{12}(\sigma_{21} + 1 - \sigma_{22} - \sigma_{22}\sqrt{T^2-4})}{2\sqrt{T^2-4}},\end{aligned}$$

where $T = \sigma_{11} + \sigma_{22}$. Since $I - \mathcal{S}$ is non-singular, there exists a unique solution for c_1, c_2. □

Equation (5.2) also shows the cohomology classification. Suppose $\{c_1, c_2\}$ and $\{c_1', c_2'\}$ are solutions for the equations with $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively. Then $(c_1' - c_1, c_2' - c_2) \in \left(\frac{1}{q}\mathbb{Z}\right)^2$ if and only if $(m_1' - m_1, m_2' - m_2) \in \operatorname{Coker}(\mathcal{S}^T - I) \cong \operatorname{Coker}(\mathcal{S} - I)$. This happens if and only if $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1',m_2')}$.

Remark 5.3. (1) Note that any lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ of Sol_1^4 projects to the standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3.

(2) In Lemma 5.2, the c_i's are independent of choice of P because equation (5.2) has coefficients only from the matrix $\mathcal{S}$.

(3) Notice that c_3 does not show up in the presentation of the lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, so c_3 can be changed without affecting the isomorphism type of the lattice.

Notation 5.4 (Standard lattice)**.** The lattice generated by

$$\mathbf{t}_1 = \begin{bmatrix} 1 & p_{11} & c_1 \\ 0 & 1 & p_{21} \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 & p_{12} & c_2 \\ 0 & 1 & p_{22} \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 1 & 0 & c_3 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_4^{\frac{1}{q}} = \begin{bmatrix} 1 & 0 & \frac{1}{q} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $c_3 = 0$, is called a *standard lattice* of Sol_1^4.

Therefore, any lattice of Sol_1^4 is isomorphic to a standard lattice. However, a non-standard lattice (i.e., $c_3 \neq 0$) will be needed when we consider finite extensions of $\widetilde{\Gamma}_{\mathcal{S}}$, specifically, in the holonomy $\mathbb{Z}_4$ case.
The following lemma on lattices of Sol_1^4 will be needed in the next section.

Lemma 5.5. *Let* $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ *be a lattice of* Sol_1^4*, embedded as in assignment* (5.1)*.*

(a) *Let* $r_1, r_2 \in \mathbb{Q}$*. Then*

$$\mathbf{t}_1^{r_1}\mathbf{t}_2^{r_2} = \mathbf{t}_2^{r_2}\mathbf{t}_1^{r_1}\mathbf{t}_4^{r_1 r_2}.$$

(b) *Let* $a_1, a_2 \in \mathbb{Q}$*. Then, for* $\bar{A} = \pm 1$*,*

$$\mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}} = \mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v}, \text{ where } \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \mathcal{S}^{\bar{A}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \text{ and } v \in \mathbb{Q}. \tag{5.3}$$

Proof. For part (a), we compute that $[\mathbf{t}_1^{r_1}, \mathbf{t}_2^{r_2}] = \mathbf{t}_4^{r_1 r_2 \det(P)} = \mathbf{t}_4^{r_1 r_2}$.

For part (b), the definition of $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ shows that $\left[\begin{smallmatrix} l_1 \\ l_2 \end{smallmatrix}\right] = \mathcal{S}^{\bar{A}} \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$. We must show that v in (5.3) is rational.

Because a_1 and a_2 are rational, there is a positive integer n so that $na_1, na_2 \in \mathbb{Z}$. By part (a),

$$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n = \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} \cdots \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} (n \text{ times}) = \mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_4^{u'}, \text{ for some } u' \in \mathbb{Q}.$$

Therefore,

$$\mathbf{t}_3^{\bar{A}}(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n\mathbf{t}_3^{-\bar{A}} = \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_4^{u'}\mathbf{t}_3^{-\bar{A}}$$

$$\begin{aligned}&= \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_3^{-\bar{A}}\mathbf{t}_4^{u'}\\ &= \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{u} \text{ for some } n_1, n_2 \in \mathbb{Z}, \text{ and some } u \in \mathbb{Q},\end{aligned}$$

where the last equality follows from that na_1 and na_2 are integers, together with the relations in Lemma 5.2.

On the other hand, we have that

$$\begin{aligned}\mathbf{t}_3^{\bar{A}}(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n\mathbf{t}_3^{-\bar{A}} &= \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}}\cdots\mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}}(n \text{ times})\\ &= \mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v}\cdots\mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v}(n \text{ times})\\ &= \mathbf{t}_1^{nl_1}\mathbf{t}_2^{nl_2}\mathbf{t}_4^{nv+w} \text{ for some } w \in \mathbb{Q},\end{aligned}$$

where v is from (5.3) and the last equality follows from part (a).
Consequently, we have

$$\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{u} = \mathbf{t}_1^{nl_1}\mathbf{t}_2^{nl_2}\mathbf{t}_4^{nv+w}.$$

This forces $n_1 = nl_1$ and $n_2 = nl_2$. Therefore, $nv + w = u$. Since $n \in \mathbb{Z}$, $u, w \in \mathbb{Q}$, it follows that $v \in \mathbb{Q}$. □

6. Crystallographic groups of $\mathbf{Sol_1^4}$

Let $\Pi \subset \mathrm{Sol_1}^4 \rtimes C$ be a crystallographic group of $\mathrm{Sol_1}^4$, where C is a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol_1}^4)$. As all maximal compact subgroups of $\mathrm{Sol_1}^4$ are conjugate, we can assume that C is the maximal compact subgroup

$$D_4 = \left\langle \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \right\rangle$$

of $\mathrm{Aut}(\mathrm{Sol_1}^4)$ (Proposition 3.1), the action of which on $\mathrm{Sol_1}^4$ is described in Proposition 1.2. As noted in Proposition 3.1, $\mathrm{Sol_1}^4$ satisfies generalization of Bieberbach's Theorems. Furthermore, as shown below, we can conjugate Π in $\mathrm{Aff}(\mathrm{Sol_1}^4)$ so that the lattice inside Π is some $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, embedded in $\mathrm{Sol_1}^4$ as in assignment (5.1).

Proposition 6.1. (1) *Any crystallographic group Π' of $\mathrm{Sol_1}^4$ can be conjugated in $\mathrm{Aff}(\mathrm{Sol_1}^4)$ to $\Pi \subset \mathrm{Sol_1}^4 \rtimes D_4$ so that*

$$\Pi \cap \mathrm{Sol_1}^4 = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle,$$

where

$$\mathbf{t}_1 = \begin{bmatrix}1 & p_{11} & c_1\\ 0 & 1 & p_{21}\\ 0 & 0 & 1\end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix}1 & p_{12} & c_2\\ 0 & 1 & p_{22}\\ 0 & 0 & 1\end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix}1 & 0 & c_3\\ 0 & \lambda & 0\\ 0 & 0 & 1\end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(2) *The holonomy group Φ is generated by at most two elements of D_4, and thus Π is generated by $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$ and at most two isometries of the form $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, for $A \in D_4$ and real numbers a_i.*

Proof. Let $\widetilde{\Gamma} = \Pi \cap \mathrm{Sol}_1^4$. This lattice must meet the center of Sol_1^4 in a lattice: $\widetilde{\Gamma} \cap \mathcal{Z}(\mathrm{Sol}_1^4)$ is a lattice of $\mathcal{Z}(\mathrm{Sol}_1^4)$, say generated by $\mathbf{t}_4^{\frac{1}{q}}$. Also $\widetilde{\Gamma} \cap \mathrm{Nil}$ is a lattice of the nilradical Nil, so we can find generators $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4^{\frac{1}{q}} \rangle$ of this lattice as given in the statement. The remaining one generator for the lattice $\widetilde{\Gamma}$ must project down to a generator of the quotient $\widetilde{\Gamma}/\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4^{\frac{1}{q}} \rangle \cong \mathbb{Z}$. It must be of the form

$$\mathbf{t}_3'' = \begin{bmatrix} 1 & a & c_3 \\ 0 & \lambda & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Conjugation by $\begin{bmatrix} 1 & \frac{a}{1-\lambda} & 0 \\ 0 & \lambda & -\frac{b\lambda}{1-\lambda} \\ 0 & 0 & 1 \end{bmatrix}$ maps $\mathbf{t}_3''$ to the form of $\mathbf{t}_3$. Note $\widetilde{\Gamma}/\mathcal{Z}(\widetilde{\Gamma})$ is a lattice of Sol^3, isomorphic to $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$, for $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{tr}(\mathcal{S}) > 2$, where $P = (p_{ij})$ diagonalizes $\mathcal{S}$. As in the case of Sol^3 (Proposition 2.3), we can assume $\det(P) = 1$, so that $[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4$. Therefore, any lattice is conjugate to a lattice $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$ of the desired form. □

Henceforth we will assume all Sol_1^4-crystallographic groups are embedded in $\mathrm{Sol}_1^4 \rtimes D_4$ as in Proposition 6.1. However, we will see that we can always take $c_3 = 0$, except possibly when the holonomy of Π, Φ, is $\mathbb{Z}_4$. Because lattices of Sol_1^4 project to lattices of Sol^3, the projections $\mathrm{Sol}_1^4 \to \mathrm{Sol}^3$ and $\mathrm{Aut}(\mathrm{Sol}_1^4) \to \mathrm{Aut}(\mathrm{Sol}^3)$ induce a projection $\mathrm{Sol}_1^4 \rtimes D_4 \to \mathrm{Sol}^3 \rtimes D_4$ which carries a Sol_1^4-crystallographic group Π to a Sol^3-crystallographic group Q. Furthermore, when Π is embedded in $\mathrm{Sol}_1^4 \rtimes D_4$ as in Proposition 6.1, the lattice $\widetilde{\Gamma}_{\mathcal{S}} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ projects to a standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3. That is, we have the following commuting diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \frac{1}{q}\mathbb{Z} = \langle \mathbf{t}_4^{\frac{1}{q}} \rangle & = & \frac{1}{q}\mathbb{Z} = \langle \mathbf{t}_4^{\frac{1}{q}} \rangle & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Gamma}_{\mathcal{S}} & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Our goal is finding all crystallographic groups Π of Sol_1^4 which project down to Q. In general, it is *not* true that there exists Π fitting the above commutative diagram of exact sequences without making the kernel $\langle \mathbf{t}_4 \rangle$ finer to $\langle \mathbf{t}_4^{1/q} \rangle$. That is, even though $\widetilde{\Gamma}_{\mathcal{S}}$ always exists, for Π to exist, sometimes the kernel $\mathbb{Z} = \langle \mathbf{t}_4 \rangle$ needs to be "inflated" to $\frac{1}{q}\mathbb{Z} = \langle \mathbf{t}_4^{1/q} \rangle$. It turns out that, after appropriate inflation, an extension Π always exists.

