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UPPER SEMICONTINUITY OF PULLBACK ATTRACTORS FOR NON-AUTONOMOUS GENERALIZED 2D PARABOLIC EQUATIONS

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Abstract. This paper is concerned with a generalized 2D parabolic equation with a nonautonomous perturbation

$$-\Delta u_t + \alpha^2\Delta^2 u_t + \mu\Delta^2 u + \nabla\cdot\overrightarrow{F}(u) + B(u,u) = \epsilon g(x,t).$$

Under some proper assumptions on the external force term g, the upper semicontinuity of pullback attractors is proved. More precisely, it is shown that the pullback attractor $\{A_\epsilon(t)\}_{t\in\mathbb{R}}$ of the equation with $\epsilon > 0$ converges to the global attractor A of the equation with $\epsilon = 0$.

1. Introduction

This work is concerned with the upper semicontinuity of pullback attractors for non-autonomous generalized 2D parabolic equations. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $(x_1, x_2) \in \Omega$. Consider a non-autonomous generalized 2D parabolic equation

$$\begin{aligned}
&-\Delta u_t + \alpha^2\Delta^2 u_t + \mu\Delta^2 u + \nabla\cdot\overrightarrow{F}(u) + B(u,u) = \epsilon g(x,t) \text{ in } \Omega\times[\tau,\infty),\\
&u = \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega\times[\tau,\infty), \qquad (1.1)\\
&u(x,\tau) = u_\tau(x) \text{ in } \Omega,
\end{aligned}$$

where ϵ is a small positive parameter, $u_t = \frac{\partial u}{\partial t}$, α, μ are positive constants, $\overrightarrow{F}$ is a nonlinear vector function, g is an external forcing term with $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, $B(u,v) = \frac{\partial u}{\partial x_1}\frac{\partial\Delta v}{\partial x_2} - \frac{\partial u}{\partial x_2}\frac{\partial\Delta v}{\partial x_1}$ and ν is the unit outward normal vector to $\partial\Omega$.

When $\epsilon = 1$ and $g(x,t) = g(x)$, that is, g is independent of time t, Polat [9] established the existence of a global attractor to the autonomous problem (1.1)

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in unbounded domain, and then Park and Park [7] showed the existence of a pullback attractor for the nonautonomous case (1.1) with $\epsilon = 1$. They used the technique of uniform estimates on the tails of solutions to prove the asymptotic compactness of the solution operator. This technique was developed by Wang [12] to investigate the behavior of reaction-diffusion equations in unbounded domains.

On the other hand, several authors have studied the upper and lower semicontinuity of attractors of perturbed dynamical systems for the autonomous case [1, 4, 6] and for the nonautonomous case [2, 3, 10, 13]. This continuous property implies some stability of attractors for the corresponding equations with some perturbations. Caraballo et al. [2] introduced a theorem on the upper semicontinuity on random attractors, and then applied the result to Navier-Stokes equations and a reaction-diffusion problem with additive noise. The authors of [5, 11, 14] considered the asymptotically regular properties of semi-folws/processes given by hyperbolic wavelike equations. Wang and Qin [13] established a technical method to verify the pullback asymptotically compactness by applying the theory of [2] and using ideas given in [5, 11, 14], and then proved the upper semicontinuous property of pullback attractors for nonclassical diffusion equations by adapting the method to overcome some difficulty generated by the equations which are similar to hyperbolic equations. Motivated by these works, we show the upper semicontinuity of pullback attractors for problem (1.1). Though the technique closely follows the arguments of [13] with some necessary modification due to the nature of the problem treated here, it is interesting to investigate whether there is the similar upper semicontinuity result as in [13] for the non-autonomous system (1.1). Owing to properties of the bilinear term $B(u, u)$, the estimates are delicate. This difficulty is overcome by using some embedding relations of spaces.

The plan of this paper is as follows. In Section 2, we give some abstract results concerning pullback attractors for non-autonomous dynamical systems and technique methods to verify the upper semicontinuity of pullback attractors. In Section 3, we derive some estimates of solutions, and then prove the upper semicontinuity of pullback attractors.

2. Preliminaries and abstract results

In this section, we give some known results about the upper semicontinuity of pullback attractors. These results and the related basic definitions can be found in [2, 13] and references therein.

Let X be a Banach space with norm $||\cdot||_X$ and metric $d_X(\cdot,\cdot)$. A two-parameter family of mappings $\{U(t,\tau)\}_{t\geq\tau}$ is said to be a continuous process in X if

(i) $U(t,s)U(s,\tau) = U(t,\tau)$ for $\tau \leq s \leq t,\ \tau \in \mathbb{R}$,

(ii) $U(\tau,\tau) = Id$ (identity operator in X) for $\tau \in \mathbb{R}$,

(iii) $U(t,\tau): X \to X$ is continuous for $\tau \le t$.
We denote by $dist_X(B_1, B_2)$ the Hausdorff semidistance in X defined by

$$dist_X(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d_X(x,y) \quad \text{for } B_1, B_2 \subset X.$$

Definition 2.1. A family of subsets $\mathcal{D} = \{D(t)\}_{t \in \mathbb{R}}$ is said to be pullback absorbing with respect to the continuous process $\{U(t,\tau)\}_{t \ge \tau}$ if, for every $t \in \mathbb{R}$ and all bounded subset $D \subset X$, there exists $T(t,D) > 0$ such that

$$U(t, t-\tau)D \subset D(t) \quad \text{for all} \quad \tau \ge T(t,D).$$

Definition 2.2. A family of compact set $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is said to be a pullback attractor if it satisfies
(i) $U(t,\tau)A(\tau) = A(t)$ for all $t \ge \tau$;
(ii) $\lim_{\tau \to \infty} dist_X(U(t, t-\tau)D, A(t)) = 0$ for all bounded subset $D \subset X$.

