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SOLVING OPERATOR EQUATIONS $AX = Y$ AND $Ax = y$ IN ALG$\mathcal{L}^{\dagger}$

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Abstract. In this paper the following is proved: Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and X and Y be operators acting on a Hilbert space $\mathcal{H}$. If $XE = EX$ for each $E \in \mathcal{L}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ if and only if $\sup \left\{ \frac{\|YEf\|}{\|XEf\|} : f \in \mathcal{H},\ E \in \mathcal{L} \right\} = K < \infty$ and $YE = EYE$.

Let x and y be non-zero vectors in $\mathcal{H}$. Let P_x be the orthogonal projection on $sp(x)$. If $EP_x = P_xE$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$.

(2) $< f, Ey > y =< f, Ey > Ey$ for each $E \in \mathcal{L}$ and $f \in \mathcal{H}$.

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1. Introduction

Interpolation problems have been developed by many mathematicians since Douglas considered a problem to find a bounded operator A satisfying $AX = Y$ for two operators X and Y acting on a Hilbert space $\mathcal{H}$ in 1966 [1, 2, 3, 4, 5, 6]. Douglas used the range inclusion property of operators to show necessary and sufficient conditions for the existence of an operator A such that $AX = Y$. A condition for the operator A to be a member of $\mathcal{A}$ which is a specified subalgebra of $\mathcal{B}(\mathcal{H})$ can be given. In this paper, authors investigated to find sufficient and necessary conditions that there exists an operator A in Alg$\mathcal{L}$ satisfying $AX = Y$ for operators X and Y acting on a Hilbert space $\mathcal{H}$ and there exists an operator B in Alg$\mathcal{L}$ satisfying $Bx = y$ for two vectors x and y in $\mathcal{H}$. And authors investigated

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the above interpolation problems for finitely or countably many operators and vectors.

The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem of Douglas.

Theorem 1.1 (R.G. Douglas [1]). *Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) rangeY^* $\subseteq$ range X^*

(2) $Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$

(3) there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.

Moreover, if (1), (2) and (3) are valid, then there exists a unique operator A so that

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$

(b) kerY^* = kerA^* and

(c) rangeA^* $\subseteq$ rangeX^-.

We need to look at the proof of Theorem A carefully. Then we know that the image of A on $\overline{rangeX}^{\perp}$ is 0 from the proof of (3) by (2).

2. The Equation $AX = Y$ in Alg$\mathcal{L}$

Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$ containing the trivial projections 0 and I. The symbol Alg$\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. A lattice $\mathcal{L}$ is a *commutative subspace lattice*, or CSL, if the projections in $\mathcal{L}$ commute; in this case, Alg$\mathcal{L}$ is called a *CSL algebra*. Let $x_1, \cdots, x_n$ be vectors of $\mathcal{H}$. Then $sp(\{x_1, \cdots, x_n\}) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \ : \ \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}\ \}$. Let M be a subset of $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{C}$ be the set of complex numbers.

Let $\mathcal{L}$ be a subspace lattice and A, X and Y be operators acting on a Hilbert space $\mathcal{H}$ such that $AX = Y$. If $XE = EX$, then $\|YEf\| = \|AXEf\| = \|AEXf\| \leq \|A\|\|XEf\|$ for all $E \in \mathcal{L}$ and for all f in $\mathcal{H}$. If we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup\left\{\frac{\|YEf\|}{\|XEf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} \leq \|A\|.$$

Theorem 2.1. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and X and Y be operators acting on the Hilbert space $\mathcal{H}$. If $XE = EX$ for each E in $\mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.

(2) $\sup\left\{\dfrac{\|YEf\|}{\|XEf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$ *and* $YE = EYE$ *for each* E *in* $\mathcal{L}$.

