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SOME PROPERTIES OF (m, n)-POTENT CONDITIONS

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Abstract. In this paper, we will consider the notions of (m, n)-potent conditions in near-rings, in particular, a near-ring R with left bipotent or right bipotent condition. We will derive some properties of near-rings with $(1, n)$ and $(n, 1)$-potent conditions where n is a positive integer, and then some properties of near-rings with (m, n)-potent conditions. Also, we may discuss the behavior of R-subgroups in $(1, n)$-potent or $(n, 1)$-potent near-rings..

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1. Introduction

The concept of Von Neumann regularity of near-rings have been studied by many authors Beidleman [2], Choudhari [3], Heatherly [4], Ligh, Mason [5], Murty, and Szeto [9]. Their main results are appeared in the book of Pilz [8].

In 1980, Mason [5] introduced the notions of left regularity, right regularity and strong regularity of near-rings.

In 1985, Ohori [7] investigated the characterization of π-regularity and strong π-regularity of rings.

A *near-ring* R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with a zero element 0, $(R, \cdot)$ is a semigroup and $(a + b)c = ac + bc$ for all a, b, c in R.

A *near-field* is a unitary near-ring $(F, +, \cdot)$ where $(F^* = F \setminus \{0\}, \cdot)$ is a group [8].

A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric.* An element d in R is called *distributive* if $d(a + b) = da + db$ for all a and b in R.

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We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in R \mid a0 = a\}$ which is called the *constant part* of R. The set of all distributive elements in R is denoted by R_d.

In 1979, Jat and Choudhari defined a near-ring R to be left bipotent (resp. right bipotent) if $Ra = Ra^2$ (resp. $aR = a^2R$) for each a in R. Also, we can define a near-ring R as subcommutative if $aR = Ra$ for all a in R like as in ring theory. Obviously, every commutative near-ring is subcommutative. From these above two concepts it is natural to investigate the near-ring R with the properties $aR = Ra^2$ (resp. $a^2R = Ra$) for each a in R. We say that such is a near-ring with $(1, 2)$-potent conditions (resp. a near-ring with $(2, 1)$-potent conditions). Thus, from this motivation, we can extend a general concept of a near-ring R with (m, n)-potent conditions.

First, we will derive properties of near-ring with $(1, 2)$ and $(2, 1)$-potent conditions, also $(1, n)$ and $(n, 1)$-potent conditions where n is a positive integer. Any homomorphic image of (m, n)-potent near-ring is also (m, n)-potent.

Next, we will find some properties of regular near-rings with (m, n)-potent conditions. Also, we will discuss the behavior of R-subgroups in $(1, n)$-potent or $(n, 1)$-potent near-rings.

For the rest of basic concepts and results on near-rings, we will refer to [8].

2. Results on (m, n)-potent conditions in Near-Rings

Let R and S be two near-rings. Then a mapping f from R to S is called a *near-ring homomorphism* if (i) $f(a + b) = f(a) + f(b)$, (ii) $f(ab) = f(a)f(b)$. We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism as in ring theory [1].

We say that a near-ring R has the *insertion of factors property (briefly, IFP)* provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$. A near-ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. On the other hand, we say that R has the *reversible IFP* in case R has the IFP and is reversible.

Also, we say that R is reduced if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. McCoy [6] proved that R is reduced iff for each a in R, $a^2 = 0$ implies $a = 0$.

A (two-sided) *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R.

Let $(G, +)$ be a group (not necessarily abelian) with the identity element o. In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $(f + g)x = fx + gx$ for all $x \in G$ and the product $f \cdot g$ by the rule $(f \cdot g)x = f(gx)$ for all $x \in G$, here, for convenience we write the image of

f at a variable x, fx instead of $f(x)$, then $(M(G),+,\cdot)$ becomes a near-ring. It is called the *self map near-ring on the group* G. Also, if we can define the set

$$M_0(G) := \{f \in M(G) \mid fo = o\},$$

then $(M_0(G),+,\cdot)$ is a zero symmetric near-ring, and

$$M_c(G) := \{f \in M(G) \mid f: \ constant\},$$

then $(M_0(G),+,\cdot)$ is a constant near-ring. $(G,+)$ is abelian if and only if $(M(G),+)$ is abelian.