The abstract kernel of $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ is given by, for $A \in \Phi$,

$$\mu(\alpha) : \Gamma_{\mathcal{S}} \to \Gamma_{\mathcal{S}}, \text{ where } \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in Q.$$

Here $\mu(\alpha)$ denotes conjugation in $\mathrm{Sol}^3 \rtimes D_4$. Suppose in Proposition 6.1, we have fixed the c_i, as well as set $q = 1$, thus fixing the lattice

$$\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle \hookrightarrow \mathrm{Sol}_1^4.$$

For any generator $A \in \Phi$, let

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A) = (a, A).$$

We consider the effect that conjugation by α has on $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$. Note that conjugation by α is independent of a_4. We have the relations:

$$\begin{aligned}
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2}\mathbf{t}_4^{v_1}, \text{ where } \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} 1 \\ 0 \end{bmatrix},\\
&\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{v_2}, \text{ where } \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} 0 \\ 1 \end{bmatrix},\\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix},\\
&\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4^{\hat{A}}.
\end{aligned}$$

We will need the following lemma on the v_i.

Lemma 6.2. *The numbers v_1 and v_2 are rational. Furthermore, we can adjust c_3 so that v_3 is rational.*

Proof. Note that the image of $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$ under conjugation by α,

$$\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}) = \alpha\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}\alpha^{-1},$$

is a lattice of Sol_1^4 lifting the standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3.

All such lifts are given in Lemma 5.2. In equation (5.2), we see that for any two solutions c_1, c_2 and c_1', c_2', both $c_1' - c_1$ and $c_2' - c_2$ must be rational. Thus v_1 and v_2 are rational numbers.

From Proposition 1.2, $A \in \Phi \subseteq D_4$ can be viewed as an element of $\mathrm{GL}(2, \mathbb{Z})$. The induced action of A on $\mathcal{Z}(\mathrm{Sol}_1^4)$ is multiplication by $\hat{A} = \det(A)$, and the induced action of A on $\mathrm{Sol}_1^4/\mathrm{Nil} \cong \mathbb{R}$ is multiplication by $\bar{A}$. We need to understand the action of A on the generator $\mathbf{t}_3$ of $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$. Let $\hat{\mathbf{t}}_3$ denote $\mathbf{t}_3$ with the $(1,3)$-slot set to be zero, so that $\mathbf{t}_3 = \hat{\mathbf{t}}_3\mathbf{t}_4^{c_3}$:

$$A(\mathbf{t}_3) = A(\hat{\mathbf{t}}_3\mathbf{t}_4^{c_3}) = A(\hat{\mathbf{t}}_3)A(\mathbf{t}_4^{c_3}) = \hat{\mathbf{t}}_3^{\bar{A}}\mathbf{t}_4^{\hat{A}c_3} = (\hat{\mathbf{t}}_3^{\bar{A}}\mathbf{t}_4^{\bar{A}c_3})(\mathbf{t}_4^{-\bar{A}c_3}\mathbf{t}_4^{\hat{A}c_3})$$

$$= \mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}.$$

In order to show

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{6.1}$$

we need only consider two cases, either $a_3 = \frac{1}{2}$ or $a_3 = 0$.

First, consider the case when $a_3 = \frac{1}{2}$. Then A must be diagonal, so that $\bar{A} = +1$. By Corollary 4.3, we can take $a_1 = a_2 = 0$ so that $\alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)$, so

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_3^{\frac{1}{2}}A(\mathbf{t}_3)\mathbf{t}_3^{-\frac{1}{2}} = \mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}\mathbf{t}_3^{-\frac{1}{2}} = \mathbf{t}_3\mathbf{t}_4^{(\hat{A}-1)c_3}.$$

Since $\hat{A} = \pm 1$, there is a choice of c_3 which makes $(\hat{A}-1)c_3 \in \mathbb{Q}$.
Now consider the case $a_3 = 0$, so that $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)$. We compute:

$$\begin{aligned}\alpha\mathbf{t}_3\alpha^{-1} &= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\mathbf{t}_3)\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} = \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\left(\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}\right)\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\ &= \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1}\mathbf{t}_3^{-\bar{A}}\right)\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}.\end{aligned}$$

Now by Lemma 5.5, and using that a_1, a_2 are rational, we have

$$\begin{aligned}\left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1}\mathbf{t}_3^{-\bar{A}}\right)\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3} &= \left(\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{u}\right)\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3} \\ &= \mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{u+(\hat{A}-\bar{A})c_3}\end{aligned}$$

for a rational number u. Equating this with equation (6.1), we obtain

$$\mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3} = \mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{u+(\hat{A}-\bar{A})c_3}.$$

Now $w_1 = b_1$ and $w_2 = b_2$ is forced. Therefore, $v_3 = u + (\hat{A}-\bar{A})c_3$. Because $\hat{A} = \pm 1$, $\bar{A} = \pm 1$, and u is rational, c_3 can always be chosen so that v_3 is rational. □

Proposition 6.3. *Let $Q \hookrightarrow \mathrm{Sol}^3 \rtimes D_4$ be a crystallographic group of Sol^3 with lattice $\Gamma_{\mathcal{S}}$. Then there exists a lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}\rangle$ of Sol_1^4, projecting to $\Gamma_{\mathcal{S}}$, for which the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ induces $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$.*

Proof. For any integer $q > 0$, we add a finer generator of the central direction to the group $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$ to obtain $\langle \widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}, \mathbf{t}_4^{\frac{1}{q}}\rangle = \widetilde{\Gamma}_{(\mathcal{S};q,qn_1,qn_2)}$.

Now, for each generator $A \in \Phi$, the v_i in Proposition 6.2 are rational. Therefore, for q large enough, $\widetilde{\Gamma}_{(\mathcal{S};q,qn_1,qn_2)}$ is invariant under conjugation by $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, for each $A \in \Phi$. As this conjugation is independent of lift of $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in \mathrm{Sol}^3 \rtimes D_4$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A) \in \mathrm{Sol}_1^4 \rtimes D_4$, with $m_1 = qn_1$ and $m_2 = qn_2$, we obtain an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$. □

Proposition 6.4. *Let $Q \hookrightarrow \mathrm{Sol}^3 \rtimes D_4$ be a crystallographic group of Sol^3 containing lattice $\Gamma_{\mathcal{S}}$. Assume that the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ induces $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$. Then for some $p > 0$, there exists Π which fits the following commuting diagram:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \frac{1}{pq}\mathbb{Z} & = & \frac{1}{pq}\mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Gamma}_{(\mathcal{S};pq,pm_1,pm_2)} & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Proof. Since the center of $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is $\frac{1}{q}\mathbb{Z}$ and Φ is finite, $H^3(\Phi; \frac{1}{q}\mathbb{Z})$ is finite. This means the obstruction class to the existence of the extension vanishes if we use $\frac{1}{pq}\mathbb{Z}$ for the coefficients, for some $p > 0$. That is, it vanishes inside $H^3(\Phi; \frac{1}{pq}\mathbb{Z})$. Thus, with such pq, the center of $\widetilde{\Gamma}_{(\mathcal{S};pq,pm_1,pm_2)}$ is $\frac{1}{pq}\mathbb{Z}$, and an extension Π exists. □

So we can assume that after appropriate inflation, there exists an extension Π with lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ for some $q > 0$. The Seifert Construction will show that such an abstract extension actually embeds in $\mathrm{Sol}_1{}^4 \rtimes D_4$ as a crystallographic group. By taking pq as a new q, we have:

Theorem 6.5. *Let $\widetilde{\Gamma}_{\mathcal{S}} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ be a lattice of $\mathrm{Sol}_1{}^4$, and*

$$1 \longrightarrow \widetilde{\Gamma}_{\mathcal{S}} \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1$$

be an extension of $\widetilde{\Gamma}_{\mathcal{S}}$ by a finite group Φ from Proposition 6.4. Then there exists an injective homomorphism

$$\theta : \Pi \to \mathrm{Sol}_1{}^4 \rtimes D_4 \subset \mathrm{Sol}_1{}^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1{}^4)$$

carrying $\widetilde{\Gamma}_{\mathcal{S}}$ onto a standard lattice. Such θ is unique up to conjugation by an element of $\mathrm{Sol}_1{}^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1{}^4)$.

Proof. This is a consequence of the Seifert construction, since $\mathrm{Sol}_1{}^4$ is completely solvable. We can apply [11, Theorem 7.3.2] with $G = \mathrm{Sol}_1{}^4$ and $W = \{\text{point}\}$. Since Φ is finite, the homomorphism $\Pi \to \mathrm{Out}(\widetilde{\Gamma}) \to \mathrm{Out}(\mathrm{Sol}_1{}^4)$ has finite image in $\mathrm{Out}(\mathrm{Sol}_1{}^4)$, and it lifts back to a finite subgroup C of $\mathrm{Aut}(\mathrm{Sol}_1{}^4)$.