Proof. Assume that $\sup\left\{\dfrac{\|YEf\|}{\|XEf\|} : f \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$ and $YE = EYE$ for each E in $\mathcal{L}$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(XE) = YE = EYE$ by Theorem A. In particular, if $E = I$, then we have an operator A_I in $\mathcal{B}(\mathcal{H})$ such that $A_I X = Y$. So $A_E(XE) = A_I XE = EA_I XE$ for each E in $\mathcal{L}$. Since $EX = XE$ for each $E \in \mathcal{L}$, $A_I XE = EA_I EX$. Hence $A_I E = EA_I E$ on $\overline{rangeX}$. Let h be in $\overline{rangeX}^{\perp}$. Since $EX = XE$ for each E in $\mathcal{L}$, $< Eh, Xf >=< h, EXf >=< h, XEf >= 0$. So $Eh \in \overline{rangeX}^{\perp}$. By the definition of A_I, $(A_I E)h = 0 = (EA_I E)h$. Hence $A_I E = EA_I E$ on $\overline{rangeX}^{\perp}$. So A_I is an operator in Alg$\mathcal{L}$. □

Assume that $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for each $i = 1, \cdots, n$. Then $Y_i Ef_i = AX_i Ef_i$ for each $i = 1, \cdots, n$, $E \in \mathcal{L}$ and each f_i in $\mathcal{H}$. Hence

$$\begin{aligned}\|\sum_{i=1}^{n} Y_i Ef_i\| &= \|\sum_{i=1}^{n} AX_i Ef_i\| \\ &\le \|A\|\|\sum_{i=1}^{n} X_i Ef_i\|\end{aligned}$$

for all $E \in \mathcal{L}$ and all f_i in $\mathcal{H}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequality above may be stated in the form

$$\sup\left\{\frac{\|\sum_{i=1}^{n} Y_i Ef_i\|}{\|\sum_{i=1}^{n} X_i Ef_i\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} \le \|A\|.$$

Theorem 2.2. *Let* $X_1, \cdots, X_n$ *and* $Y_1, \cdots, Y_n$ *be bounded operators acting on* $\mathcal{H}$. *If* $X_i E = EX_i$ *for each* E *in* $\mathcal{L}$ *and* i *in* $\{1, 2, \cdots, n\}$, *then the following are equivalent.*

(1) There exists an operator A *in Alg*$\mathcal{L}$ *such that* $AX_i = Y_i$ *for* $i = 1, 2, \cdots, n$.

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^{n} Y_i Ef_i\|}{\|\sum_{i=1}^{n} X_i Ef_i\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$ *and* $Y_i E = EY_i E$ *for each* $i = 1, \cdots, n$ *and* E *in* $\mathcal{L}$.

Proof. Assume that $\sup\left\{\dfrac{\|\sum_{i=1}^{n} Y_i Ef_i\|}{\|\sum_{i=1}^{n} X_i Ef_i\|} : f_i \in \mathcal{H},\ E \in \mathcal{L}\right\} = K < \infty$ and $Y_i E = EY_i E$ for each $i = 1, \cdots, n$ and $E \in \mathcal{L}$. Let E be in $\mathcal{L}$ and

$$\mathcal{M}_E = \left\{\sum_{i=1}^{n} X_i Ef_i \ : \ f_i \in \mathcal{H}\right\}.$$

Define $A_E : \mathcal{M}_E \to \mathcal{H}$ by $A_E(\sum_{i=1}^n X_iEf_i) = \sum_{i=1}^n Y_iEf_i$. Then A_E is well-defined and bounded linear. Extend A_E on $\overline{\mathcal{M}_E}$ continuously. Define $A_Ef = 0$ for each $f \in \overline{\mathcal{M}_E}^{\perp}$. Then $A_E : \mathcal{H} \to \mathcal{H}$ is a bounded linear and $A_EEX_i = Y_iE$ for $i = 1, \cdots, n$. If $E = I$, then $A_IX_i = Y_i$ for $i = 1, \cdots, n$. Since $EX_i = X_iE$ and $Y_iE = EY_iE$ for each $i = 1, \cdots, n$, $A_EX_iE = A_IX_iE = A_IEX_i$ and $A_EX_iE = EA_IX_iE = EA_IEX_i$. Hence $A_IE = EA_IE$ on $\overline{\mathcal{M}_E}$. Let h be in $\overline{\mathcal{M}_E}^{\perp}$. Then since $EX_i = X_iE$ for each $i = 1, \cdots, n$, $< Eh, X_if >=< h, EX_if >=< h, X_iEf >= 0$ for each $f \in \mathcal{H}$. So

$$\left\langle Eh\ , \sum_{i=1}^n X_if_i \right\rangle = 0\ .$$

By the definition of A_I, $A_IEh = 0 = EA_IEh$ for each $E \in \mathcal{L}$. Hence $A_IE = EA_IE$ on $\overline{\mathcal{M}_E}^{\perp}$. So A_I is an operator in Alg$\mathcal{L}$ $\square$

We can generalize the above Theorem to the countable case easily.