Let $S(t) : X \to X$, $t \in \mathbb{R}^+$, be a C_0-semigroup defined on X. Suppose that there exists a global attractor A for $S(t)$. We perturb the semigroup by a nonautonomous term depending on a small parameter $\epsilon \in (0, \epsilon_0]$, so that we obtain a continuous process $U_\epsilon(\cdot,\cdot)$ driven by the nonautonomous dynamical system.

Theorem 2.1 ([2])**.** *Assume that the following conditions hold*:
(i) *for each* $t \in \mathbb{R}$, $\tau \in \mathbb{R}^+$, *and* $x \in X$,

$$(H_1) \qquad \lim_{\epsilon \to 0^+} d_X(U_\epsilon(t, t-\tau)x, S(\tau)x) = 0 \quad \textit{uniformly on bounded sets of } X;$$

(ii) *for any* $\epsilon \in (0, \epsilon_0]$, *there exists a pullback attractor* $\mathcal{A}_\epsilon = \{A_\epsilon(t)\}_{t \in \mathbb{R}}$;
(iii) *there exists a compact set* $K \subset X$ *such that*

$$(H_2) \qquad \lim_{\epsilon \to 0^+} dist_X(A_\epsilon(t), K) = 0 \quad \textit{for any } t \in \mathbb{R}.$$

Then, $\mathcal{A}_\epsilon$ *and* A *have the upper semicontinuity, that is*

$$\lim_{\epsilon \to 0^+} dist_X(A_\epsilon(t), A) = 0 \quad \textit{for any } t \in \mathbb{R}.$$

Theorem 2.2 ([13])**.** *Let the family* $\mathcal{D}_\epsilon = \{D_\epsilon(t)\}_{t \in \mathbb{R}}$ *be pullback absorbing for* $U_\epsilon(\cdot,\cdot)$, *and for each* $\epsilon \in (0, \epsilon_0]$, $\mathcal{K}_\epsilon = \{K_\epsilon(t)\}_{t \in \mathbb{R}}$ *be a family of compact sets in* X. *Suppose* $U_\epsilon(\cdot,\cdot) = U_{1,\epsilon}(\cdot,\cdot) + U_{2,\epsilon}(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \times X \to X$ *satisfies*
(i) *for any* $t \in \mathbb{R}$ *and* $\epsilon \in (0, \epsilon_0]$,

$$||U_{1,\epsilon}(t, t-\tau)x||_X \le \Phi(t,\tau) \quad \textit{for all } x \in D_\epsilon(t-\tau), \;\; \tau > 0,$$

where $\Phi(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *with* $\lim_{\tau \to \infty} \Phi(t,\tau) = 0$ *for each* $t \in \mathbb{R}$;
(ii) *for any* $t \in \mathbb{R}$, $\epsilon \in (0, \epsilon_0]$ *and* $T \ge 0$, $\cup_{0 \le \tau \le T} U_{2,\epsilon}(t, t-\tau)D_\epsilon(t-\tau)$ *is bounded, and for any* $t \in \mathbb{R}$, *there exists a* $T_{t,\mathcal{D}_{\epsilon_0}} > 0$, *which is independent of* ϵ, *such that*

$$U_{2,\epsilon}(t, t-\tau)D_\epsilon(t-\tau) \subset K_\epsilon(t) \quad \textit{for all } \tau \ge T_{t,\mathcal{D}_{\epsilon_0}}, \;\; \epsilon \in (0, \epsilon_0],$$

and there exists a compact set $K \subset X$ *such that*

$$(H_2)' \qquad \lim_{\epsilon \to 0^+} dist_X(K_\epsilon(t), K) = 0 \quad \textit{for any } t \in \mathbb{R}.$$

Then for each $\epsilon \in (0, \epsilon_0]$, *there exists a pullback attractor* $\mathcal{A}_\epsilon = \{A_\epsilon(t)\}_{t\in\mathbb{R}}$ *and* (H_2) *holds.*

3. Upper semicontinuity of pullback attractors

In this section, we establish the relationship between the pullback attractors $\mathcal{A}_\epsilon = \{A_\epsilon(t)\}_{t\in\mathbb{R}}$ for the perturbed equation (1.1) with $\epsilon > 0$ and the global attractor A for the unperturbed equation (1.1) with $\epsilon = 0$.

We denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ the usual Lebesque and Sobolev spaces, respectively. For simplicity, we denote $||\cdot||_{W^{m,p}(\Omega)}$ and $||\cdot||_{L^p(\Omega)}$ by $||\cdot||_{m,p}$ and $||\cdot||_p$, respectively. Moreover, we denote $||\cdot||_2$ by $||\cdot||$. $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$ and $||\cdot||$ the induced norm. Let $A = \Delta^2$ with domain $H_0^2(\Omega)\cap H^4(\Omega)$, and consider a family of Hilbert spaces $D(A^{\frac{s}{2}})$, $s \in \mathbb{R}$, with the scalar product and the norm

$$(\cdot,\cdot)_{D(A^{\frac{s}{2}})} = (A^{\frac{s}{2}}\cdot, A^{\frac{s}{2}}\cdot), \ \ ||\cdot||_{D(A^{\frac{s}{2}})} = ||A^{\frac{s}{2}}\cdot||.$$

Moreover we recall (see e.g. [8, 13]) that

(i) The embedding $D(A^{\frac{s}{2}}) \hookrightarrow D(A^{\frac{r}{2}})$ is continuous for $0 < r < s$;

(ii) The embedding $D(A^{\frac{s}{2}}) \hookrightarrow L^{\frac{4}{2-4s}}$ is continuous for $0 \le s < \frac{1}{2}$.

For convenience, we set

$$\mathcal{H}^s = D(A^{\frac{s}{2}}), \ \ s \in \mathbb{R},$$

then $\mathcal{H}^0 = L^2(\Omega)$, $\mathcal{H}^1 = H^2(\Omega)\cap H_0^1(\Omega)$ and $\mathcal{H}^2 = H^4(\Omega)\cap H_0^2(\Omega)$.
Throughout this article, we will use the following inequalities:

$$\text{Sobolev inequality}: \ ||u||_p \le c||u||_{1,2} \quad \text{for } u \in H^1(\Omega) \text{ and } p \ge 2;$$

$$\text{Ladyzhenskaya inequality}: \ ||u||_4 \le c||u||^{\frac{1}{2}}||u||_{1,2}^{\frac{1}{2}} \ \text{ for } \ u \in H^1(\Omega).$$

Let λ and λ_1 be the constants with

$$(3.1) \qquad ||u||^2 \le \frac{1}{\lambda_1}||\Delta u||^2, \ \ ||\nabla u||^2 \le \frac{1}{\lambda}||\Delta u||^2 \quad \text{for } u \in H_0^2(\Omega).$$

We put

$$\delta = \min\{\frac{\lambda\mu}{2}, \frac{\mu}{2\alpha^2}\}.$$

For the nonlinear vector function $\overrightarrow{F}(s) = (F_1(s), F_2(s))$, we denote

$$f_i(s) = F_i'(s), \ \ \mathcal{F}_i(s) = \int_0^s F_i(r)dr,$$

$$\overrightarrow{f}(s) = (f_1(s), f_2(s)) \quad \text{and } \overrightarrow{\mathcal{F}}(s) = (\mathcal{F}_1(s), \mathcal{F}_2(s)) \ \ \forall s \in \mathbb{R}.$$

We assume that $F_i (i = 1, 2)$ is a smooth function satisfying

$$(3.2) \qquad F_i(0) = 0, \ \ |F_i(s)| \le c_1|s| + c_2|s|^2 \quad \text{for } s \in \mathbb{R},$$

$$|f_i(s)| \le c_3 + c_4|s| \quad \text{and } |\mathcal{F}_i(s)| \le c_5|s|^2 + c_6|s|^3 \quad \text{for } s \in \mathbb{R}, \tag{3.3}$$

where c_i, $i = 1, 2, \ldots, 6$, is a positive constant.

For the external force $g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, we assume that there exist $\beta \ge 0$ and $0 \le \eta < \frac{\delta}{2}$ such that

$$||g(t)||^2 \le \beta e^{\eta|t|}. \tag{3.4}$$

This infers that for all $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} e^{\delta s}||g(s)||^2 ds < \infty, \quad \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\delta r}||g(r)||^2 dr ds < \infty \tag{3.5}$$

and

$$\int_{-\infty}^{t} \Big(\int_{-\infty}^{s} e^{\frac{\delta}{2}r}||g(r)||^2 dr\Big)^2 ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{3.6}$$

Now, we recall the existence of global attractors for autonomous system (1.1) with $\epsilon = 0$.