But this C can be conjugated into $D_4 \subset \mathrm{Aut}(\mathrm{Sol}_1^4)$, a maximal compact subgroup. Consequently, we have a commuting diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{\Gamma}_{\mathcal{S}} & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}_1^4 \rtimes D_4 & \longrightarrow & D_4 & \longrightarrow & 1 \end{array}$$

The homomorphism $\mathit{\Pi} \to \mathrm{Sol}_1^4 \rtimes D_4$ is injective since the abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$ from Proposition 6.4 is injective. The essence of the argument is showing that the cohomology set $H^2(\Phi; \mathrm{Sol}_1^4)$ is trivial for any finite group Φ. The uniqueness is a result of [11, Corollary 7.7.4]. It also comes from $H^1(\Phi; \mathrm{Sol}_1^4) = 0$. □

After inflation, the Seifert Construction produces a crystallographic group of Sol_1^4. Often we can assume that $c_3 = 0$, that is, $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is a standard lattice of Sol_1^4. Recall that $\mathrm{Aut}(\mathrm{Sol}_1^4) = \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$ (Proposition 1.2), where $\hat{k} \in \mathbb{R}$ acts by

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^u x & z + ku \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.$$

We have the following:

Theorem 6.6. *For all holonomy groups, except $\mathbb{Z}_4$, a crystallographic group $\mathit{\Pi}$ of Sol_1^4 embeds into $\mathrm{Sol}_1^4 \rtimes D_4$ in such a way that $\mathit{\Pi} \cap \mathrm{Sol}_1^4$ is a standard lattice ($c_3 = 0$).*

Proof. Let e denote the identity element of Sol_1^4. For the statement concerning c_3, conjugation by $(e, \hat{k})$ with $k = -\frac{c_3}{\ln \lambda}$ sets $c_3 = 0$ in $\mathbf{t}_3$. However, this conjugation moves D_4 to $\hat{k} D_4 \hat{k}^{-1}$.

Suppose every $A \in \Phi$ satisfies $\bar{A}\hat{A} = +1$. Since such A commute with $\hat{k}$, conjugation by $(e, \hat{k})$ leaves the holonomy group Φ inside D_4 while setting $c_3 = 0$ in $\mathbf{t}_3$. This applies to, from the list of Theorem 3.3, all the groups lifting Sol^3-crystallographic groups of type `(2a)`, `(2b)`, `(3)`, `(3i)`, `(6a)`, `(6ai)`, `(6b)`, and `(6bi)`.

Suppose Φ contains $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then Corollary 4.3 and Lemma 6.7 below show that a generator α of $\mathit{\Pi}$ projecting to $A \in \Phi$ can be conjugated to $\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)$ (so that $a_1 = a_2 = a_4 = 0$). Then, we shall show that $\mathbf{t}_3 = \hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3}$ can be replaced by $\hat{\mathbf{t}}_3$ (where $\hat{\mathbf{t}}_3$ is $\mathbf{t}_3$ with $c_3 = 0$).

$$\begin{aligned} \alpha^2 &= (\mathbf{t}_3^{\frac{1}{2}}, A)^2 = ((\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}}, A)^2 = (\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}} A((\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}}) \\ &= \hat{\mathbf{t}}_3^{\frac{1}{2}} \mathbf{t}_4^{\frac{c_3}{2}} \cdot \hat{\mathbf{t}}_3^{\frac{1}{2}} \mathbf{t}_4^{-\frac{c_3}{2}} = \hat{\mathbf{t}}_3. \end{aligned}$$

Thus $\hat{\mathbf{t}}_3 = \alpha^2 \in \mathit{\Pi}$, and we can take $\hat{\mathbf{t}}_3$ instead of $\mathbf{t}_3$ as a generator for the same group (which is apparently redundant since α is in the group already). This shows that $\mathbf{t}_4^{c_3} = \alpha^{-2}\mathbf{t}_3 \in \Pi$ must be a multiple of $\frac{1}{q}$, and we can take $c_3 = 0$. From the list in Theorem 3.3, the groups `(1)`, `(5)`, `(7)` and `(7i)` contain such an A in the holonomy.

The only case that is not covered by these two cases is when $\Phi = \mathbb{Z}_4$ (type `(4)` in the list), which is discussed below in our main classification (Theorem 6.13). □

Lemma 6.7. *If* $\det(A) = -1$*, by conjugation,* a_4 *can be made* 0*.*

Proof. Suppose $\det(A) = -1$. Conjugation by $\mathbf{t}_4^{-\frac{a_4}{2}}$ fixes the lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, and moves $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)$. □

Proposition 6.8 (Fixing a_4, b_4)**.** *Consider the commuting diagram in* Proposition 6.4. *Given* Q *and integers* q, m_1, m_2*, we had* $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$*. The only thing that remains for the construction of* Π *is fixing* a_4, b_4*. As is known, all the extensions* Π *in the short exact sequence*

$$1 \to \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} \to \Pi \to \Phi \to 1$$

are classified by $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})) = H^2(\Phi;\mathbb{Z})$*. When* $\Phi = \langle A\rangle$*,*

$$H^2(\mathbb{Z}_p;\mathbb{Z}) = \begin{cases} 0, & \text{if } \hat{A} = -1; \\ \mathbb{Z}_p, & \text{if } \hat{A} = 1, \end{cases}$$

see [12, Theorem 7.1, p. 122].

In actual calculation, this becomes an equation

$$\alpha^p = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{k_4}$$

for integers n_i *and* $k_4 = \frac{i}{q}$*,* $i = 0, 1, \ldots, p-1$*.*

Remark 6.9. When $\Phi = \langle A, B\rangle$ is not cyclic, $\hat{A} = \hat{B} = +1$ never happens, so we can set one of a_4, b_4 to zero. Thus, $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is cyclic for all Φ.

6.10 (**Detecting Torsion in $\mathbf{Sol_1^4}$-Crystallographic Groups).** Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of $\mathrm{Sol_1}^4$ (which projects to a lattice $\Gamma_{\mathcal{S}}$ of Sol^3), the short exact sequence

$$1 \to \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}) \to \widetilde{\Gamma}_{\mathcal{S}} \to \Gamma_{\mathcal{S}} \to 1$$

induces an S^1-bundle over the solvmanifold $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$,

$$S^1 \to \widetilde{\Gamma}_{\mathcal{S}}\backslash(\mathrm{Sol_1}^4) \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3.$$

The following two lemmas will be useful for determining when a $\mathrm{Sol_1}^4$-crystallographic group is torsion free.

Lemma 6.11. *Let* $\widetilde{\Gamma}_{\mathcal{S}}$ *be a lattice of* $\mathrm{Sol_1}^4$*, projecting to a standard lattice* $\Gamma_{\mathcal{S}}$ *of* Sol^3*, and suppose that for* $\alpha \in \mathrm{Sol_1}^4 \rtimes D_4$*, the group* $\Pi = \langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ *is crystallographic. Let* $\bar{\alpha}$ *denote the projection of* α *to* $\mathrm{Sol}^3 \rtimes D_4$*. When the automorphism part of* α *acts as a reflection on the center of* $\mathrm{Sol_1}^4$*,* Π *is torsion free if and only if* $\langle\Gamma_{\mathcal{S}}, \bar{\alpha}\rangle \subset \mathrm{Sol}^3 \rtimes D_4$ *is torsion free.*

Proof. Evidently, if $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ is torsion free, then Π must be torsion free. For the converse, suppose that $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ has torsion. In this case, the action of $\bar{\alpha}$ on the solvmanifold $\Gamma_{\mathcal{S}} \backslash \mathrm{Sol}^3$ must fix a point. Observe that the action of α on the solvmanifold $\widetilde{\Gamma}_{\mathcal{S}} \backslash \mathrm{Sol_1}^4$ is S^1 fiber preserving. Therefore, a circle fiber is left invariant under the action of α. Since α acts as reflection on the fiber, α must fix a point. Since the action of α fixes a point on $\widetilde{\Gamma}_{\mathcal{S}} \backslash \mathrm{Sol_1}^4$, the action of Π fixes a point on $\mathrm{Sol_1}^4$. Thus, Π has torsion. □

Lemma 6.12. *Let Π be a crystallographic group of $\mathrm{Sol_1}^4$ with lattice $\widetilde{\Gamma}_{\mathcal{S}}$. If $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A) \in \Pi$ satisfies $a_3 = \frac{1}{2}$ and $\bar{A} = 1$, then $\gamma\alpha$ is infinite order for any $\gamma \in \widetilde{\Gamma}_{\mathcal{S}}$.*

Proof. Note that A is necessarily of order 2. Let $\mathrm{pr} : \mathrm{Sol_1}^4 \to \mathbb{R}$ denote the quotient homomorphism of $\mathrm{Sol_1}^4$ by its nil-radical Nil. Write $\gamma \in \widetilde{\Gamma}_{\mathcal{S}}$ as $\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{n_4}$. Application of pr to $(\gamma\alpha)^2$ yields

$$\mathrm{pr}(\gamma\alpha)^2 = 2n_3 + 1,$$

from which we infer $\gamma\alpha$ is of infinite order. □

We are now ready to give our main classification of $\mathrm{Sol_1}^4$-crystallographic groups. Following Proposition 6.1, a crystallographic group

$$\Pi \subset \mathrm{Sol_1}^4 \rtimes D_4$$

of $\mathrm{Sol_1}^4$ is generated by a lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$ of $\mathrm{Sol_1}^4$, together with at most two generators of the form

$$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A), (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B),$$

where A, B generate the holonomy group $\Phi \subset D_4$. The $\mathrm{Sol_1}^4$-crystallographic group Π projects to a Sol^3-crystallographic group Q. We view Q as an extension

$$1 \to \mathbb{Z}^2 \to Q \to \mathbb{Z}_\Phi \to 1,$$

and Theorem 6.6 classifies all possible $\mathbb{Z}_\Phi$ and abstract kernels $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$. We organize the $\mathrm{Sol_1}^4$-crystallographic groups according to which $\mathbb{Z}_\Phi$ and $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ in Theorem 6.6 they project to. Theorem 6.13 also classifies Sol^3-crystallographic groups, by projecting from $\mathrm{Sol_1}^4 \rtimes D_4$ to $\mathrm{Sol}^3 \rtimes D_4$.

Theorem 6.13 (Classification of $\mathrm{Sol_1}^4$-Crystallographic Groups)**.** *The following is a complete list of crystallographic groups Π of $\mathrm{Sol_1}^4$, generated by a lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ of $\mathrm{Sol_1}^4$, together with at most two generators of the form*

$$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A), (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B).$$

They are organized according to which $\mathbb{Z}_\Phi$ and $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ they project to (see Theorem 6.6). This determines the exponents a_3, b_3.

We find equations describing $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$, and thus classifying $\mathbf{a} = \left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$, $\mathbf{b} = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$. *In general, $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ depends on $\mathcal{S}$.*

By Proposition 6.4 and Theorem 6.5, for sufficiently large q, an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$ is induced, with vanishing obstruction to the existence of Π in $H^3(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}))$. The exponents on $\mathbf{t}_4$, a_4 and b_4, are classified by the group $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}))$.

