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THIRD ORDER HANKEL DETERMINANT FOR CERTAIN UNIVALENT FUNCTIONS

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Abstract. The estimate of third Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

of the analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, for which $\Re(1 + zf''(z)/f'(z)) > -1/2$ are investigated. The corrected version of a known results [2, Theorem 3.1 and Theorem 3.3] are also obtained.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathbb{D})$ normalized by the condition $f(0) = 0 = f'(0) - 1$ and having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$. We denote by $\mathcal{R}$ a subclass of $\mathcal{A}$ consisting of functions f which satisfy $\Re(f'(z)) > 0,\ z \in \mathbb{D}$. Functions in $\mathcal{R}$ are known to be close-to-convex (and hence univalent) in $\mathbb{D}$. Further, a function $f \in \mathcal{A}$ is called starlike (with respect to the origin 0), if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ whose members are starlike in $\mathbb{D}$. It is well known that $f \in \mathcal{S}^*$ satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{2}$$

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Further, let $\mathcal{F}$ be the class of functions $f \in \mathcal{A}$ that are locally univalent and satisfying the inequality

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}. \tag{3}$$

It is well known that functions in the class $\mathcal{F}$ are close-to-convex (and hence univalent) in the unit disk. The class $\mathcal{F}$ plays an important role in the discussion on certain extremal problems for the classes of complex-valued and sense-preserving harmonic convex functions and some other related problems in determining univalence criteria for sense-preserving harmonic mappings (see [26]).

For $f \in \mathcal{A}$ of the form (1), the classical *Fekete-Szegö functional* $\Phi_\lambda(f) = a_3 - \lambda a_2^2$ plays an important role in the function theory. A classical problem settled by Fekete and Szegö [9] is to find for each $\lambda \in [0,1]$, the maximum value of the $|\Phi_\lambda(f)|$ over the function $f \in \mathcal{S}$. By applying the *Löewner* method they proved that

$$\max_{f\in\mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 1+2\exp\{-2\lambda/(1-\lambda)\}, & \lambda \in [0,1) \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating $\max_{f\in\mathbb{F}} |\Phi_\lambda(f)|$ for various compact subfamilies $\mathbb{F}$ of $\mathcal{A}$, as well as λ being an arbitrary real or complex number, was also considered by many authors (see e.g. [1, 5, 12, 13, 14, 20]).

The Hankel determinants $H_{q,n}(f)$ of Taylor's coefficients of functions $f \in \mathcal{A}$ of the form (1), is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \tag{4}$$

where $a_1 = 1$ and $n, q \in \mathbb{N} = \{1, 2, \ldots\}$. The Hankel determinants $H_{q,n}(f)$ are useful, for example, in showing that a function of bounded characteristic in $\mathbb{D}$, *i.e.*, a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [6]. Noonan and Thomas [22] studied the growth rate of the second Hankel determinant of an areally mean p-valent function. Pommerenke [25] proved that the Hankel determinants of univalent functions satisfy $|H_{q,n}(f)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and K depends only on q. Later, Hayman [10] proved that $|H_{2,n}(f)| < A\, n^{1/2}$ (A is an absolute constant) for areally mean univalent functions. Ehrenborg studied Hankel determinant of the exponential polynomials [8] and Noor studied Hankel determinant for the close-to-convex functions [23].

Note that, $H_{2,1}(f) = \Phi_1(f)$ is the *Fekete-Szegö functional.* Recently many authors have studied the problem of calculating $\max_{f\in\mathbb{F}} |H_{2,2}(f)|$ for various subfamilies $\mathbb{F} \subset \mathcal{A}$ (see e.g. [4, 11, 15, 16]). The third Hankel determinant

$H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{5}$$
$$= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

Recently, Babalola [2] has studied $\max_{f\in\mathbb{F}} |H_{3,1}(f)|$ when $\mathbb{F}$ are the classes $\mathcal{R}$, $\mathcal{S}^*$. Also, Raza and Malik [27] have obtained the upper bound on $|H_{3,1}(f)|$ for a subclass of $\mathcal{A}$ associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