Theorem 3.1 (see e.g. [9])**.** *Assume that* (3.2)-(3.6) *hold. Then, problem* (1.1) *possesses a unique global solution u^ϵ satisfying $u^\epsilon \in C([\tau, \infty), \mathcal{H}^1)$ for every $\epsilon \ge 0$, $\tau \in \mathbb{R}$ and $u_\tau \in \mathcal{H}^1$, and the solution is continuous with respect to the initial condition u_τ in $\mathcal{H}^1$.*

Moreover, the semigroup $\{S(t)\}_{t\in\mathbb{R}}$ generated by problem (1.1) *with $\epsilon = 0$ possesses a global attractor A in $\mathcal{H}^1$.*

We construct a continuous process on $\mathcal{H}^1$ generated by problem (1.1). Define the solution operator $U_\epsilon(t, \tau)$ on $\mathcal{H}^1$ as $U_\epsilon(t, \tau)u_\tau = u^\epsilon(t, \tau; u_\tau)$, where $u^\epsilon(t, \tau; u_\tau)$ is the solution of (1.1) with initial data $u_\tau \in \mathcal{H}^1$ at time τ. Then for each $\epsilon > 0$, the solution operator $U_\epsilon(t, \tau)$ forms a continuous process on $\mathcal{H}^1$. We decompose the solution $U_\epsilon(t, \tau)u_\tau$ of (1.1) as follows:

$$U_\epsilon(t, \tau)u_\tau = U_{1,\epsilon}(t, \tau)u_\tau + U_{2,\epsilon}(t, \tau)u_\tau,$$

where $U_{1,\epsilon}(t, \tau)u_\tau = v(t)$ and $U_{2,\epsilon}(t, \tau)u_\tau = w(t)$ solve the following equations

$$\begin{cases} -\Delta v_t + \alpha^2\Delta^2 v_t + \mu\Delta^2 v + \nabla\cdot\overrightarrow{F}(v) + B(v, v) = 0 \text{ in } \Omega\times[\tau, \infty), \\ v = \frac{\partial v}{\partial\nu} = 0 \text{ on } \partial\Omega\times[\tau, \infty), \\ v(x, \tau) = u_\tau(x) \text{ in } \Omega, \end{cases} \tag{3.7}$$

and

$$\begin{cases} -\Delta w_t + \alpha^2\Delta^2 w_t + \mu\Delta^2 w + \nabla\cdot\{\overrightarrow{F}(u^\epsilon) - \overrightarrow{F}(v)\} + B(u^\epsilon, u^\epsilon) - B(v, v) \\ = \epsilon g(x, t) \text{ in } \Omega\times[\tau, \infty), \\ w = \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega\times[\tau, \infty), \\ w(x, \tau) = 0 \text{ in } \Omega. \end{cases} \tag{3.8}$$

We derive some estimates of solutions to problem (1.1), (3.7) and (3.8) by employing the technique in [13]. For convenience, hereafter we denote c an arbitrary positive constant independent of t, τ and ϵ, which may be different from line to line or even in the same line.

Lemma 3.1. *Assume that* (3.2)-(3.6) *hold. Then, for any* $\epsilon > 0$, *the process* $U_\epsilon(\cdot,\cdot)$ *associated with problem* (1.1) *has a pullback absorbing set* $\mathcal{D}_\epsilon = \{D_\epsilon(t)\}_{t\in\mathbb{R}}$ *in* $\mathcal{H}^1$.

Proof. Taking inner product of the first equation of (1.1) with u^ϵ in $L^2(\Omega)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\nabla u^\epsilon||^2+\alpha^2||\Delta u^\epsilon||^2)+\mu||\Delta u^\epsilon||^2\\&+\int_\Omega(\nabla\cdot\overrightarrow{F}(u^\epsilon))u^\epsilon dx+(B(u^\epsilon,u^\epsilon),u^\epsilon)=\epsilon(g,u^\epsilon).\end{aligned}$$

Since

$$\int_\Omega(\nabla\cdot\overrightarrow{F}(u^\epsilon))u^\epsilon dx=-\int_\Omega\overrightarrow{F}(u^\epsilon)\cdot\nabla u^\epsilon dx=-\int_\Omega\nabla\cdot\overrightarrow{\mathcal{F}}(u^\epsilon)dx=0,$$
$$(B(u^\epsilon,u^\epsilon),u^\epsilon)=0$$

and

$$\epsilon(g,u^\epsilon)\le\frac{\mu}{2}||\Delta u^\epsilon||^2+\frac{\epsilon^2||g||^2}{2\lambda_1\mu},$$

it holds that

$$\frac{d}{dt}(||\nabla u^\epsilon||^2+\alpha^2||\Delta u^\epsilon||^2)+\mu||\Delta u^\epsilon||^2\le\frac{\epsilon^2||g||^2}{\lambda_1\mu}.$$

Owing to $\delta=\min\{\frac{\lambda\mu}{2},\frac{\mu}{2\alpha^2}\}$, we have

$$\frac{d}{dt}(||\nabla u^\epsilon||^2+\alpha^2||\Delta u^\epsilon||^2)+\delta(||\nabla u^\epsilon||^2+\alpha^2||\Delta u^\epsilon||^2)\le\frac{\epsilon^2||g||^2}{\mu\lambda_1}.$$

Multiplying this by $e^{\delta t}$ and integrating it over $(t-\tau,t)$, we get

$$\begin{aligned}&||\nabla u^\epsilon(t)||^2+\alpha^2||\Delta u^\epsilon(t)||^2\\&\le e^{-\delta\tau}(||\nabla u^\epsilon(t-\tau)||^2+\alpha^2||\Delta u^\epsilon(t-\tau)||^2)+\frac{\epsilon^2e^{-\delta t}}{\mu\lambda_1}\int_{t-\tau}^t e^{\delta s}||g(s)||^2ds.\end{aligned}\tag{3.9}$$