Theorem 2.3. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. If $X_iE = EX_i$ for each E in $\mathcal{L}$ and i in $\mathbb{N}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^m Y_iEf_i\|}{\|\sum_{i=1}^m X_iEf_i\|} : f_i \in \mathcal{H},\ E \in \mathcal{L} \text{ and } m \in \mathbb{N}\right\} = K < \infty$ *and* $Y_iE = EY_iE$ *for each* $i = 1, \cdots$ *and* $E \in \mathcal{L}$.

Proof. Assume that $\sup\left\{\dfrac{\|\sum_{i=1}^m Y_iEf_i\|}{\|\sum_{i=1}^m X_iEf_i\|} : f_i \in \mathcal{H},\ E \in \mathcal{L} \text{ and } m \in \mathbb{N}\right\} = K < \infty$ and $Y_iE = EY_iE$ for each $i = 1, \cdots$. Let E be in $\mathcal{L}$ and

$$\mathcal{N}_E = \left\{\sum_{i=1}^m X_iEf_i\ :\ f_i \in \mathcal{H} \text{ and } m \in \mathbb{N}\right\}.$$

Define $A_E : \mathcal{N}_E \to \mathcal{H}$ by $A_E(\sum_{i=1}^m X_iEf_i) = \sum_{i=1}^m Y_iEf_i$. Then A_E is well-defined and bounded linear. Extend A_E on $\overline{\mathcal{N}_E}$ continuously. Define $A_Ef = 0$ for each $f \in \overline{\mathcal{N}_E}^{\perp}$. Then $A_E : \mathcal{H} \to \mathcal{H}$ is a bounded linear and $A_EEX_i = Y_iE$ for $i = 1, \cdots$. If $E = I$, then $A_IX_i = Y_i$ for $i = 1, \cdots$. Since $EX_i = X_iE$ and $Y_iE = EY_iE$ for each $i = 1, \cdots$, $A_EX_iE = A_IX_iE = A_IEX_i$ and $A_EX_iE = EA_IX_iE = EA_IEX_i$. Hence $A_IE = EA_IE$ on $\overline{\mathcal{N}_E}$. Let h be in $\overline{\mathcal{N}_E}^{\perp}$. Then since $EX_i = X_iE$ for each $i = 1, \cdots$, $< Eh, X_if >=< h, EX_if >=< h, X_iEf >= 0$ for each $f \in \mathcal{H}$. So

$$\left\langle Eh\ , \sum_{i=1}^n X_if_i \right\rangle = 0$$

for each $n \in \mathbb{N}$. By the definition of A_I, $A_IEh = 0 = EA_IEh$ for each $E \in \mathcal{L}$. Hence $A_IE = EA_IE$ on $\overline{\mathcal{N}_E}^{\perp}$. So A_I is an operator in Alg$\mathcal{L}$. $\square$

3. The Equation $Ax = y$ in Alg$\mathcal{L}$

Let x and y be non-zero vectors in a Hilbert space $\mathcal{H}$. Let $X = x \otimes y$ and $Y = y \otimes y$. Then for f in $\mathcal{H}$ and $E \in \mathcal{L}$,

$$\begin{aligned} \|YEf\| &= \|(y \otimes y)Ef\| \\ &= \| < Ef, y > y\| \\ &= \| < f, Ey > y\|, \\ \|XEf\| &= \|(x \otimes y)Ef\| \\ &= \| < Ef, y > x\| \\ &= \| < f, Ey > x\|. \end{aligned}$$

If for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then for f in $\mathcal{H}$ and $E \in \mathcal{L}$,

$$\frac{\|YEf\|}{\|XEf\|} = \frac{\| < f, Ey > y\|}{\| < f, Ey > x\|}$$

is

$$\frac{\|y\|}{\|x\|} \text{ or } 0 .$$

Hence $\sup \left\{ \frac{\|YEf\|}{\|XEf\|} : f \in \mathcal{H} \text{ and } E \in \mathcal{L} \right\} = \frac{\|y\|}{\|x\|}$, $YEf =< f, Ey > y$ and $EYEf =< f, Ey > Ey$ for each f in $\mathcal{H}$ and each $E \in \mathcal{L}$.