In all cases, except, $\Phi = \mathbb{Z}_4$, we can take $c_3 = 0$ in the lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ of Π (Theorem 6.6). In the $\mathbb{Z}_4$ holonomy case, we have two different (up to isomorphism) choices for c_3.

Whenever the holonomy group contains an automorphism of Sol_1^4 which is represented by an off-diagonal matrix, the orbifold $\Pi\backslash\mathrm{Sol}_1^4$ is non-orientable. We give precise criterion for Π to be torsion free. When Π is torsion free, $\Pi\backslash\mathrm{Sol}_1^4$ is an infra-solvmanifold of Sol_1^4.

By projecting each Sol_1^4-crystallographic group Π to a crystallographic group $Q \subset \mathrm{Sol}^3 \rtimes D_4$, we also obtain a classification of Sol^3-crystallographic groups.

(0) $\Phi = $ *trivial*

$$\Pi = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}.$$

- *Torsion free.*

(1) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
$\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, *and* $\mathcal{S} = nK + I$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle.$$

- *$H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ is trivial so that* $\mathbf{a} = \mathbf{0}$.
- *$H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.*
- *Both Q and Π are torsion free.*

(2a) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
$\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)\rangle.$$

- $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I-\mathcal{S})\}/\{2\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I-\mathcal{S})\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. *There are two choices for a_4, the solutions of* $\alpha^2 = \mathbf{t}_4^{\frac{i}{q}}$ $(i = 0, 1)$.
- *Q has torsion, Π is torsion free when $i = 1$ and q is even.*

(2b) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
$\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$, *and* $\mathcal{S} = nK - I$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)\rangle.$$

- *$H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$ is trivial so that* $\mathbf{a} = \mathbf{0}$.

• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ *or* $\frac{1}{2q}$.
• *Both* Q *and* Π *are torsion free.*

(3) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
$\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{12} = -\sigma_{21}$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle.$$

• $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I-\mathcal{S}), a_2 \equiv -a_1\}/$ $\left\{\begin{bmatrix} v_1 - v_2 \\ v_2 - v_1 \end{bmatrix} \mid \mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})\right\} \subseteq \mathbb{Z}_2$.
• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ *is trivial.*
• *Both* Q *and* Π *have torsion.*

(3i) $\Phi = \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
$\varphi(\bar{\alpha}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and* $\sigma_{11} = \sigma_{22}$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle.$$

• $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I-\mathcal{S}), 2a_1 \equiv 0\}/$ $\left\{\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix} \mid \mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})\right\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$.
• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ *is trivial.*
• *Both* Q *and* Π *are torsion free if and only if* $a_1 \equiv \frac{1}{2}$ *and* $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ *for any* $n \in \mathbb{Z}$.

(4) $\Phi = \mathbb{Z}_4 : A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_4 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.
$\varphi(\bar{\alpha}) = A$, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$ *and symmetric.*

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)\rangle.$$

• *There are two choices for* c_3 *in* $\mathbf{t}_3$. *They are solutions of* $d = 0$ *or* $d = \frac{1}{q}$ *for* c_3, *where* $\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{(1-\sigma_{22})a_1+\sigma_{12}a_2}\mathbf{t}_2^{\sigma_{21}a_1+(1-\sigma_{11})a_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{d}$. *Each corresponds to a distinct abstract kernel* $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$.
• $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})\}/\{(I - A)\mathbf{a} \mid \mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})\} \subseteq \mathbb{Z}_2$.
• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. *There are 4 choices for* a_4, *the solutions of* $\alpha^4 = \mathbf{t}_4^{\frac{i}{q}}$ $(i = 0, 1, 2, 3)$.
• Q *has torsion,* Π *is torsion free precisely when* $i = 1, 3$ *and* q *is even.*

(5) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 : A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$.
$\varphi(\bar{\alpha}) = -K$, $\varphi(\bar{\beta}) = B$ (1)+(2a)
$\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, *and* $\mathrm{tr}(K) = n > 0$.

$$\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4}, B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{b}\mid\mathbf{b}\in\mathrm{Coker}(I+K)\}/\{2\mathbf{b}\mid\mathbf{b}\in\mathrm{Coker}(I+K)\}\subseteq\mathbb{Z}_2\times\mathbb{Z}_2$.

• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. *There are two choices for* b_4*, the solutions of* $\beta^2=\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$.

• *Q has torsion, Π is torsion free precisely when $i=1$ and q is even.*

(6a) $\Phi=\mathbb{Z}_2\times\mathbb{Z}_2: A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $B=\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=(\mathbb{Z}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar\alpha=(\mathbf{t}_3^0,A),\bar\beta=(\mathbf{t}_3^0,B)\rangle$.
$\varphi(\bar\alpha)=A$, $\varphi(\bar\beta)=B$ (3)+(2a)
$\mathcal{S}\in\mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S})>2$ *and* $\sigma_{12}=-\sigma_{21}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(\mathbf{a},\mathbf{b})\mid\mathbf{a},\mathbf{b}\in\mathrm{Coker}(I-\mathcal{S}),a_2\equiv-a_1,b_1-b_2-2a_1\equiv0\}/\left\{\left(\begin{bmatrix}v_1-v_2\\v_2-v_1\end{bmatrix},2\mathbf{v}\right)\mid\mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\right\}$.

• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. *There are two choices for* b_4*, the solutions of* $\beta^2=\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$.

• *Both Q and Π have torsion.*

(6ai) $\Phi=\mathbb{Z}_2\times\mathbb{Z}_2: A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $B=\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=(\mathbb{Z}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar\alpha=(\mathbf{t}_3^0,A),\bar\beta=(\mathbf{t}_3^0,B)\rangle$.
$\varphi(\bar\alpha)=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$, $\varphi(\bar\beta)=B$ (3i)+(2a)
$\mathcal{S}\in\mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S})>2$ *and* $\sigma_{11}=\sigma_{22}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{(\mathbf{a},\mathbf{b})\mid\mathbf{a},\mathbf{b}\in\mathrm{Coker}(I-\mathcal{S}),2a_1\equiv0,2b_2-2a_2\equiv0\}/\left\{\left(\begin{bmatrix}0\\2v_2\end{bmatrix},2\mathbf{v}\right)\mid\mathbf{v}\in\mathrm{Coker}(I-\mathcal{S})\right\}$.

• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. *There are two choices for* b_4*, the solutions of* $\beta^2=\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$.

• *Q has torsion, Π is torsion free if and only if $i=1$, q is even, and* $a_1\equiv\frac{1}{2}$, $a_2\equiv b_2+\frac{1}{2}$, $b_1\not\equiv\frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}+\frac{1}{2}$, $b_2\not\equiv\frac{(\sigma_{11}+1)(2m+1)}{2\sigma_{12}}+\frac{1}{2}$ *for any* $m,n\in\mathbb{Z}$.

(6b) $\Phi=\mathbb{Z}_2\times\mathbb{Z}_2: A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $B=\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=\mathbb{Z}\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar\alpha=(\mathbf{t}_3^0,A),\bar\beta=(\mathbf{t}_3^{\frac{1}{2}},B)\rangle$.
$\varphi(\bar\alpha)=A$, $\varphi(\bar\beta)=-K$ (3)+(2b)
$\mathcal{S}=nK-I$, *where* $K\in\mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K)=n>2$*;* $k_{12}=-k_{21}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\mid\mathbf{a}\in\mathrm{Coker}(I+K),a_2\equiv-a_1\}/\left\{\begin{bmatrix}v_1-v_2\\v_2-v_1\end{bmatrix}\mid\mathbf{v}\in\mathrm{Coker}(I+K)\right\}\subseteq\mathbb{Z}_2$.

• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. *There are two choices for* b_4, *the solutions of* $\beta^2=\mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$.
• *Both* Q *and* Π *have torsion.*

(6bi) $\Phi=\mathbb{Z}_2\times\mathbb{Z}_2 : A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],\ B=\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=\mathbb{Z}\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}},B)\rangle$.
$\varphi(\bar{\alpha})=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right],\ \varphi(\bar{\beta})=-K$ (3i)+(2b)
$\mathcal{S}=nK-I$, *where* $K\in\mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K)=n>2$; $k_{11}=k_{22}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\mid\mathbf{a}\in\mathrm{Coker}(I+K),2a_1\equiv 0\}/\left\{\begin{bmatrix}0\\2v_2\end{bmatrix}\mid\mathbf{v}\in\mathrm{Coker}(I+K)\right\}\subseteq\mathbb{Z}_2\times\mathbb{Z}_2$.
• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. *There are two choices for* b_4, *the solutions of* $\beta^2=\mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$.
• *Both* Q *and* Π *are torsion free if and only if* $a_1=\frac{1}{2}$ *and* $a_2\not\equiv\frac{(k_{11}-1)(2n+1)}{2k_{12}}$ *for any* $n\in\mathbb{Z}$.

(7) $\Phi=\mathbb{Z}_4\rtimes\mathbb{Z}_2 : A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],B=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=(\mathbb{Z}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}},B)\rangle$.
$\varphi(\bar{\alpha})=A,\ \varphi(\bar{\beta})=-K$ (*includes* (6a)) (3)+(1)
$\mathcal{S}=nK+I$, $K\in\mathrm{GL}(2,\mathbb{Z})$, $\det(K)=-1$, $\mathrm{tr}(K)>0$; $k_{12}=-k_{21}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\mid\mathbf{a}\in\mathrm{Coker}(I-\mathcal{S}),a_2\equiv -a_1\}/\left\{\begin{bmatrix}v_1-v_2\\v_2-v_1\end{bmatrix}\mid\mathbf{v}\in\mathrm{Coker}(I+K)\right\}$.
• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_4$. *There are 4 choices for* b_4, *the solutions of* $(\beta\alpha)^4=\mathbf{t}_4^{\frac{j}{q}}$ $(j=0,1,2,3)$.
• *Both* Q *and* Π *have torsion.*

(7i) $\Phi=\mathbb{Z}_4\rtimes\mathbb{Z}_2 : A=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],B=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$,
$\mathbb{Z}_\Phi=(\mathbb{Z}\times\mathbb{Z}_2)\rtimes\mathbb{Z}_2=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}},B)\rangle$.
$\varphi(\bar{\alpha})=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right],\ \varphi(\bar{\beta})=-K$ (*includes* (6ai)) (3i)+(1)
$\mathcal{S}=nK+I$, $K\in\mathrm{GL}(2,\mathbb{Z})$, $\det(K)=-1$, $\mathrm{tr}(K)=n>0$, $k_{11}=k_{22}$.

$$\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A),\beta=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4},B)\rangle.$$

• $H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))=\{\mathbf{a}\mid\mathbf{a}\in\mathrm{Coker}(I-\mathcal{S}),2a_1\equiv 0\}/\left\{\begin{bmatrix}0\\2v_2\end{bmatrix}\mid\mathbf{v}\in\mathrm{Coker}(I+K)\right\}$.
• $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_4$. *There are 4 choices for* b_4, *the solutions of* $(\beta\alpha)^4=\mathbf{t}_4^{\frac{j}{q}}$ $(j=0,1,2,3)$.
• Q *has torsion,* Π *is torsion free if and only if* $j=1,3$, q *is even, and* $a_1=\frac{1}{2}$ *and* $a_2=-\frac{k_{21}+1}{2k_{11}}+\frac{i}{k_{11}}$ *for* $i=0,\ldots,k_{11}-1$.