Let

$$r_\epsilon(t)=\frac{2\epsilon^2e^{-\delta t}}{\mu\lambda_1}\int_{-\infty}^t e^{\delta s}||g(s)||^2ds\tag{3.10}$$

and

$$D_\epsilon(t)=\{u\in\mathcal{H}^1:||u||^2_{\mathcal{H}^1}\le r_\epsilon(t)\},\tag{3.11}$$

then $\mathcal{D}_\epsilon=\{D_\epsilon(t)\}_{t\in\mathbb{R}}$ is pullback absorbing for $U_\epsilon(\cdot,\cdot)$ in $\mathcal{H}^1$. Indeed, for any $t\in\mathbb{R}$, any bounded set $B\subset\mathcal{H}^1$ and $u_{t-\tau}\in B$, (3.9) ensures the existence of $T_{t,B}>0$ satisfying

$$||U_\epsilon(t,t-\tau)u_{t-\tau}||^2_{\mathcal{H}^1}\le r_\epsilon(t)\quad\text{for }\tau\ge T_{t,B}.\tag{3.12}$$

This completes the proof. $\square$

Lemma 3.2. *Let $D_\epsilon(t)$ be given in* (3.11). *Then, for any $t \in \mathbb{R}$, the solution of* (3.7) *satisfies*

$$||U_{1,\epsilon}(t,t-\tau)u_{t-\tau}||^2_{\mathcal{H}^1} \leq c\epsilon^2 e^{-\delta t}\int_{-\infty}^{t-\tau} e^{\delta s}||g(s)||^2 ds \text{ for } \tau \geq 0,\ u_{t-\tau} \in D_\epsilon(t-\tau).$$

Proof. Taking the inner product of (3.7) with v in $L^2(\Omega)$, we have

$$\frac{d}{dt}(||\nabla v||^2 + \alpha^2||\Delta v||^2) + \delta(||\nabla v||^2 + \alpha^2||\Delta v||^2) \leq 0.$$

Multiplying this by $e^{\delta t}$ and integrating it over $(t-\tau,t)$, we derive

$$||\nabla v(t)||^2 + \alpha^2||\Delta v(t)||^2 \leq e^{-\delta\tau}(||\nabla v(t-\tau)||^2 + \alpha^2||\Delta v(t-\tau)||^2).$$

For $v(t-\tau) = u_{t-\tau} \in D_\epsilon(t-\tau)$, it follows that

$$||U_{1,\epsilon}(t,t-\tau)u_{t-\tau}||^2_{\mathcal{H}^1} \leq ce^{-\delta\tau}\epsilon^2 e^{-\delta(t-\tau)}\int_{-\infty}^{t-\tau} e^{\delta s}||g(s)||^2 ds.$$

This finishes the proof. $\square$

Lemma 3.3. *Let $\mathcal{D}_\epsilon = \{D_\epsilon(t)\}_{t\in\mathbb{R}}$ be given in* (3.11). *For any $t \in \mathbb{R}$, there exists $T_{t,\mathcal{D}_{\epsilon_0}} > 0$ and $I_\epsilon(t)$ such that*

$$||U_{2,\epsilon}(t,t-\tau)u_{t-\tau}||^2_{\mathcal{H}^{1+\sigma}} \leq I_\epsilon(t) \text{ for } \tau \geq T_{t,\mathcal{D}_{\epsilon_0}} \text{ and } u_{t-\tau} \in D_\epsilon(t-\tau), \tag{3.13}$$

where $0 < \sigma < \frac{1}{6}$.

Proof. Multiplying (3.8) by $A^\sigma w$ in $L^2(\Omega)$, we get

$$\begin{aligned}&\frac{d}{dt}\Big\{||A^{\frac{1+2\sigma}{4}}w||^2 + \alpha^2||A^{\frac{1+\sigma}{2}}w||^2\Big\} + 2\mu||A^{\frac{1+\sigma}{2}}w||^2\\
&= 2(\nabla\cdot\overrightarrow{F}(v) - \nabla\cdot\overrightarrow{F}(u^\epsilon), A^\sigma w) + 2(B(v,v) - B(u^\epsilon,u^\epsilon), A^\sigma w) + 2(\epsilon g, A^\sigma w).\end{aligned} \tag{3.14}$$