The class of *Carathéodory functions* $\mathcal{P}$, is the class of functions $p \in \mathcal{H}(\mathbb{D})$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{6}$$

having a positive real part in $\mathbb{D}$. Following are the well known results for the functions belonging to the class $\mathcal{P}$:

Lemma 1.1 ([7])**.** *If $p \in \mathcal{P}$ is of the form* (6)*, then*

$$|c_n| \le 2, \quad n \in \mathbb{N}. \tag{7}$$

The inequality (7) *is sharp and the equality holds for the function*

$$\varphi(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n.$$

Lemma 1.2 ([18, 19])**.** *If $p \in \mathcal{P}$ is of the form* (6)*, then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{8}$$

and

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z \tag{9}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

2. Main results

We first provide the corrected form of the results in [2, Theorem 3.1 and Theorem 3.2], given in Theorem 2.1 and Theorem 2.2 below.

Theorem 2.1. *Let the function $f \in \mathcal{R}$ of the form* (1)*. Then*

$$|a_2a_3 - a_4| \le \frac{1}{2}. \tag{10}$$

The inequality (10) *is sharp and the equality is attended by the function*

$$f(z) = \int_0^z \frac{1+\zeta^3}{1-\zeta^3} d\zeta. \tag{11}$$

Proof. If $f \in \mathcal{R}$ of the form (1), then $f'(z) = p(z)$, where $p \in \mathcal{P}$ of the form (6). Equating the coefficients of the series expansion of f' and p, we get

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}c_2 \quad \text{and} \quad a_4 = \frac{1}{4}c_3. \tag{12}$$

Hence

$$|a_2a_3 - a_4| = \left|\frac{1}{6}c_1c_2 - \frac{1}{4}c_3\right|. \tag{13}$$

Using Lemma 1.2 in (13) for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we get

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{48}\left|4c_1\{c_1^2 + x(4-c_1^2)\} - 3\{c_1^3 + 2c_1x(4-c_1^2) - c_1x^2(4-c_1^2)\right. \\ &\quad \left. +2(1-|x|^2)(4-c_1^2)z\}\right| \\ &= \frac{1}{48}\left|c_1^3 + (4-c_1^2)(-2c_1x + 3c_1x^2 - 6(1-|x|^2)z\right|. \end{aligned}$$

By Lemma 1.1, we have $|c_1| \leq 2$. Therefore, letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Thus applying the triangle inequality with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_3 - a_4| &\leq \frac{1}{48}\left[c^3 + (4-c^2)(6 + 2c\mu + 3\mu^2(c-2))\right] \\ &= F(c,\mu). \end{aligned} \tag{14}$$

Let $\Omega = \{(c,\mu) : 0 \leq c \leq 2, 0 \leq \mu \leq 1\}$. To find the maximum value of F over the region Ω we use the Hessian matrix method. For this, differentiate F with respect to μ and c and set them equal to zero;

$$\frac{\partial F}{\partial \mu} = \frac{1}{24}\left[(4-c^2)(c + 3\mu(c-2))\right] = 0, \tag{15}$$

$$\frac{\partial F}{\partial c} = \frac{1}{48}\left[8\mu + 12\mu^2 + 12(\mu^2-1)c + 3(1-2\mu-3\mu^2)c^2\right] = 0. \tag{16}$$

Solving (15) and (16) with the help of the mathematica software, we get the critical points

$(-2, -(1+2\sqrt{7})/6)$, $(-2, (-1+2\sqrt{7})/6)$, $(0,0)$, $(2,-3/4)$ and $(8/3,-4/3)$.

Observe that, the only critical point lying in Ω is $(0,0)$. At this critical point $(0,0)$, we find that

$$\frac{\partial^2 F}{\partial \mu^2} = -1 < 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial \mu^2}\frac{\partial^2 F}{\partial c^2} - \left(\frac{\partial^2 F}{\partial \mu\, \partial c}\right)^2 = \frac{2}{9} > 0.$$

Therefore $F(c,\mu)$ has a local maximum at $(0,0)$.