Proof. Consider the descriptions of

$$H^1(\Phi; \mathrm{Coker}(I-\mathcal{S})) \cong Z^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))/B^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$$

in Remark 4.2. In our computations below, we use that the condition

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathrm{Coker}(I-S) = (I-S)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$$

is equivalent to $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0} \bmod \mathbb{Z}^2$.

In cases `(2a)`, `(2b)` and `(4)`, $\Phi = \mathbb{Z}_p$, $p = 2$ or 4. Since $\det(A) = +1$, α^p has $\mathbf{t}_4$ component ${\mathbf{t}_4}^{p\cdot a_4+\ell}$, where ℓ is independent of a_4. We then have p choices for a_4 (modulo $\frac{1}{q}\mathbb{Z}$). Namely, the solutions of

$$p\cdot a_4 + \ell = \tfrac{1}{q}, \ldots, \tfrac{p-1}{q},$$

each corresponding to a different class in $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$. In fact, the number ℓ is always a rational number, and hence so is a_4 (or b_4). The remaining cases when $\Phi = \mathbb{Z}_2\times\mathbb{Z}_2$ or D_4 are similar. We set one of exponents on $\mathbf{t}_4$ by Lemma 6.7, and apply the above technique to find the remaining exponent on $\mathbf{t}_4$.

`(0)` See Theorem 5.1.

`(1)` Corollary 4.3 shows $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ is trivial, and thus we can take $a_1 = a_2 = 0$. Since $\hat{A} = \det(A) = -1$, Lemma 6.7 implies a_4 can be conjugated to zero. So, $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$. By Lemma 6.12, both Π and Q are torsion free.

`(2a)` In this case $\varphi(\bar{\alpha}) = -I$. Now $\mathbf{a}$ must satisfy $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$ taken modulo $(I-\varphi(\bar{\alpha}))\mathbf{a} = 2\mathbf{a}$, since the cocycle condition in Remark 4.2, $(I+\varphi(\bar{\alpha}))\mathbf{a} = \mathbf{0} \in \mathbb{Z}^2$, is satisfied independently of $\mathbf{a}$. Note that all elements of $H^1(\Phi; \mathrm{Coker}(I-S))$ are of order 2, and is generated by at most 2 elements. Therefore, $H^1(\Phi; \mathrm{Coker}(I-S))$ is isomorphic to a subgroup of $\mathbb{Z}_2\times\mathbb{Z}_2$.

There are two choices for a_4, the solutions of $\alpha^2 = \mathbf{t}_4^{\frac{i}{q}}$ $(i = 0, 1)$. Indeed, α^2 projects to the identity on Sol^3. Therefore, Π is torsion free only when $i = 1$ and q is even (see classification of crystallographic groups of Nil, case 2, p. 160, [5]), and Q always has torsion.

`(2b)` By Corollary 4.3, we can take $a_1 = a_2 = 0$ so that $\alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)$. Then $\alpha^2 = \mathbf{t}_3\mathbf{t}_4^{2a_4}$. Therefore, $a_4 = 0$ or $\frac{1}{2q}$. By Lemma 6.12, both Π and Q are torsion free.

`(3)` From Remark 4.2, $\mathbf{a}$ must satisfy $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$, and

$$(I+\varphi(\bar{\alpha}))\mathbf{a} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\mathbf{a} \equiv \mathbf{0} \text{ modulo } (I-\varphi(\bar{\alpha}))\mathbf{v} = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]\mathbf{v} \text{ for } (I-\mathcal{S})\mathbf{v} \equiv \mathbf{0}.$$

Computing, we obtain $a_2 \equiv -a_1$, modulo $\left[\begin{smallmatrix} v_1-v_2 \\ v_2-v_1 \end{smallmatrix}\right]$. Applying the coboundary operator to the cocycles yields:

$$(I-\varphi(\bar{\alpha}))\begin{bmatrix} a_1 \\ -a_1 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ -2a_1 \end{bmatrix},$$

which implies that $\mathbf{a}$ has order at most 2 and so $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ is either $\mathbb{Z}_2$ or trivial, depending on $\mathrm{Coker}(I-\mathcal{S})$. By Lemma 6.7, we may assume $a_4 = 0$, equivalently, $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ vanishes, so that $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)$.

Direct computation shows that the projection of Π to a Sol^3-crystallographic group, Q, always has torsion. Note that $a_2 \equiv -a_1$, and

$$\begin{aligned}\alpha^2 &= (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}, A)^2 = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}\cdot A(\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}), I)\\ &= (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}\cdot \mathbf{t}_2^{a_1}\mathbf{t}_1^{-a_1}, I)\\ &= (e, I).\end{aligned}$$

On Sol_1^4, $\hat{A} = -1$, so A acts as reflection on $\mathcal{Z}(\mathrm{Sol}_1^4)$. Lemma 6.11 applies to show that Π always has torsion.

(3i) From Remark 4.2, $\mathbf{a}$ must satisfy $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$,

$$(I+\varphi(\bar{\alpha}))\mathbf{a} = \left[\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right]\mathbf{a} \equiv \mathbf{0} \text{ modulo } (I-\varphi(\bar{\alpha}))\mathbf{v} = \left[\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right]\mathbf{v}, \text{ for } (I-\mathcal{S})\mathbf{v} \equiv \mathbf{0},$$

that is, $2a_1 \equiv \mathbf{0}$ (so $a_1 \equiv 0$ or $\frac{1}{2}$), modulo $\left[\begin{smallmatrix}0\\2v_2\end{smallmatrix}\right]$. This implies that

$$H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$$

is isomorphic to a subgroup of $\mathbb{Z}_2\times\mathbb{Z}_2$. By Lemma 6.7, we may assume $a_4 = 0$, that is, $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ vanishes. Therefore, $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$.

Lemma 6.11 applies to show that Π is torsion free precisely when the Sol^3-crystallographic group Q is torsion free, which is equivalent to the action of Q on Sol^3 having no fixed points. By Lemma 4.4, $Q\backslash\mathrm{Sol}^3$ is $T^2\times I$ with $T^2\times\{0\}$ identified to itself by the affine involution of T^2 $\left(\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right], \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\right)$ and $T^2\times\{1\}$ identified to itself by the affine involution $\left(\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right], \left[\begin{smallmatrix}\sigma_{11}&-\sigma_{12}\\\sigma_{21}&-\sigma_{11}\end{smallmatrix}\right]\right)$. Both of these involutions act freely on the torus precisely when $a_1 \equiv \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n\in\mathbb{Z}$.

(4) This is the only case where a non-standard lattice is present, that is $c_3 \neq 0$.

By Remark 4.2, $\mathbf{a}$ must satisfy $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $\mathrm{Im}(I-\varphi(\bar{\alpha}))$. Note that $\det(I-\varphi(\bar{\alpha})) = \det\left(\left[\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right]\right) = 2$, which implies that $H^1(\Phi; \mathrm{Coker}(I-S))$ is either $\mathbb{Z}_2$ or the trivial group.

We compute that

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{(1-\sigma_{22})a_1+\sigma_{12}a_2}\mathbf{t}_2^{\sigma_{21}a_1+(1-\sigma_{11})a_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{u_4+2c_3}.$$

By Proposition 6.2, u_4 must be rational. We have two choices for c_3 (modulo $\frac{1}{q}\mathbb{Z}$, as $\mathbf{t}_4^{\frac{1}{q}}$ is a generator of the lattice), $c_3 = -\frac{u_4}{2}, -\frac{u_4}{2}+\frac{1}{2q}$, so that $u_4+2c_3 = 0$ or 1. Unless c_3 is a multiple of $\frac{1}{q}$, the corresponding lattice is non-standard.

For a_4, we have

$$\alpha^4 = \mathbf{t}_4^{4a_4-(a_1-a_2)^2+v_4}.$$

Then there are 4 choices for a_4, $a_4 = \frac{(a_1-a_2)^2-v_4+i}{4q}$ $(i = 0,1,2,3)$. These are the solutions of $\alpha^4 = \mathbf{t}_4^{\frac{i}{q}}$ $(i = 0,1,2,3)$.

From this, Q must always have torsion. For $i = 0, 2$, Π has torsion. To see this when $i = 2$, note that

$$(\mathbf{t}_4^{-\frac{1}{q}}\alpha^2)^2 = \mathbf{t}_4^{-\frac{2}{q}}\mathbf{t}_4^{\frac{2}{q}} = e.$$

For $i = 1, 3$ and q even (see classification of crystallographic groups of Nil, case 10, p. 163, [5]), Π is torsion free.

`(5)` By Corollary 4.3, we take $a_1 = a_2 = 0$. We need $\mathbf{b}$ to satisfy $(I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$. Then the cocycle conditions for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Remark 4.2 show that we must have $(I - \varphi(\bar{\alpha}))\mathbf{b} = (I + K)\mathbf{b} \equiv \mathbf{0}$. In fact, since $(I - \mathcal{S}) = (I - K)(I + K)$, this condition implies $(I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$. Since we have already fixed $a_1 = a_2 = 0$, for the coboundary in Remark 4.2, we take $\mathbf{b}$ modulo $(I - \varphi(\bar{\beta}))\mathbf{v} = 2\mathbf{v}$ only when $\mathbf{v}$ satisfies $(I - \varphi(\bar{\alpha}))\mathbf{v} = (I + K)\mathbf{v} \equiv \mathbf{0}$.