We can obtain the following theorem by Theorem 2.1.

Theorem 3.1. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let x and y be non-zero vectors in $\mathcal{H}$. Let P_x be the orthogonal projection on $sp(x)$. If $EP_x = P_xE$ for each $E \in \mathcal{L}$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$.

(2) $< f, Ey > y =< f, Ey > Ey$ for each $E \in \mathcal{L}$ and $f \in \mathcal{H}$.

Let $x_i, y_i (i = 1, \cdots, n)$ be non-zero vectors in $\mathcal{H}$. Let $X_i = x_i \otimes y_i$ and $Y_i = y_i \otimes y_i$. Then for f_i in $\mathcal{H}$ and $E \in \mathcal{L}$

$$\begin{aligned} \|\sum_{i=1}^{n} Y_i E f_i\| &= \|\sum_{i=1}^{n} (y_i \otimes y_i) E f_i\| \\ &= \|\sum_{i=1}^{n} < E f_i, y_i > y_i\| \\ &= \|\sum_{i=1}^{n} < f_i, E y_i > y_i\|, \\ \|\sum_{i=1}^{n} X_i E f_i\| &= \|\sum_{i=1}^{n} (x_i \otimes y_i) E f_i\| \end{aligned}$$

$$= \| \sum_{i=1}^{n} < Ef_i, y_i > x_i \|$$
$$= \| \sum_{i=1}^{n} < f_i, Ey_i > x_i \|.$$

Hence $\dfrac{\| \sum_{i=1}^n Y_i E f_i \|}{\| \sum_{i=1}^n X_i E f_i \|} = \dfrac{\| \sum_{i=1}^n < f_i, Ey_i > y_i \|}{\| \sum_{i=1}^n < f_i, Ey_i > x_i \|}$ and $Y_i E f =< f, Ey_i > y_i$ and $< EY_i E f >=< f, Ey_i > Ey_i$ for each $E \in \mathcal{L}$, $f \in \mathcal{H}$ and $i = 1, \cdots, n$.

We can obtain the following theorem by Theorem 2.2.

Theorem 3.2. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. Let P_{x_i} be the orthogonal projection on sp(x_i). If $EP_{x_i} = P_{x_i}E$ for each $E \in \mathcal{L}$ and $i = 1, \cdots, n$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots, n$.

(2) $\sup \left\{ \dfrac{\| \sum_{i=1}^n < f_i, Ey_i > y_i \|}{\| \sum_{i=1}^n < f_i, Ey_i > x_i \|} : \ f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} = K_0 < \infty$ *and* $< f, Ey_i > y_i =< f, Ey_i > Ey_i$ *for each* $E \in \mathcal{L}$, $f \in \mathcal{H}$ *and* $i = 1, \cdots, n$.

We can extend Theorem 3.2 to countably infinite vectors and get the following theorem from Theorem 2.3.

Theorem 3.3. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let $\{x_i\}$ and $\{y_i\}$ be vectors in $\mathcal{H}$ for $i \in \mathbb{N}$. Let P_{x_i} be the orthogonal projection on sp(x_i). If $EP_{x_i} = P_{x_i}E$ for each $E \in \mathcal{L}$ and $i = 1, 2, \cdots$, then the following are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots$.

(2) $\sup \left\{ \dfrac{\| \sum_{i=1}^n < f_i, Ey_i > y_i \|}{\| \sum_{i=1}^n < f_i, Ey_i > x_i \|} : \ f_i \in \mathcal{H}, \ E \in \mathcal{L}, \ n \in \mathbb{N} \right\} = K_0 < \infty$ *and* $< f, Ey_i > y_i =< f, Ey_i > Ey_i$ *for each* $E \in \mathcal{L}$, $f \in \mathcal{H}$ *and* $i = 1, 2, \cdots$.

