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The Normality of Meromorphic Functions with Multiple Zeros and Poles Concerning Sharing Values

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Abstract. In this paper we study the problem of normal families of meromorphic functions concerning shared values. Let F be a family of meromorphic functions in the plane domain $D \subseteq \mathbb{C}$ and n, k be two positive integers such that $n \geq k+1$, and let a, b be two finite complex constants such that $a \neq 0$. Suppose that (1) $f + a(f^{(k)})^n$ and $g + a(g^{(k)})^n$ share b in D for every pair of functions $f, g \in F$; (2) All zeros of f have multiplicity at least $k+2$ and all poles of f have multiplicity at least 2 for each $f \in F$ in D; (3) Zeros of $f^{(k)}(z)$ are not the b points of $f(z)$ for each $f \in F$ in D. Then F is normal in D. And some examples are provided to show the result is sharp.

1. Introduction and Main Results

In this paper, we denote by $\mathbb{C}$ the whole complex plane. A function f is called meromorphic if it is analytic in a domain $D \subset \mathbb{C}$ except at possible isolated poles. A function f is called normal if there exists a positive number M such that $f^\sharp(z) \leq M$ for all $z \in D$, where $f^\sharp(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ denotes the spherical derivative of f. For $a \in \mathbb{C}$, set $\overline{E}_f(a) = \{z \in D : f(z) = a\}$. We say that two meromorphic functions f and g share the value a provided that $\overline{E}_f(a) = \overline{E}_g(a)$ in D. When $a = \infty$ the zeros of $f - a$ mean the poles of f (see [4]). Let F be a family of meromorphic functions in a domain $D \subseteq \mathbb{C}$. We say that F is normal in D if every sequence $\{f_n\} \subseteq F$ contains a subsequence which converges spherically uniformly on the compact subsets of D (see [8,11]).

In 1992, W. Schwick [9] obtained a connection between normality criteria and sharing values. He proved the theorem as follows.

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Theorem A. *Let F be a family of meromorphic functions on a domain D and a_1, a_2, a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3 for every $f \in F$, then F is normal in D.*

Since then many results in this direction have been obtained. In 2011, D. W. Meng and P. C. Hu [7] proved the following normality criteria.

Theorem B. *Take a positive integer k and a complex number $a(\neq 0)$. Let F be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that each $f \in F$ has only zeros of multiplicity at least $k+1$. For each pair $f, g \in F$, if $ff^{(k)}$ and $gg^{(k)}$ share a, then F is normal in D.*

Recently, G. Datt and S. Kumar [4] obtained the following result.

Theorem C. *Let F be a family of meromorphic functions defined in a domain D such that for each $f \in F$ satisfies the followings :*

(1) Zeros of $f(z)$ are of multiplicity at least 3 in D and poles of $f(z)$ are of multiplicity at least 2.

(2) Zeros of $f'(z)$ are not the b points of $f(z)$, where b is a non-zero constant.
If for each pair of functions $f, g \in F$, $f + (f')^n$ and $g + (g')^n$ share the value b, then F is normal in D.

Let f be a meromorphic function in $D \subset \mathbb{C}$ and $a \in \mathbb{C} - \{0\}$ and $n(\geq 2), k$ are two positive integers, we define

$$D(f) = f + a(f^{(k)})^n$$

a non-linear differential polynomial. It is natural to ask whether Theorem C can be improved by the idea of $D(f) = f + a(f^{(k)})^n$. In this paper, we study the problem and obtain the following result.

Theorem 1. *Let F be a family of meromorphic functions in the plane domain $D \subseteq \mathbb{C}$ and n, k be two positive integers such that $n \geq k+1$, and let a, b be two finite complex constants such that $a \neq 0$. Suppose that*

(1) Zeros of f have multiplicity at least $k+2$ and poles of f have multiplicity at least 2 for each $f \in F$ in D;

(2) Zeros of $f^{(k)}(z)$ are not the b points of $f(z)$ for each $f \in F$ in D.
If $D(f)$ and $D(g)$ share b in D for every pair of functions $f, g \in F$, then F is normal in D.