The continuous embedding $D(A^{\frac{1+\sigma}{2}}) \hookrightarrow D(A^\sigma)$, $D(A^{\frac{1+\sigma}{2}}) \hookrightarrow D(A^{\frac{1}{4}+\sigma})$, the condition (3.2), Sobolev inequality and (3.1) give

$$\begin{aligned}&2(\epsilon g, A^\sigma w) \leq 2\epsilon||g||||A^\sigma w|| \leq c\epsilon^2||g||^2 + \frac{\mu}{3}||A^{\frac{1+\sigma}{2}}w||^2,\\
&2(\nabla\cdot\overrightarrow{F}(v) - \nabla\cdot\overrightarrow{F}(u^\epsilon), A^\sigma w)\\
&= 2(\overrightarrow{F}(v) - \overrightarrow{F}(u^\epsilon), \nabla A^\sigma w)\\
&\leq c||\overrightarrow{F}(u^\epsilon) - \overrightarrow{F}(v)||||A^{\frac{1}{4}+\sigma}w||\\
&\leq c(||u^\epsilon||^2 + ||u^\epsilon||_4^4 + ||v||^2 + ||v||_4^4) + \frac{\mu}{3}||A^{\frac{1+\sigma}{2}}w||^2\\
&\leq c\ (||\Delta u^\epsilon||^2 + ||\Delta u^\epsilon||^4 + ||\Delta v||^2 + ||\Delta v||^4) + \frac{\mu}{3}||A^{\frac{1+\sigma}{2}}w||^2.\end{aligned}$$

Using Sobolev inequality, the embeddings

$$D(A^{\frac{1}{6}}) \hookrightarrow L^6(\Omega),\ D(A^{\frac{1+\sigma}{2}}) \hookrightarrow D(A^{\frac{5}{12}+\sigma})$$

and $w = u^\epsilon - v$, we find

$$\begin{aligned}
&2(B(v,v) - B(u^\epsilon, u^\epsilon), A^\sigma w)\\
=&\ -2(B(v,w), A^\sigma w) - 2(B(w,v), A^\sigma w) - 2(B(w,w), A^\sigma w)\\
=&\ 2\int_\Omega \Big(\frac{\partial v}{\partial x_2}\frac{\partial \Delta w}{\partial x_1} - \frac{\partial v}{\partial x_1}\frac{\partial \Delta w}{\partial x_2}\Big)A^\sigma w dx + 2\int_\Omega \Big(\frac{\partial w}{\partial x_2}\frac{\partial \Delta v}{\partial x_1} - \frac{\partial w}{\partial x_1}\frac{\partial \Delta v}{\partial x_2}\Big)A^\sigma w dx\\
&+ 2\int_\Omega \Big(\frac{\partial w}{\partial x_2}\frac{\partial \Delta w}{\partial x_1} - \frac{\partial w}{\partial x_1}\frac{\partial \Delta w}{\partial x_2}\Big)A^\sigma w dx\\
=&\ 2\int_\Omega \Big(\frac{\partial v}{\partial x_1}\Delta w\frac{\partial A^\sigma w}{\partial x_2} - \frac{\partial v}{\partial x_2}\Delta w\frac{\partial A^\sigma w}{\partial x_1}\Big)dx\\
&+ 2\int_\Omega \Big(\frac{\partial w}{\partial x_1}\Delta v\frac{\partial A^\sigma w}{\partial x_2} - \frac{\partial w}{\partial x_2}\Delta v\frac{\partial A^\sigma w}{\partial x_1}\Big)dx\\
&+ 2\int_\Omega \Big(\frac{\partial w}{\partial x_1}\Delta w\frac{\partial A^\sigma w}{\partial x_2} - \frac{\partial w}{\partial x_2}\Delta w\frac{\partial A^\sigma w}{\partial x_1}\Big)dx\\
\le&\ 4\Big(||\nabla v||_3||\Delta w|| + ||\nabla w||_3||\Delta v|| + ||\nabla w||_3||\Delta w||\Big)||A^{\frac{1}{4}+\sigma}w||_6\\
\le&\ c\Big(||\Delta v||||\Delta w|| + ||\Delta w||^2\Big)||A^{\frac{5}{12}+\sigma}w||\\
\le&\ c\Big(||\Delta v||^4 + ||\Delta u^\epsilon||^4\Big) + \frac{\mu}{3}||A^{\frac{1+\sigma}{2}}w||^2.
\end{aligned}$$

Substituting these into (3.14), we see that

$$\begin{aligned}
&\frac{d}{dt}\{||A^{\frac{1+2\sigma}{4}}w||^2 + \alpha^2||A^{\frac{1+\sigma}{2}}w||^2\} + \delta(||A^{\frac{1+2\sigma}{4}}w||^2 + \alpha^2||A^{\frac{1+\sigma}{2}}w||^2)\\
\le&\ c\epsilon^2||g(t)||^2 + c\Big(||\Delta u^\epsilon||^2 + ||\Delta u^\epsilon||^4 + ||\Delta v||^2 + ||\Delta v||^4\Big).
\end{aligned}$$

Multiplying this by $e^{\delta t}$ and integrating it over $(t-\tau, t)$, we obtain

$$\begin{aligned}
(3.15)\qquad &||A^{\frac{1+2\sigma}{4}}w(t)||^2 + \alpha^2||A^{\frac{1+\sigma}{2}}w(t)||^2\\
\le&\ ce^{-\delta t}\int_{t-\tau}^t e^{\delta s}(||\Delta u^\epsilon(s)||^2 + ||\Delta u^\epsilon(s)||^4 + ||\Delta v(s)||^2 + ||\Delta v(s)||^4)ds\\
&+ c\epsilon^2 e^{-\delta t}\int_{t-\tau}^t e^{\delta s}||g(s)||^2 ds,
\end{aligned}$$

here we used the fact that $w(t-\tau) = 0$ in Ω.