A near-ring R [8] is called *simple* if it has no non-trivial ideal, that is, R has no ideals except 0 and R. Also, R is called *R-simple* if R has no R-subgroups except $R0$ and R.

A near-ring R is called *left regular (resp. right regular)* if for each a in R, there exists an element x in R such that

$$a = xa^2 (resp.\ a = a^2x).$$

A near-ring R is called *strongly left regular* if R is left regular and regular, similarly, we can define strongly right regular. A strongly left regular and strongly right regular near-ring is called *strongly regular near-ring.*

A near-ring R is called *left κ-regular (resp. right κ-regular)* if for each a in R, there exists an element x in R such that

$$a^n = xa^{n+1} (resp.\ a^n = a^{n+1}x)$$

for some positive integer n. A left κ-regular and right κ-regular near-ring is called *κ-regular near-ring.*

An integer group $(\mathbb{Z}_2,+)$ modulo 2 with the multiplication rule: $0\cdot 0 = 0\cdot 1 = 0$, $1\cdot 0 = 1\cdot 1 = 1$ is a near-field. Obviously, this near-field is isomorphic to $M_c(\mathbb{Z}_2)$. All other near-fields are zero-symmetric. Consequently, we get the following important statement.

Lemma 2.1 ([8]). *Let R be a near-field. Then $R \cong M_c(\mathbb{Z}_2)$ or R is zero-symmetric.*

In our subsequent discussion of near-fields, we will exclude the silly near-field $M_c(\mathbb{Z}_2)$ of order 2. Evidently, every near-field is simple.

Lemma 2.2 ([8]). *Let R be a near-ring. Then the following statements are equivalent:*

(1) *R is a near-field.*
(2) *$R_d \neq 0$ and for each nonzero element a in R, $Ra = R$.*
(3) *R has a left identity and R is R-simple.*

From now on, we give the new concept of an (m,n)-potent near-ring, and then illustrate this notion with suitable examples.

We say that a near-ring R has the *(m,n)-potent condition* if for all a in R, there exist positive integers m,n such that $a^mR = Ra^n$. We shall refer to such a near-ring as an *(m,n)-potent near-ring.*

Obviously, every (m, n)-potent near-ring is zero-symmetric. On the other hand, from the Lemmas 2.1 and 2.2, we obtain the following examples (1), (2).

Examples 2.3. (1) Every near-field is an (m, n)-potent near-ring for all positive integers m, n.

(2) The direct sum of near-fields is an (m, n)-potent near-ring for all positive integers m, n.
(3) Every subcommutative near-ring is an $(1, 1)$-potent near-ring.
(4) Every Boolean subcommutative near-ring is an (m, n)-potent near-ring for all positive integers m, n.
(5) Let $R = \{0, a, b, c\}$ be an additive Klein 4-group. This is a near-ring with the following multiplication table (p. 408 [8]):

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	c	a
b	0	c	a	b
c	0	a	b	c

This near-ring have several $(1, 1)$, $(1, 4)$, $(2, 2)$, $(2, 4)$, $(3, 3)$, $(4, 1)$, $(4, 2)$ and $(4, 4)$-potent conditions, but it is not a Boolean near-ring.

A near-ring R is called *left S-unital (resp. right S-unital)* if for each a in R, $a \in Ra$ (resp. $a \in aR$).

Lemma 2.4. *Let R be a zero-symmetric and reduced near-ring. Then R has the reversible IFP.*

Proof. Suppose that a, b in R such that $ab = 0$. Then, since R is zero-symmetric, we have

$$(ba)^2 = baba = b0a = b0 = 0.$$

Reducedness implies that $ba = 0$.