We now look the critical points on the boundary of Ω. At $L_1 = \{(2,\mu) : 0 \leq \mu \leq 1\}$, we have $F(2,\mu) = 1/6$, which is a constant. At $L_2 = \{(0,\mu) : 0 \leq \mu \leq 1\}$, we have $F(0,\mu) = (1-\mu^2)/2$, which gives the same critical point $(0,0)$. At $L_3 = \{(c,1) : 0 \leq c \leq 2\}$, we have $F(c,1) = (5c - c^3)/12$, which gives another critical point $(\sqrt{5/3}, 1)$. At $L_4 = \{(c, 0) : 0 \leq c \leq 2\}$, we have $F(c, 0) = (c^3 - 6c^2 + 24)/48$, giving the same critical point $(0, 0)$. Observe that

$$F(2, \mu) < F(\sqrt{5/3}, 1) < F(0, 0).$$

Thus the local maximum at $(0, 0)$ is also the global maximum on Ω. Hence

$$\max_{\Omega} F(c, \mu) = F(0, 0) = 1/2.$$

To show the sharpness, set $c_1 = x = 0, z = 1$ in (8) and (9), to get $c_2 = 0$ and $c_3 = 2$. Using these values in (13), we find that the inequality (10) is sharp and it can be seen easily that the equality in (10) is attended by the function f given in (11). This completes the proof. □

It is well known that, if $f \in \mathcal{R}$ is of the form (1), then $|a_n| \leq 2/n$, $n = 2, 3, \ldots$, [21], $|a_3 - a_2^2| \leq 2/3$ [3], and $|a_2a_4 - a_3^2| \leq 4/9$ [11]. Using these coefficient bounds and Theorem 2.1, we get

$$\begin{aligned}|H_{3,1}(f)| &\leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2| \\ &\leq \frac{2}{3}.\frac{4}{9} + \frac{2}{4}.\frac{1}{2} + \frac{2}{5}.\frac{2}{3} = \frac{439}{540}.\end{aligned}$$

Thus, we state that:

Theorem 2.2. *Let the function $f \in \mathcal{R}$ of the form* (1)*. Then*

$$|H_{3,1}(f)| \leq \frac{439}{540}.$$

Remark 2.3. Babalola in [2, Theorem 3.3] proved that, *if $f \in \mathcal{S}^*$ is of the form* (1)*, then $|a_2a_3 - a_4| \leq 2$. This inequality is sharp and the equality is attended for the Koebe function $k(z) = z/(1 - z)^2$ and its rotation.* While observing its proof, we see, that the author's claim about $F'(\rho) > 0$ is not correct. From the method used in Theorem 2.1, we can easily see that the result in [2, Theorem 3.3] is correct and its proof is similar to that of Theorem 2.1 above. This can easily be worked out, and therefore, we skip giving details in this regard.

Theorem 2.4. *Let the function $f \in \mathcal{F}$ of the form* (1)*. Then*

$$|a_3 - a_2^2| \leq \frac{1}{2}. \tag{17}$$

The inequality (17) *is sharp.*

Proof. If $f \in \mathcal{F}$ of the form (1), then we may write

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2}p(z) - \frac{1}{2}.$$

Substituting the series expansion of $f''(z)$, $f'(z)$ and $p(z)$ and equating the coefficients, we get

$$a_2 = \frac{3}{4}c_1, \quad a_3 = \frac{1}{8}(3c_1^2 + 2c_2), \quad a_4 = \frac{1}{64}(9c_1^3 + 18c_1c_2 + 8c_3). \tag{18}$$

Using these values of coefficients and Lemma 1.2 for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we get

$$|a_3 - a_2^2| = \frac{1}{16}\left|-c_1^2 + 2x(4-c_1^2)\right|. \tag{19}$$

By Lemma 1.1, we may assume $c_1 = c \in [0,2]$. Applying the triangle inequality in (19) with $\mu = |x|$, we obtain

$$|a_3 - a_2^2| \leq \frac{1}{16}\left[c^2 + 2\mu(4-c^2)\right] = H_1(c,\mu).$$

Differentiating H_1 with respect to μ, we get

$$\frac{\partial H_1}{\partial \mu} = \frac{1}{8}(4-c^2) \geq 0 \quad \text{for} \quad 0 \leq \mu \leq 1.$$