Since $\det(A) = -1$, we may assume $a_4 = 0$ by Lemma 6.7. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$, just like in case `(2a)`. That is, $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. Indeed, β has order 2 when projected to $\mathrm{Sol}^3 \rtimes D_4$, and hence Q always has torsion.

Note that $\gamma\alpha$ and $\gamma\alpha\beta$ are of infinite order for all $\gamma \in \widetilde{\Gamma}_{\mathcal{S}}$ by Lemma 6.12. Like case `(2a)`, Π is torsion free precisely when $\beta^2 = \mathbf{t}_4^{\frac{1}{q}}$ and q is even.

`(6a)` This is a combination of cases `(3)`+`(2a)`.

We have $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $(I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$. Also, $\mathbf{a}$ and $\mathbf{b}$ must satisfy the cocycle conditions in Remark 4.2. Note that $(I + \varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$ forces $a_2 \equiv -a_1$, whereas

$$(I - \varphi(\bar{\alpha}))\mathbf{b} - (I - \varphi(\bar{\beta}))\mathbf{a} \equiv \mathbf{0}$$

forces $b_1 - b_2 - 2a_1 \equiv 0$, $-b_1 + b_2 - 2a_2 \equiv 0$. Since $a_2 \equiv -a_1$, the second equation is redundant. We take $\mathbf{a}$ and $\mathbf{b}$ modulo $(I - \varphi(\bar{\alpha}))\mathbf{v}$ and $(I - \varphi(\bar{\beta}))\mathbf{v}$, respectively, where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$. By Lemma 6.7, we may assume $a_4 = 0$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$. That is, $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$.

As Π contains a subgroup of type `(3)`, both Q and Π always have torsion.

`(6a`i`)` Similar to case `(6a)`, this is a combination of `(3`i`)`+`(2a)`. The description of $H^1(\Phi, \mathrm{Coker}(I - \mathcal{S}))$ follows just like in case `(6a)`.

There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$. That is, $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. Since β^2 projects to the identity on Sol^3, Q always has torsion.

For Π to be torsion free, the subgroups $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$, $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$, and $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$, where

$$\alpha\beta = \left(\mathbf{t}_1^{a_1+b_1}\mathbf{t}_2^{a_2-b_2}\mathbf{t}_4^{b_4'}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right),$$

must all be torsion free. The group $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ is torsion free precisely when b_4 satisfies $\beta^2 = \mathbf{t}_4^{\frac{1}{q}}$ and q is even.

By Lemma 6.11, $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\beta\rangle$ are torsion free precisely when their projections to Sol^3, $\langle\Gamma_{\mathcal{S}},(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2},A)\rangle$ and $\langle\Gamma_{\mathcal{S}},(\mathbf{t}_1^{a_1+b_1}\mathbf{t}_2^{a_2-b_2},AB)\rangle$ are torsion free. Similar to case (3i), by computing when the appropriate affine involutions on T^2 in Lemma 4.4 have no fixed points, we obtain the conditions $a_1=\frac{1}{2}$, $a_2=b_2+\frac{1}{2}$, $b_1\not\equiv\frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}+\frac{1}{2}$ $b_2\not\equiv\frac{(\sigma_{11}+1)(2m+1)}{2\sigma_{12}}+\frac{1}{2}$ for any $m,n\in\mathbb{Z}$.

(6b) This is a combination of (3)+(2b).

By Corollary 4.3, we can take $b_1=b_2=0$. The cocycle conditions in Remark 4.2 force $(I+\varphi(\bar\alpha))\mathbf{a}\equiv\mathbf{0}$ as well as $(I-\varphi(\bar\beta))\mathbf{a}=(I+K)\mathbf{a}\equiv\mathbf{0}$, so that $\mathbf{a}\in\mathrm{Coker}(I+K)$. Since $b_1,b_2=0$ is fixed, we can take $\mathbf{a}$ modulo $(I-\varphi(\bar\alpha))\mathbf{v}$ only when $(I-\varphi(\bar\beta))\mathbf{v}=(I+K)\mathbf{v}\equiv\mathbf{0}$, that is, only for $\mathbf{v}\in\mathrm{Coker}(I+K)$.

Note that we can take $a_4=0$ by Lemma 6.7, and there are two choices for b_4, the solutions of $\beta^2=\mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}$ $(i=0,1)$. Hence $H^2(\Phi,\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$. Both Q and Π always have torsion, as they contain a subgroup of type (3).

(6bi) This is a combination of (3i)+(2b).

By Corollary 4.3, we can take $b_1=b_2=0$. The computation of

$$H^1(\Phi;\mathrm{Coker}(I-\mathcal{S}))$$

is identical to that of (6b). In this case, we use $\varphi(\bar\alpha)=\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$ rather than $\varphi(\bar\alpha)=A$. Note that we take $a_4=0$ by Lemma 6.7, and there are two choices for b_4, the solutions of $\beta^2=\mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}},(i=0,1)$. Thus $H^2(\Phi,\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$.

By Lemma 6.11, Π is torsion free precisely when the Sol^3-crystallographic group Q is torsion free, which is equivalent to Q acting freely on Sol^3. By Lemma 4.4, $Q\backslash\mathrm{Sol}^3$ is $T^2\times I$ with $T^2\times\{0\}$ identified to itself by the affine involution of T^2 $\left(\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\right)$, and $T^2\times\{1\}$ identified to itself by the affine involution $\left(\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right],\left[\begin{smallmatrix}-k_{11}&k_{12}\\-k_{21}&k_{11}\end{smallmatrix}\right]\right)$. Both of these involutions act freely on the torus precisely when $a_1=\frac{1}{2}$ and $a_2\not\equiv\frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n\in\mathbb{Z}$.

(7) This is a combination (3)+(1). which includes (6a).

By Corollary 4.3, we can take $b_1=b_2=0$. For $(I-\mathcal{S})\mathbf{a}\equiv\mathbf{0}$, the only cocycle condition that $\mathbf{a}$ must satisfy is $(I+\varphi(\bar\alpha))\mathbf{a}\equiv\mathbf{0}$, which forces $a_2\equiv-a_1$. However, we have fixed $b_1=b_2=0$. Therefore, when computing the coboundaries, we can take $\mathbf{a}$ modulo $(I-\varphi(\bar\alpha))\mathbf{v}$ only for $\mathbf{v}$ that satisfies $(I-\varphi(\bar\beta))\mathbf{v}=(I+K)\mathbf{v}\equiv\mathbf{0}$. Note that $(I+K)\mathbf{v}\equiv\mathbf{0}$ actually implies $(I-\mathcal{S})\mathbf{v}\equiv\mathbf{0}$ since $(I-\mathcal{S})=(I-K)(I+K)$.

We may take $a_4=0$ by Lemma 6.7. The computation

$$(\beta\alpha)^4=\mathbf{t}_4^{4b_4+\ell}$$

shows that there are 4 choices for b_4, the solutions of $(\beta\alpha)^4=\mathbf{t}_4{}^{\frac{j}{q}}$ $(j=0,1,2,3)$. Hence $H^2(D_4;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_4$. Both Π and Q contain a subgroup of type (3), and so both always have torsion.

(7i) This is a combination of (3i)+(1), which includes (6ai).

By Corollary 4.3, we can take $b_1 = b_2 = 0$. The description for

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$$

follows as in case (7), using $\varphi(\bar\alpha) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ rather than $\varphi(\bar\alpha) = A$.

Like case (7), by Lemma 6.7, we take $a_4 = 0$, and there are 4 choices for b_4, the solutions of $(\beta\alpha)^4 = \mathbf{t}_4{}^{\frac{j}{q}}$ $(j = 0, 1, 2, 3)$, so that $H^2(D_4; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$.

For Π to be torsion free, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\alpha\rangle$ is necessarily torsion free. This forces b_4 to satisfy $(\beta\alpha)^4 = \mathbf{t}_4^{\frac{j}{q}}$ $(j = 1, 3)$, and q even. Note that $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\alpha\rangle$, are torsion free by Lemma 6.12.

Thus the only remaining subgroups of Π to consider are $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\alpha\beta\rangle$, where

$$\beta\alpha\beta = \left(\mathbf{t}_1^{-k_{11}a_1 - k_{12}a_2}\mathbf{t}_2^{-k_{21}a_1 - k_{11}a_2}\mathbf{t}_4^{2b_4+v}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right).$$

By Lemma 6.11, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\alpha\beta\rangle$ are torsion free precisely when their projections to Sol^3, $\langle \Gamma_{\mathcal{S}}, (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$ and $\langle \Gamma_{\mathcal{S}}, (\mathbf{t}_1^{-k_{11}a_1 - k_{12}a_2}\mathbf{t}_2^{-k_{21}a_1 - k_{11}a_2}A, BAB)\rangle$, are torsion free.

By Proposition 4.4, we just need to ensure that the appropriate affine maps are fixed point free on T^2, and this occurs precisely when

$$a_1 = \frac{1}{2}, \quad a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}, \tag{6.2}$$

$$\frac{-k_{21}}{2} - k_{11}a_2 \equiv \frac{1}{2}, \tag{6.3}$$

$$\frac{-k_{11}}{2} - k_{12}a_2 \not\equiv \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}. \tag{6.4}$$

Now we claim that the second part of condition (6.2) and the condition (6.4) are redundant. That is, they follow from (6.3).