Next, assume that for all a, b, x in R with $ab = 0$. Then

$$(axb)^2 = axbaxb = ax0xb = ax0 = 0$$

This implies $axb = 0$, by reducedness. Hence R has the reversible IFP. □

Theorem 2.5. *Let R be an $(n, n+2)$-potent reduced near-ring, for some positive integer n. Then R is a left κ-regular near-ring.*

Proof. Suppose R is an $(n, n+2)$-potent reduced near-ring. Then for any a in R, we have that

$$a^n R = Ra^{n+2}$$

This implies that $a^{n+1} \in a^n R = Ra^{n+2}$. Hence there exists x in R such that $a^{n+1} = xa^{n+2}$, that is, $(a^n - xa^{n+1})a = 0$. From Lemma 2.4, we see that $a(a^n -$

$xa^{n+1}) = 0$. Also, we can compute that $a^n(a^n - xa^{n+1}) = 0$ and $xa^{n+1}(a^n - xa^{n+1}) = 0$. Thus from the equation

$$(a^n - xa^{n+1})^2 = a^n(a^n - xa^{n+1}) - xa^{n+1}(a^n - xa^{n+1}) = 0 - 0 = 0$$

and reducedness, we see that $a^n = xa^{n+1}$. Consequently, R is a left κ-regular near-ring. □

Corollary 2.6. *Let R be an $(1,3)$-potent reduced near-ring. Then R is a left regular near-ring.*

Theorem 2.7. *Let R be an $(1,2)$-potent near-ring. Then we have the following statements:*

(1) *If R is reduced, then R is a left S-unital near-ring.*
(2) *If R is right S-unital, then R is a left regular and reduced near-ring.*

Proof. Since R is an $(1,2)$-potent near-ring, consider the equality, $aR = Ra^2$ for each a in R.

(1) From $a^2 \in aR = Ra^2$, there exists x in R such that $a^2 = xa^2$. This implies that $(a - xa)a = 0$. Since R is zero-symmetric and reduced, Lemma 2.4 guarantees that $a(a - xa) = 0$ and $xa(a - xa) = 0$. Hence we have the equation

$$(a - xa)^2 = a(a - xa) - xa(a - xa) = 0 - 0 = 0$$

Reducedness implies that $(a - xa) = 0$, that is, $a = xa$, for some $x \in R$. Therefore R is left S-unital.

(2) Since R is right S-unital and has $(1,2)$-potent condition, for each $a \in R$, $a \in aR = Ra^2$. Thus $a = xa^2$, for some $x \in R$. Also, in this equation, $a^2 = 0$ implies that $a = 0$. Hence R is a left regular and reduced near-ring. □

Proposition 2.8. *Let R be an $(2,1)$-potent near-ring. Then we have the following statements:*

(1) *If $R = R_d$ is reduced, then R is a right S-unital near-ring.*
(2) *If R is left S-unital, then R is a right regular and reduced near-ring.*

Proof. This proof is an analogue of the proof in Proposition 2.7. □

Theorem 2.9. *Every homomorphic image of an (m,n)-potent near-ring is also an (m,n)-potent near-ring.*

Proof. Let R be an (m,n)-potent near-ring and let $f : R \longrightarrow R'$ be a near-ring epimorphism. Consider an equality $a^mR = Ra^n$, for all $a \in R$, where m, n are positive integers.

We must show that for all $a' \in R'$, $a'^mR' = R'a'^n$, for some positive integers m, n. Let a', $x' \in R'$. Then there exist a, $x \in R$ such that $a' = f(a)$ and $x' = f(x)$. So we get the following equations:

$$a'^m x' = f(a)^m f(x) = f(a^m)f(x) = f(a^m x) = f(ya^n) = f(y)f(a)^n = f(y)a'^n,$$

where $a^m x \in a^m R = Ra^n$, so that there exist $y \in R$ such that $a^m x = ya^n$. This implies that ${a'}^m R' \subset R'{a'}^n$.

In a similar fashion, we obtain that $R'{a'}^n \subset {a'}^m R'$. Therefore our desired result is completed. □

Finally, we may discuss the behavior of R-subgroups of $(1, n)$-potent near-ring as following.

Proposition 2.10. *Every left R-subgroup of an $(1, n)$-potent near-ring R is an R-subgroup.*

Proof. Let A be a left R-subgroup of R. Then we see that $RA \subset A$. To show that $AR \subset A$, let $ar \in AR$, where $a \in A,\ r \in R$. Since R has $(1, n)$-potent condition, we have $ar \in aR = Ra^n$. This implies that

$$ar = sa^n = (sa^{n-1})a \in Ra \subset RA \subset A,$$

for some s in R. Hence A is an R-subgroup of R. □

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