Example 1. Let $D = \{z : |z| < 1\}, n, k \in \mathbb{N}$ and $F = \{f_n(z)\}$, where

$$f_n(z) = nz^{k+1}, \quad z \in D, \quad n = 1, 2, \ldots.$$

Obviously, $f_n + (f_n^{(k)})^{k+1} = [n + (n(k+1)!)^{k+1}]z^{k+1}$. So for each pair m, n, $f_n + (f_n^{(k)})^{k+1}$ and $f_m + (f_m^{(k)})^{k+1}$ share the value 0 in D, however, F fails to be normal in D since $f_n^\sharp\left(\frac{1}{\sqrt[k+1]{n}}\right) = \frac{\sqrt[k+1]{n}(k+1)}{2} \to \infty$ as $n \to \infty$.

Example 1 shows that Theorem 1 is not valid when all zeros of f have multiplicity $k+1$, so the condition that f has only zeros with multiplicity $k+2$ is best possible for Theorem 1.

2. Some Lemmas

In order to prove our theorems, we need the following preliminary results.

Suppose that f is non-constant and meromorphic and k is a positive integer. Set $M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$ and $\gamma_M = \Sigma_{j=0}^{k} n_j$, where $n_0, n_1, \ldots, n_k$ are non-negtive integers, then $M[f]$ is called a differential monomial of f, and γ_M the degree of $M[f]$. Suppose that $M_j[f]$ are differential monomials of f with degree $\gamma_{M_j}(j = 1, \ldots, n)$. Set $Q[f] = \Sigma_{j=1}^{n} a_j(z)M_j[f]$ and $\gamma_Q = max_{1\le j\le n}\gamma_{M_j}$. Then $Q[f]$ is said to be a differential polynomial of f with degree γ_Q if the coefficients $a_j(z)(j = 1, \ldots, n)$ satisfy $T(r, a_j(z)) = S(r, f)$. If $\gamma_{M_1} = \gamma_{M_2} = \cdots = \gamma_{M_n}$, then $Q[f]$ is called a homogeneous differential polynomial of f. In addition, we shall use the following standard notations of Nevanlinna′s Theory and its some fundamental results (see [8,11]). In particular, $S(r, f) = o(T(r, f)) \quad (r \to \infty)$ except for a finite linear measure of the set of the value r.

The following result is due to Pang and Zalcman [6] (cf. [2]).

Lemma 1.([2],[6]) *Let F be a family of meromorphic functions in the unit disc $\triangle \subseteq \mathbb{C}$ and let k be a positive integer. Suppose that all zeros of f have multiplicity at least k for every $f \in F$, and suppose that there exists a number $A \ge 1$ such that $|f^{(k)}(z)| \le A$ whenever $f(z) = 0$. If F is not normal at $z_0 \in \triangle$, then for any $0 \le \alpha \le k$, there exist*
(1) a number $r \in (0, 1)$,
(2) a sequence of complex numbers $z_n \to z_0, |z_n| \le r$,
(3) a sequence of functions $f_n \in F$,
(4) a sequence of positive numbers $\rho_n \to 0$
such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n\xi)$ converges locally uniformly (with respect to spherical metric) to a non-constant meromorphic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^\sharp(\xi) \le g^\sharp(0) = kA + 1$.

Remark 1. In Lemma 1, if $0 \le \alpha < k$, then the hypothesis of $f^{(k)}$ can be dropped, and $kA + 1$ can be replaced by an arbitrary positive number (see [2]).

Lemma 2.([3]) *A normal function has order at most two. A normal entire function is of exponential type, and thus has order at most one.*

Lemma 3.([12]) *Let $f = \frac{P}{Q}$ be a rational function and Q be non constant. Then $(f^{(k)})_\infty \le (f)_\infty - k$, where k is a positive integer, $(f)_\infty = \deg(P) - \deg(Q)$.*

Lemma 4.([10]) *Let f be a transcendental meromorphic function, and let a be a nonzero finite complex number and n, k be two positive integers such that $n \ge k+1$, then $f + a(f^{(k)})^n$ assumes every finite complex value infinitely often.*

Lemma 5. *Let n, k be two positive integers such that $n \geq k+1$, and let f be a non-constant rational function such that all zeros of f have multiplicity at least $k+2$ and poles (if exists) of f are of multiplicity at least 2, then $D(f) = f + a(f^{(k)})^n$ has at least two distinct zeros.*

Proof. We consider the following cases.

Case 1. $D(f) = f + a(f^{(k)})^n$ has exactly one zero z_0 (say).