From (6.3), we have

$$a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{p}{k_{11}}, \ p \in \mathbb{Z}. \tag{6.5}$$

With $a_1 = \frac{1}{2}$ and above a_2 with $p = 0, \ldots, k_{11} - 1$, using that $\det(K) = -1$ and $K^2 = \mathcal{S}$, one can compute that the remaining criteria are satisfied. In fact, we compute the term in (6.2)

$$\begin{aligned}\frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}} &= \frac{(k_{11}^2 + k_{12}k_{21} + 1)(2n+1)}{4k_{11}k_{12}} \\ &= \frac{2k_{12}k_{21}(2n+1)}{4k_{11}k_{12}} = \frac{k_{21}(2n+1)}{2k_{11}}.\end{aligned}$$

Now, for some $m \in \mathbb{Z}$, suppose we had

$$a_2 = \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}} + m,$$

as opposed to (6.2). Then we would have

$$-\frac{k_{21}+1}{2k_{11}}+\frac{p}{k_{11}}=\frac{k_{21}(2n+1)}{2k_{11}}+m.$$

Clearing up, we get

$$-1+2p=2k_{21}(n+1)+2mk_{11},$$

a contradiction for any integers p, n, m, as they are of different parity. Thus, (6.2) holds.

For (6.4), using (6.5), we get

$$\begin{aligned}\frac{-k_{11}}{2}-k_{12}a_2&=\frac{-k_{11}}{2}-k_{12}\left(-\frac{k_{21}+1}{2k_{11}}+\frac{p}{k_{11}}\right)\\&=\frac{-k_{11}^2+k_{12}k_{21}+k_{12}-2k_{12}p}{2k_{11}}\\&=\frac{1+k_{12}-2k_{12}p}{2k_{11}}.\end{aligned}$$

Now suppose we had

$$\frac{-k_{11}}{2}-k_{12}a_2=\frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}+m$$

for some $m\in\mathbb{Z}$. Then we would have

$$\frac{1+k_{12}-2k_{12}p}{2k_{11}}=\frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}+m=\frac{2k_{11}k_{12}(2n+1)}{2(k_{11}^2+k_{12}k_{21}-1)}+m.$$

Clearing up, we get

$$1-2k_{12}p=2(nk_{12}+mk_{11}),$$

a contradiction for any integers p, n, m, as they are of different parity. Thus, (6.4) holds automatically.

Consequently, with $a_1=\frac{1}{2}$, $a_2=-\frac{k_{21}+1}{2k_{11}}+\frac{p}{k_{11}}$ for $p=0,\dots,k_{11}-1$, and $(\beta\alpha)^4=\mathbf{t}_4^{\frac{j}{q}}$ $(j=1,3)$, Π is torsion free.

This completes the proof of Theorem 6.13. □

7. Examples

We can embed Sol^3 and Sol_1^4 into $\mathrm{Aff}(3)$ and $\mathrm{Aff}(4)$, respectively so that our Sol^3 and Sol_1^4-orbifolds, $Q\backslash\mathrm{Sol}^3$ and $\Pi\backslash\mathrm{Sol}_1^4$, have complete affinely flat structures. Below we use the embedding $\mathrm{Aff}(n)=\mathbb{R}^n\rtimes\mathrm{GL}(n,\mathbb{R})\hookrightarrow\mathrm{GL}(n+1,\mathbb{R})$. See [13] for the more general question.

One can check the following correspondence is an injective homomorphism of Lie groups, $\mathrm{Sol}_1^4 \longrightarrow \mathrm{Aff}(4)$,

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & -\frac{1}{2}e^{-u}y & \frac{e^u x}{2} & 0 & z-\frac{xy}{2} \\ 0 & e^{-u} & 0 & 0 & x \\ 0 & 0 & e^u & 0 & y \\ 0 & 0 & 0 & 1 & u \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.1}$$

Moreover, the automorphisms

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_1^4)$$

can also be embedded as

$$\begin{bmatrix} ad & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -bc & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively, where a, b, c, d are ± 1. Note that, if we remove the first row and the first column from $\mathrm{Aff}(4)$, we get a representation of Sol^3 into $\mathrm{Aff}(3)$.

If we write the element $(\mathbf{a}, A) \in \mathrm{Sol}_1^4 \rtimes D_4$ by the product $\mathbf{a} \cdot A$, then the group operation of $\mathrm{Sol}_1^4 \rtimes D_4$ is compatible with the matrix product in this affine group. The action of A on $\mathbf{a}$ is by conjugation. That is,

$$\begin{aligned} (\mathbf{a} \cdot A)(\mathbf{b} \cdot B) &= \mathbf{a}A\mathbf{b}B \\ &= \mathbf{a}(A\mathbf{b}A^{-1}) \cdot AB \\ &= (\mathbf{a}, A) \cdot (\mathbf{b}, B). \end{aligned}$$

We have embedded $\mathrm{Isom}(\mathrm{Sol}_1^4)$ into $\mathrm{Aff}(4)$ in such a way that any lattice acts on $\mathbb{R}^4$ properly discontinuously. Therefore all of our infra-Sol_1^4-orbifolds will have an *affine structure*. Note that not every nilpotent Lie group admits an affine structure [11, p. 227].

With $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$, $\mathrm{tr}(\mathcal{S}) > 2$, and appropriate P and Δ, so that $P\mathcal{S}P^{-1} = \Delta$, we can lift $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} \subset \mathbb{R}^2 \rtimes_{\mathcal{S}} \mathbb{R}$ to a lattice of Sol_1^4 as in the proof of Theorem 5.1. The image of our lattice in $\mathrm{Aff}(5)$ under the embedding (7.1) is complicated. When we conjugate it by

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & p_{11} & p_{12} & 0 & 0 \\ 0 & p_{21} & p_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1},$$

we get a much better representation of the group as shown below. Note that c_3 will have no effect on the presentation of our lattice. Since $\det(P) = 1$,

$[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4$.

$$\mathbf{e}_1 = \left(\begin{bmatrix}1\\0\end{bmatrix}, 0\right) \longmapsto \mathbf{t}_1 = \begin{bmatrix}1 & p_{11} & c_1\\0 & 1 & p_{21}\\0 & 0 & 1\end{bmatrix} \longmapsto \begin{bmatrix}1 & 0 & \frac{1}{2} & 0 & c_1 - \frac{\sigma_{21}}{2\sqrt{T^2-4}}\\0 & 1 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{e}_2 = \left(\begin{bmatrix}0\\1\end{bmatrix}, 0\right) \longmapsto \mathbf{t}_2 = \begin{bmatrix}1 & p_{12} & c_2\\0 & 1 & p_{22}\\0 & 0 & 1\end{bmatrix} \longmapsto \begin{bmatrix}1 & -\frac{1}{2} & 0 & 0 & c_2 - \frac{\sigma_{12}}{2\sqrt{T^2-4}}\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{e}_3 = \left(\begin{bmatrix}0\\0\end{bmatrix}, 1\right) \longmapsto \mathbf{t}_3 = \begin{bmatrix}1 & 0 & c_3\\0 & \lambda & 0\\0 & 0 & 1\end{bmatrix} \longmapsto \begin{bmatrix}1 & 0 & 0 & 0 & c_3\\0 & \sigma_{11} & \sigma_{12} & 0 & 0\\0 & \sigma_{21} & \sigma_{22} & 0 & 0\\0 & 0 & 0 & 1 & \ln(\lambda)\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{t}_4 = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \longmapsto \begin{bmatrix}1 & 0 & 0 & 0 & 1\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

where $T = \mathrm{tr}(\mathcal{S})$.

Example 7.1 ((4) Non-standard lattice)**.** This is an example where c_3 can be non-zero (Theorem 6.13, case (4)). Here $A = \left[\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right]$, so that the holonomy $\Phi = \mathbb{Z}_4$.

Let $\mathcal{S} = \left[\begin{smallmatrix}1 & 2\\2 & 5\end{smallmatrix}\right]$. Then $\lambda = 3 + 2\sqrt{2}$, and with

$$P = \begin{bmatrix} -\frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{1}{2}\sqrt{2-\sqrt{2}}\\ -\frac{1}{\sqrt{2(2+\sqrt{2})}} & -\frac{1}{2}\sqrt{2+\sqrt{2}}\end{bmatrix},$$

our crystallographic group $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{1/q}, \alpha\rangle$, where $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A) \in \mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$, has a representation into $\mathrm{Aff}(4)$:

$$\mathbf{t}_1 = \begin{bmatrix}1 & 0 & \frac{1}{2} & 0 & m_1 - \frac{m_2}{2} - \frac{3}{2}\\0 & 1 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix}1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}(-m_1 - 1)\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix},$$

$$\mathbf{t}_3=\begin{bmatrix}1&0&0&0&c_3\\0&1&2&0&0\\0&2&5&0&0\\0&0&0&1&\ln\left(3+2\sqrt{2}\right)\\0&0&0&0&1\end{bmatrix},\ \mathbf{t}_4=\begin{bmatrix}1&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$

$$(a,A)=\begin{bmatrix}1&-\frac{a_1}{2}&-\frac{a_2}{2}&0&\frac{1}{2}(2a_4-a_2(m_1+1)+a_1(a_2+2m_1-m_2-3))\\0&0&1&0&a_1\\0&-1&0&0&a_2\\0&0&0&-1&0\\0&0&0&0&1\end{bmatrix}.$$

Π has presentation

$$[\mathbf{t}_1,\mathbf{t}_2]=\mathbf{t}_4,\quad \text{and } \mathbf{t}_4 \text{ is central, } \mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1}=\mathbf{t}_1\mathbf{t}_2^2\mathbf{t}_4^{m_1},\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1}=\mathbf{t}_1^2\mathbf{t}_2^5\mathbf{t}_4^{m_2},$$
$$\alpha\mathbf{t}_1\alpha^{-1}=\mathbf{t}_2^{-1}\mathbf{t}_4^{\frac{1}{2}(-4-2a_1+m_1-m_2)},\ \alpha\mathbf{t}_2\alpha^{-1}=\mathbf{t}_1\mathbf{t}_4^{\frac{1}{2}(2-2a_2-3m_1+m_2)},$$
$$\alpha\mathbf{t}_3\alpha^{-1}=\mathbf{t}_1^{-4a_1+2a_2}\mathbf{t}_2^{2a_1}\mathbf{t}_3^{-1}\mathbf{t}_4^{5a_1^2+2c_3+a_1(-5+5m_1-2m_2)+a_2(3-a_2-2m_1+m_2)},$$
$$\alpha\mathbf{t}_4\alpha^{-1}=\mathbf{t}_4,\ \alpha^4=\mathbf{t}_4^{-a_1^2+4a_4-a_2(2+a_2+2m_1)+2a_1(-3+a_2+2m_1-m_2)}.$$

Since $(I-\mathcal{S})^{-1}=\left[\begin{smallmatrix}1&-\frac{1}{2}\\-\frac{1}{2}&0\end{smallmatrix}\right]$, $\mathrm{Coker}(I-\mathcal{S})=\mathbb{Z}_2\times\mathbb{Z}_2\cong\left\langle\left[\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}\right]\right\rangle$. Therefore, the equation $(I-\mathcal{S})\mathbf{a}\equiv\mathbf{0}$ has 4 solutions modulo $\mathbb{Z}^2$;

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}0\\0\end{bmatrix},\ \tfrac{1}{2}\begin{bmatrix}1\\0\end{bmatrix},\ \tfrac{1}{2}\begin{bmatrix}0\\1\end{bmatrix},\ \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}.$$

Recall that we had no other conditions on $\mathbf{a}$ in Theorem 6.13 case (4). The coboundary is

$$\mathrm{Im}(I-\varphi(\bar{\alpha}))=\mathrm{Im}\begin{bmatrix}1&-1\\1&1\end{bmatrix}=\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}\right\}.$$

Thus, we have only have to consider two cases

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}.$$

For simplicity, we shall assume $m_1=m_2=0$.