Hence, H_1 is an increasing function of μ on $[0,1]$. Therefore

$$\max_{0\leq\mu\leq 1} H_1(c,\mu) = H_1(c,1) = \frac{1}{16}(8-c^2) = \mathcal{H}(c).$$

It is clear that $\mathcal{H}(c)$ is a decreasing function of $c\,(0 \leq c \leq 2)$, hence the maximum value of $H_1(c,\mu)$ is attended at the point $(0,1)$, that is,

$$\max_{\Omega} H_1(c,\mu) = H_1(0,1) = \frac{1}{2}.$$

To show the sharpness of (17), choose $c_1 = 0$ and $x = 1$ in (8) and (9), we get $c_2 = 2$ and $c_3 = 0$. Using these values in (19) we find that inequality (17) is sharp. This completes the proof. □

Theorem 2.5. *Let the function $f \in \mathcal{F}$ of the form* (1). *Then*

$$|a_2a_3 - a_4| \leq \frac{9}{4\sqrt{15}}.$$

Proof. Using the values of a_2, a_3 and a_4 from (18) and using (8) and (9) for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we get

$$|a_2a_3 - a_4| = \frac{1}{64}\left|4c_1^3 + (4-c_1^2)\{-7c_1x + 2c_1x^2 - 4(1-|x|^2)z\}\right|. \tag{20}$$

By Lemma 1.1, we have $|c_1| \leq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Thus applying the triangle inequality in (20) with $\mu = |x|$, we obtain

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{1}{64}\left[4c^3 + (4-c^2)(7c\mu + 2c\mu^2 + 4 - 4\mu^2)\right]\\ &= H_2(c,\mu).\end{aligned}$$

Differentiating H_2 with respect to μ and c, we get

$$\begin{aligned}\frac{\partial H_2}{\partial \mu} &= \frac{1}{64}\left[(4-c^2)(7c + 4c\mu - 8\mu)\right],\\ \frac{\partial H_2}{\partial c} &= \frac{1}{64}\left[12c^2 + 28\mu + 8\mu^2 - 21c^2\mu - 6c^2\mu^2 - 8c + 8c\mu^2\right].\end{aligned}$$

Solving $\frac{\partial H_2}{\partial \mu} = 0$ and $\frac{\partial H_2}{\partial c} = 0$, we find that the critical points of H_2 are

$$(-2, -(7+\sqrt{177})/8), \quad (-2, (-7+\sqrt{177})/8),$$
$$(-44/81, -77/206), \quad (0,0) \quad \text{and} \quad (2, 4/7).$$

Observe that $(0,0)$ and $(2,4/7)$ are the only critical points laying inside Ω, but at both points

$$\frac{\partial^2 H_2}{\partial \mu^2}\frac{\partial^2 H_2}{\partial c^2} - \left(\frac{\partial^2 H_2}{\partial \mu \partial c}\right)^2 < 0.$$

Hence, $H_2(c,\mu)$ does not attain extremum at $(0,0)$ and $(2,4/7)$.

Next, we examine the critical points at the boundary of Ω. We find that, along $L_1 = \{(2,\mu) : 0 \leq \mu \leq 1\}$, we have $H_2(2,\mu) = 1/2$, which is a constant and another critical points at the boundary are only $(2/3, 0)$ and $(6/\sqrt{15}, 1)$. Since $H_2(2/3,0) < H_2(2,\mu) < H_2(6/\sqrt{15},1)$, we get

$$\max_{\Omega} H_2(c,\mu) = H_2(6/\sqrt{15}, 1) = \frac{9}{4\sqrt{15}}.$$

This completes the proof. □

Theorem 2.6. *Let the function $f \in \mathcal{F}$ of the form* (1). *Then*

$$|a_2a_4 - a_3^2| \leq \frac{21}{64}.$$

Proof. Using the values of a_2, a_3 and a_4 from (18) and using (8) and (9) for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we get

$$\begin{aligned} &|a_2a_4 - a_3^2| \\ &= \frac{1}{256}\left|-4c_1^4 + (4-c_1^2)\{7c_1^2x - 6c_1^2x^2 + 12c_1(1-|x|^2)z - 4x^2(4-c_1^2)\}\right|. \end{aligned}$$