From (3.12) and Lemma 3.2, there exists $T_{t,\mathcal{D}_{\epsilon_0}} > 0$ satisfying

$$\begin{aligned}
&\int_{t-\tau}^t e^{\delta s}(||\Delta u^\epsilon(s)||^2 + ||\Delta u^\epsilon(s)||^4 + ||\Delta v(s)||^2 + ||\Delta v(s)||^4)ds\\
\le&\ c\epsilon^2\int_{t-\tau}^t\int_{-\infty}^s e^{\delta r}||g(r)||^2 drds + c\epsilon^4\int_{t-\tau}^t e^{\delta s}\Big(e^{-\delta s}\int_{-\infty}^s e^{\delta r}||g(r)||^2 dr\Big)^2 ds
\end{aligned}$$

$$+ c\epsilon^2 \int_{t-\tau}^{t} \int_{-\infty}^{s-\tau} e^{\delta r}||g(r)||^2 drds + c\epsilon^4 \int_{t-\tau}^{t} e^{\delta s}\Big(e^{-\delta s} \int_{-\infty}^{s-\tau} e^{\delta r}||g(r)||^2 dr\Big)^2 ds$$
$$\leq c\epsilon^2 \int_{t-\tau}^{t} \int_{-\infty}^{s} e^{\delta r}||g(r)||^2 drds + c\epsilon^4 \int_{t-\tau}^{t} e^{\delta s}\Big(e^{-\frac{\delta s}{2}} \int_{-\infty}^{s} e^{\frac{\delta r}{2}}||g(r)||^2 dr\Big)^2 ds$$
$$\leq c\epsilon^2 \int_{-\infty}^{t} \int_{-\infty}^{s} e^{\delta r}||g(r)||^2 drds + c\epsilon^4 \int_{-\infty}^{t} \Big(\int_{-\infty}^{s} e^{\frac{\delta r}{2}}||g(r)||^2 dr\Big)^2 ds.$$

Applying this to (3.15), we deduce that

$$\begin{aligned}
&||U_{2,\epsilon}(t, t-\tau)u_{t-\tau}||^2_{\mathcal{H}^{1+\sigma}} \\
&\leq c\epsilon^2 e^{-\delta t} \int_{-\infty}^{t} e^{\delta s}||g(s)||^2 ds + c\epsilon^2 \int_{-\infty}^{t} \int_{-\infty}^{s} e^{\delta r}||g(r)||^2 drds \\
&\quad + c\epsilon^4 \int_{-\infty}^{t} \Big(\int_{-\infty}^{s} e^{\frac{\delta r}{2}}||g(r)||^2 dr\Big)^2 ds := I_\epsilon(t),
\end{aligned}$$

and obtain the desired result. □

Lemma 3.4. *Let B be a bounded subset in $\mathcal{H}^1$, $t \in \mathbb{R}$, $\tau \in \mathbb{R}^+$ and $u_0 \in B$. If $u^\epsilon(t) = U_\epsilon(t, t-\tau)u_0$ and $u(t) = S(\tau)u_0$ represent the solutions of the perturbed and the unperturbed equations with the same initial data u_0 at time $t-\tau$, respectively, then they satisfy*

$$\lim_{\epsilon\to 0+} \sup_{u_0 \in B} ||U_\epsilon(t, t-\tau)u_0 - S(\tau)u_0||_{\mathcal{H}^1} = 0. \tag{3.16}$$

Proof. Let $z^\epsilon(t) = u^\epsilon(t) - u(t)$, then z^ϵ satisfies

(3.17)
$$\begin{cases} -\Delta z^\epsilon_t + \alpha^2\Delta^2 z^\epsilon_t + \mu\Delta^2 z^\epsilon = \nabla\cdot\overrightarrow{F}(u) - \nabla\cdot\overrightarrow{F}(u^\epsilon) + B(u,u) - B(u^\epsilon, u^\epsilon) \\ +\epsilon g(x,t) \text{ in } \Omega\times[\tau,\infty), \\ z^\epsilon = \frac{\partial z^\epsilon}{\partial\nu} = 0 \text{ on } \partial\Omega\times[\tau,\infty), \\ z^\epsilon(t-\tau) = 0 \text{ in } \Omega. \end{cases}$$

Multiplying (3.17) by z^ϵ, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\nabla z^\epsilon||^2 + \alpha^2||\Delta z^\epsilon||^2) + \mu||\Delta z^\epsilon||^2 \\
&= (\nabla\cdot\overrightarrow{F}(u) - \nabla\cdot\overrightarrow{F}(u^\epsilon), z^\epsilon) + (B(u,u) - B(u^\epsilon,u^\epsilon), z^\epsilon) + \epsilon(g, z^\epsilon) \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.18}$$