Theorem 3.4. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and x and y be vectors in $\mathcal{H}$. Let P_x be the orthogonal projection on sp(x). If $EP_x = P_xE$ for each $E \in \mathcal{L}$ and* $\sup \left\{ \dfrac{\|Ey\|}{\|Ex\|} : E \in \mathcal{L} \right\} = K < \infty$, *then there exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$.*

Proof. Assume that $\sup \left\{ \dfrac{\|Ey\|}{\|Ex\|} : E \in \mathcal{L} \right\} = K < \infty$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E Ex = Ey$ by Theorem 1.1. In particular, if $E = I$, then we have an operator A_I in $\mathcal{B}(\mathcal{H})$ such that $A_I x = y$. Let's put $A_I = A$. So $A_E Ex = Ey = EAx$ for each $E \in \mathcal{L}$. Hence $A_E E = EA$ on $sp(x)$. Let h be in $sp(x)^{\perp}$. Since $EP_x = P_x E$ for each $E \in \mathcal{L}$, $< Eh, Ex >=< h, Ex >=< h, EP_x x >=< h, P_x Ex >= 0$. Hence $Eh \in sp(Ex)^{\perp}$. By the definition of A_E and A, $A_E Eh = 0 = EAh$ for each E in $\mathcal{L}$. Hence $A_E E = EA$ on $\mathcal{H}$ for each E in $\mathcal{L}$. So $A = EA$. Therefore A is in Alg$\mathcal{L}$. $\square$

Theorem 3.5. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. Let P_{x_i} be the orthogonal projection on $sp(x_i)$. If $EP_{x_i} = P_{x_i}E$ for each $E \in \mathcal{L}$ and $i = 1, \cdots, n$ and*

$$\sup \left\{ \frac{\|E(\sum_{i=1}^n \alpha_i y_i)\|}{\|E(\sum_{i=1}^n \alpha_i x_i)\|} : E \in \mathcal{L},\ \alpha_i \in \mathbb{C} \right\} < \infty,$$

then there exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, \cdots, n$.

Proof. Assume that $\sup \left\{ \frac{\|E(\sum_{i=1}^n \alpha_i y_i)\|}{\|E(\sum_{i=1}^n \alpha_i x_i)\|} : E \in \mathcal{L},\ \alpha_i \in \mathbb{C} \right\} < \infty$. Let E be in $\mathcal{L}$. Define $A_E : sp(\{Ex_1, \cdots, Ex_n\}) \to \mathcal{H}$ by $A_E(\sum_{i=1}^n \alpha_i Ex_i) = (\sum_{i=1}^n \alpha_i Ey_i)$. Then A_E is well-defined and bounded linear. Define $A_E f = 0$ for each $f \in sp(\{Ex_1, \cdots, Ex_n\})^{\perp}$. Then $A_E : \mathcal{H} \to \mathcal{H}$ is bounded linear and $A_E Ex_i = Ey_i$ for $i = 1, \cdots, n$. If $E = I$, then $A_I x_i = y_i$ for $i = 1, \cdots, n$. Let's put $A_I = A$. So $A_E Ex_i = Ey_i = EAx_i$ for each $E \in \mathcal{L}$. Hence $A_E E = EA$ on $sp(\{x_1, \cdots, x_n\})$. Let h be in $sp(\{x_1, \cdots, x_n\})^{\perp}$. Since $< Eh, Ex_i >=< h, Ex_i >= 0$, $< Eh, \sum_{i=1}^n Ex_i >= 0$. So $Eh \in sp(\{Ex_1, \cdots, Ex_n\})^{\perp}$. By the definition of A_E and A, $A_E Eh = 0 = EAh$ for each E in $\mathcal{L}$. Hence $A_E E = EA$ on $\mathcal{H}$ for each E in $\mathcal{L}$. So $A = EA$. Therefore A is in Alg$\mathcal{L}$. □

We ca generalize the above theorem for countable case.

Theorem 3.6. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $\{x_i\}$ and $\{y_i\}$ be vectors in $\mathcal{H}$. Let P_{x_i} be the orthogonal projection on $sp(x_i)$ for each $i = 1, 2, \cdots$. If $EP_{x_i} = P_{x_i}E$ for each $E \in \mathcal{L}$ and $i = 1, 2, \cdots$ and*

$$\sup \left\{ \frac{\|E(\sum_{i=1}^n \alpha_i y_i)\|}{\|E(\sum_{i=1}^n \alpha_i x_i)\|} : E \in \mathcal{L},\ \alpha_i \in \mathbb{C},\ n \in \mathbb{N} \right\} < \infty,$$

then there exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \cdots$.

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