By Lemma 1.1, we assume $c_1 = c \in [0,2]$. Applying the triangle inequality in above equation with $\mu = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{1}{256}\left[4c^4 + (4-c^2)(7c^2\mu + 2c^2\mu^2 + 12c - 12c\mu^2 + 16\mu^2)\right] \\ &= H_3(c,\mu). \end{aligned}$$

Differentiating H_3 with respect to μ and c, we get

$$\begin{aligned} \frac{\partial H_3}{\partial \mu} &= \frac{1}{256}\left[(4-c^2)(7c^2 + 4c^2\mu - 24c\mu + 32\mu)\right], \\ \frac{\partial H_3}{\partial c} &= \frac{1}{256}\left[16c^3 + 56c\mu - 16c\mu^2 + 48 - 48\mu^2 - 28c^3\mu - 8c^3\mu^2 - 36c^2 + 36c^2\mu^2\right]. \end{aligned}$$

Solving $\frac{\partial H_3}{\partial \mu} = 0$ and $\frac{\partial H_3}{\partial c} = 0$, we get the critical points are

$$(-2, -(7+\sqrt{721})/24), \quad (-2, (-7+\sqrt{721})/24), \quad \text{and} \quad (2, 2/7).$$

We observe that, $(2, 2/7)$ is the only critical point laying inside Ω, but at this point

$$\frac{\partial^2 H_3}{\partial \mu^2}\frac{\partial^2 H_3}{\partial c^2} - \left(\frac{\partial^2 H_3}{\partial \mu \partial c}\right)^2 < 0.$$

Hence H_3 does not attain extremum at (2,2/7).

Next, we examine the critical points at the boundary of Ω. We find that, along $L_1 = \{(2, \mu) : 0 \le \mu \le 1\}$, $H_3(2, \mu) = 1/4$, which is a constant and other critical points at the boundary are only $(0, 1)$ and $(\sqrt{2}, 1)$. Hence $H_3(0, 0) < H_3(2, \mu) = H_3(0, 1) < H_3(\sqrt{2}, 1)$. Therefore

$$\max_{\Omega} H_3(c, \mu) = H_3(\sqrt{2}, 1) = \frac{21}{64}.$$

This completes the proof. □

It is known that, if $f \in \mathcal{F}$ of the form (1), then $|a_n| \le \frac{n+1}{2}$ for $n \ge 2$ [26]. Using this bound and Theorem 2.4, Theorem 2.5 and Theorem 2.6, we get:

Theorem 2.7. *Let the function $f \in \mathcal{F}$ of the form* (1). *Then*

$$|H_{3,1}(f)| \le \frac{180 + 69\sqrt{15}}{32\sqrt{15}}.$$

Remark 2.8. For $f \in \mathcal{S}$, Thomas [24, p. 166] conjectured that

$$|H_{2,n}(f)| = |a_n a_{n+2} - a_{n+1}^2| \le 1, \quad n = 2, 3, \ldots.$$

Subsequently, Li and Srivastava [17, p. 1040] showed that this conjecture is not valid for $n \ge 4$, *i.e.*, conjecture is valid only for $n = 2, 3$. From the known result $|a_2 a_4 - a_3^2| \le 4/9$ (see [11]) and Theorem 2.6, we found that, if the function f is a member of the class $\mathcal{R}$ and $\mathcal{F}$, respectively and each having form (1), then

$$|H_{2,2}(f)| \le \frac{4}{9} \quad \text{and} \quad |H_{2,2}(f)| \le \frac{21}{64}.$$

Since all functions in $\mathcal{R}$ and $\mathcal{F}$ are close-to-convex and hence also univalent in $\mathbb{D}$. Therefore, the result in [11] and Theorem 2.6 validate the Thomas conjecture when $n = 2$ for the function belonging to the classes $\mathcal{R}$ and $\mathcal{F}$.

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