Making use of mean value theorem, Hölder inequality, (3.3), Ladyzhenskaya inequality and (3.1), we find that

$$\begin{aligned}
|I_1| &= |(\overrightarrow{F}(u) - \overrightarrow{F}(u^\epsilon), \nabla z^\epsilon)| \\
&= \Big|\int_\Omega \Big\{(F_1(u) - F_1(u^\epsilon)\frac{\partial z^\epsilon}{\partial x_1} + (F_2(u) - F_2(u^\epsilon)\frac{\partial z^\epsilon}{\partial x_2}\Big\}dx\Big| \\
&\leq \int_\Omega \Big\{|F_1'(\kappa u + (1-\kappa)u^\epsilon)(u-u^\epsilon)||\frac{\partial z^\epsilon}{\partial x_1}| + |F_2'(pu + (1-p)u^\epsilon)(u-u^\epsilon)||\frac{\partial z^\epsilon}{\partial x_2}|\Big\}dx
\end{aligned}$$

$$
\begin{aligned}
&=\int_\Omega\Big\{|f_1(\kappa u+(1-\kappa)u^\epsilon)(u-u^\epsilon)||\frac{\partial z^\epsilon}{\partial x_1}|+|f_2(pu+(1-p)u^\epsilon)(u-u^\epsilon)||\frac{\partial z^\epsilon}{\partial x_2}|\Big\}dx\\
&\le c(1+||u^\epsilon||+||u||)||z^\epsilon||_4||\nabla z^\epsilon||_4\\
&\le c(1+||u^\epsilon||+||u||)||z^\epsilon||^{\frac12}||\nabla z^\epsilon||||\Delta z^\epsilon||^{\frac12}\\
&\le c(1+||\Delta u^\epsilon||+||\Delta u||)||\Delta z^\epsilon||^2 \quad \text{for } 0\le\kappa,p\le 1,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= (B(u,u-u^\epsilon),z^\epsilon)+(B(u-u^\epsilon,u^\epsilon),z^\epsilon)\\
&\le 2||\nabla u||_4||\Delta z^\epsilon||||\nabla z^\epsilon||_4+2||\nabla z^\epsilon||_4||\Delta u^\epsilon||||\nabla z^\epsilon||_4\\
&\le c||\nabla u||^{\frac12}||\Delta u||^{\frac12}||\Delta z^\epsilon||||\nabla z^\epsilon||^{\frac12}||\Delta z^\epsilon||^{\frac12}+c||\nabla z^\epsilon||||\Delta z^\epsilon||||\Delta u^\epsilon||\\
&\le c(||\Delta u||+||\Delta u^\epsilon||)||\Delta z^\epsilon||^2,
\end{aligned}
$$

and

$$I_3\le \mu||\Delta z^\epsilon||^2+\frac{\epsilon^2||g||^2}{4\lambda_1\mu}.$$

Adapting these to (3.18), we observe

$$\frac{d}{dt}(||\nabla z^\epsilon||^2+\alpha^2||\Delta z^\epsilon||^2)\le c(1+||\Delta u||+||\Delta u^\epsilon||)||\Delta z^\epsilon||^2+c\epsilon^2||g||^2.$$

This yields that

$$
\begin{aligned}
&||\nabla z^\epsilon(t)||^2+\alpha^2||\Delta z^\epsilon(t)||^2\\
&\le \Big(||\nabla z^\epsilon(t-\tau)||^2+\alpha^2||\Delta z^\epsilon(t-\tau)||^2+\int_{t-\tau}^t c\epsilon^2||g(s)||^2ds\Big)\\
&\quad\cdot e^{\int_{t-\tau}^t c(1+||\Delta u(s)||+||\Delta u^\epsilon(s)||)ds}\\
&\le \Big(c\epsilon^2\int_{-\infty}^t||g(s)||^2ds\Big)e^{\int_{t-\tau}^t c(1+||\Delta u(s)||+||\Delta u^\epsilon(s)||)ds},
\end{aligned}
$$

and which implies (3.16). □

The main result of the paper reads as follows.

Theorem 3.2. *Suppose* (3.2)-(3.6) *hold. Then the pullback attractor* $\mathcal{A}_\epsilon=\{A_\epsilon(t)\}_{t\in\mathbb{R}}$ *for* (1.1) *with* $\epsilon>0$ *and the global attractor* A *for* (1.1) *with* $\epsilon=0$ *satisfy*

$$\lim_{\epsilon\to0^+} dist_X(A_\epsilon(t),A)=0 \quad \textit{for any } t\in\mathbb{R}. \tag{3.19}$$

Proof. We put $\Phi(t,\tau)=c\epsilon_0^2e^{-\delta t}\int_{-\infty}^{t-\tau}e^{\delta s}||g(s)||^2ds$. Then the condition (3.5) gives that $\lim_{\tau\to\infty}\Phi(t,\tau)=0$. Thus, for any $t\in\mathbb{R}$ and $\epsilon\in(0,\epsilon_0]$, Lemma 3.2 ensures the condition (i) of Theorem 2.2. Since the embedding $\mathcal{H}^{1+\sigma}\hookrightarrow\mathcal{H}^1$ is compact, Lemma 3.3 guarantees the condition (ii) of Theorem 2.2. By Lemma 3.4, the condition (H_1) holds. Consequently, from Lemma 3.1, Theorem 2.2 and Theorem 2.1, we obtain (3.19). □

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