With $\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $\Pi=\langle\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4^{\frac{1}{q}},\alpha\rangle$, where $\alpha=(\mathbf{t}_4^{a_4},A)$ has presentation

$$[\mathbf{t}_1,\mathbf{t}_2]=\mathbf{t}_4,\quad\text{and } \mathbf{t}_4 \text{ is central, } \mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1}=\mathbf{t}_1\mathbf{t}_2^2,\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1}=\mathbf{t}_1^2\mathbf{t}_2^5,$$
$$\alpha\mathbf{t}_1\alpha^{-1}=\mathbf{t}_2^{-1}\mathbf{t}_4^{-2},\ \alpha\mathbf{t}_2\alpha^{-1}=\mathbf{t}_1\mathbf{t}_4,\ \alpha\mathbf{t}_3\alpha^{-1}=\mathbf{t}_3^{-1}\mathbf{t}_4^{2c_3},\ \alpha\mathbf{t}_4\alpha^{-1}=\mathbf{t}_4,$$
$$\alpha^4=\mathbf{t}_4^{4a_4}.$$

The minimum q for $\widetilde{\Gamma}_{\mathcal{S}}$ is $q=1$. However, to have a torsion free crystallographic group we must take q to be even, say $q=2$. Then we have choices $a_4=0,\frac{1}{8},\frac{1}{4},\frac{3}{8}$ and $c_3=0,\frac{1}{4}$ (any combination of a_4 and c_3), with the same center. So, there are 8 distinct groups. Half of them (with $c_3=0$) have standard lattices, and the rest (with $c_3 = \frac{1}{4}$) have non-standard lattices. When $a_4 = \frac{1}{8}$ or $\frac{3}{8}$ (regardless of c_3), Π is torsion free, and $\Pi\backslash \mathrm{Sol}_1^4$ is an infra-solvmanifold of Sol_1^4 with $\mathbb{Z}_4$ holonomy.

With $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right]$, $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha\rangle$, where $\alpha = (\mathbf{t}_1^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)$ has presentation

$$
\begin{aligned}
&[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4, \quad \text{and } \mathbf{t}_4 \text{ is central, } \mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1\mathbf{t}_2^2,\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^2\mathbf{t}_2^5,\\
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-\frac{5}{2}},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4,\ \alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-2}\mathbf{t}_2\mathbf{t}_3^{-1}\mathbf{t}_4^{-\frac{5}{4}+2c_3},\\
&\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4,\ \alpha^4 = \mathbf{t}_4^{-\frac{13}{4}+4a_4}.
\end{aligned}
$$

The minimum q for $\widetilde{\Gamma}_{\mathcal{S}}$ is $q = 2$ (which comes out of $\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-\frac{5}{2}}$), and we have choices $a_4 = \frac{1}{16} + \frac{1}{2}\cdot\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\} = \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$ and $c_3 = \frac{1}{8} + \frac{1}{2}\cdot\{0, \frac{1}{2}\} = \frac{1}{8}, \frac{3}{8}$ (any combination of a_4 and c_3), with the same center. So, there are 8 distinct groups.

All these groups have non-standard lattices, because no c_3 is an integer multiple of $\frac{1}{q}$, $q = 2$. When $a_4 = \frac{3}{16}$ or $\frac{7}{16}$ (regardless of c_3), Π is torsion free, and $\Pi\backslash \mathrm{Sol}_1^4$ is an infra-solvmanifold of Sol_1^4 with $\mathbb{Z}_4$ holonomy.

Example 7.2 ((7*i*) Maximal holonomy)**.** Even if this has the maximal holonomy group D_4, it does not contain all the possible holonomy actions. For example, groups of type (6b) or (6b*i*) are not contained in this group. Let $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \langle A, B\rangle$, where

$$
A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right],\ B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right],\ \text{and } \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B).
$$

Our $\mathcal{S}$ is of the form $\mathcal{S} = nK + I$, where $K = \left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$ with $\det(K) = -1$ and $\mathrm{tr}(K) = n \neq 0$. Now for $\varphi(\bar{\alpha}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, we take $k_{11} = k_{22}$. For example, we need $K = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]$, $n = k_{11} + k_{22} = 2$, $\mathcal{S} = nK + I = \left[\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right]$. Then $\lambda = 3 + 2\sqrt{2}$, and with $P = \begin{bmatrix} -\frac{1}{\sqrt[4]{2^3}} & \frac{1}{\sqrt[4]{2}} \\ -\frac{1}{\sqrt[4]{2^3}} & -\frac{1}{\sqrt[4]{2}} \end{bmatrix}$, the equations in Lemma 5.2 yield

$$
c_1 = \frac{1}{8}(-12 + \sqrt{2} + 4m_1 - 4m_2),\ c_2 = \frac{1}{4}(-\sqrt{2} - 4m_1 + 2m_2).
$$

Recall we can take $c_3 = 0$ by Theorem 6.6. Our crystallographic group $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha, \beta\rangle$ has a representation into Aff(4):

$$
\mathbf{t}_1 = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2}(m_1 - m_2 - 3) \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\quad
\mathbf{t}_2 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}(m_2 - 2m_1) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$\mathbf{t}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ln\left(3+2\sqrt{2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{t}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(a, A) = \begin{bmatrix} -1 & -\frac{a_2}{2} & -\frac{a_1}{2} & 0 & \frac{1}{2}(a_1(a_2+m_1-m_2-3)+a_2(m_2-2m_1)) \\ 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 & a_2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(b, B) = \begin{bmatrix} -1 & 0 & 0 & 0 & b_4 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2}\ln\left(3+2\sqrt{2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$\mathrm{Coker}(I - \mathcal{S}) = (\mathbb{Z}_2)^2 = \left\{ \tfrac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tfrac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tfrac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Now

$$\varphi(\bar{\alpha}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \quad \varphi(\bar{\beta}) = -K,$$

yields

$$I + \varphi(\bar{\alpha}) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \ I + \varphi(\bar{\beta}) = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

Then $(I + \varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$ yields $2a_1 \equiv 0$, which is not a new condition. We therefore have 4 choices for $\mathbf{a}$,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \tfrac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \tfrac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The coboundary $\mathrm{Im}(I - \varphi(\bar{\alpha}))$ yields the trivial group, and hence there are 4 distinct choices for $\mathbf{a}$. The group $\varPi$ has a presentation

$$[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4, \ [\mathbf{t}_i, \mathbf{t}_4] = 1 \ (i = 1, 2, 3),$$

$$\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^3\mathbf{t}_2^2\mathbf{t}_4^{m_1}, \ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^4\mathbf{t}_2^3\mathbf{t}_4^{m_2},$$

$$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{3-a_2-m_1+m_2}, \ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-a_1},$$

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-2a_1+4a_2}\mathbf{t}_2^{2(a_1-a_2)}\mathbf{t}_3^{-1}\mathbf{t}_4^{3a_1^2-a_1(3+6a_2-3m_1+2m_2)+a_2(6+2a_2-4m_1+3m_2)},$$

$$\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4^{-1},$$

$$\beta\mathbf{t}_1\beta^{-1} = \mathbf{t}_1^{-1}\mathbf{t}_2^{-1}\mathbf{t}_4^{\frac{1}{2}(-1-2m_1+m_2)}, \ \beta\mathbf{t}_2\beta^{-1} = \mathbf{t}_1^{-2}\mathbf{t}_2^{-1}\mathbf{t}_4^{-4+m_1-m_2},$$

$$\beta\mathbf{t}_3\beta^{-1} = \mathbf{t}_3, \ \beta\mathbf{t}_4\beta^{-1} = \mathbf{t}_4^{-1},$$

$$\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_4^{-a_1(-3+a_2+m_1-m_2)},$$

$$\beta^2 = \mathbf{t}_3,$$
$$(\alpha\beta)^4 = \mathbf{t}_4^{-4b_4+a_1^2+4a_1a_2+2a_2^2-2a_1(3-m_1+m_2)-2a_2(2m_1-m_2)}.$$

Of the four choices for $\mathbf{a}$, only $a_1 = \frac{1}{2}$, $a_2 = 0$ can yield a torsion free group, and the other three choices always yield a group with torsion:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}: \quad \alpha^2 = \mathrm{id}.$$
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}: \quad \left(\mathbf{t}_2^{-1}(\alpha\beta)^2\alpha\right)^2 = \mathrm{id}.$$
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}: \quad a_2 \equiv -\tfrac{k_{21}+1}{2k_{11}} = -1.$$

Let us take $m_1 = m_2 = 0$. When $a_1 = \frac{1}{2}$, $a_2 = 0$, $q = 4$ (minimum), b_4 takes values $\frac{j}{16}$, $0 \le j \le 3$. When $b_4 = \frac{1}{16}$ or $\frac{3}{16}$, Π has torsion. However, when $b_4 = 0$ or $\frac{2}{16}$, Π is torsion free when $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right]$, because all criteria of Theorem 6.13 case (7*i*) are satisfied. In this case, $\Pi\backslash\mathrm{Sol}_1^4$ is an infra-solvmanifold of Sol_1^4 with maximal holonomy D_4.

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