Case 1.1. f is a non-constant polynomial. it is easily obtained that $D(f) = f + a(f^{(k)})^n$ has zeros since all zeros of f have multiplicity at least $k+2$. Suppose that $D(f) = f + a(f^{(k)})^n$ has exactly one zero z_0 with multiplicity l, then $D(f) = f + a(f^{(k)})^n$ has the form $D(f) = f + a(f^{(k)})^n = A(z-z_0)^l$, where A is a non-zero constant, l is a positive integer. Obviously, $l \geq k+2$ since f has only zeros with multiplicity at least $k+2$. So

$$(2.1) \qquad \begin{aligned} (D(f))^{(k)} &= f^{(k)} + a\left[\left(f^{(k)}\right)^n\right]^{(k)} \\ &= Al(l-1)\cdots(l-k+1)(z-z_0)^{l-k}. \end{aligned}$$

On the other hand, the simple calculation implies that

$$(2.2) \qquad (D(f))^{(k)} = f^{(k)} + a[(f^{(k)})^n]^{(k)} = f^{(k)}\left[1 + Q\left(f^{(k)}\right)\right],$$

where

$$\begin{aligned} Q\left(f^{(k)}\right) &= a\left(f^{(k)}\right)^{n-k-1} \frac{n!}{(n-k)!}\left(f^{(k+1)}\right)^k \\ &+ a\left(f^{(k)}\right)^{n-k-1} \frac{C_n^2 n!}{(n-k+1)!}\left(f^{(k+1)}\right)^{k-2} f^{(k+2)} \\ &+ \cdots + an\left(f^{(k)}\right)^{n-k-1}\left(f^{(k)}\right)^{k-1} f^{(2k)}, \end{aligned}$$

and $Q(f^{(k)})$ is a homogeneous differential polynomial of $f^{(k)}$ of degree $n-1$. From (2.1) and (2.2) we know that $f^{(k)}$ has exactly the same zero z_0, so f has the same zero z_0 and z_0 is the unique zero of f. Thus f has the form $f(z) = A_0(z-z_0)^p$, where A_0 is non-zero constant and p is a positive integer such that $p \geq k+2$. Thus $D(f) = f + a(f^{(k)})^n = A_0(z-z_0)^p\{1 + aA_0^{n-1}[p(p-1)\cdots(p-k+1)]^n(z-z_0)^{(n-1)p-nk}\}$ has at least two distinct zeros since $(n-1)p - nk \geq 1$ for $n \geq k+1$ and $p \geq k+2$. This is a contradiction that our assumptions. Thus $D(f) = f + a(f^{(k)})^n$ has at least two distinct zeros.

Case 1.2. f is a nonconstant rational function which is not a polynomial. Suppose that $D(f) = f + a(f^{(k)})^n$ has exactly one zero z_0 with multiplicity l. So we deduce that f has exactly one zero z_0 and then z_0 is the unique zero of f. Otherwise $f + a(f^{(k)})^n$ has at least two distinct zeros, which contradicts that our assumptions.

Put

$$f(z)=\frac{A(z-z_0)^p}{(z-z_1)^{q_1}(z-z_2)^{q_2}\cdots(z-z_t)^{q_t}}, \tag{2.3}$$

where A is a non-zero constant and $q_i \geq 2 (i=1,2,\ldots,t), p$ are positive integers such that $p \geq k+2$.

For brevity, we denote

$$q_1+q_2+\cdots+q_t=q\geq 2t.$$

From (2.3), it follows that

$$f^{(k)}(z)=\frac{A(z-z_0)^{p-k}g(z)}{(z-z_1)^{q_1+k}(z-z_2)^{q_2+k}\cdots(z-z_t)^{q_t+k}}, \tag{2.4}$$

where $g(z)$ is a polynomial and $c_{kt-1},\ldots,c_1,c_0$ are constants. Then (2.3) and (2.4) imply that $(f)_\infty=p-q$ and $(f^{(k)})_\infty=p-k+\deg(g(z))-q-kt$, It is easy to see that $\deg(g(z))\leq kt$ by Lemma 3.

From (2.3) and (2.4), then

$$\begin{aligned} D(f) &= f+a\left(f^{(k)}\right)^n \\ &= \frac{Ap_1(z)+aA^np_2(z)}{(z-z_1)^{n(q_1+k)}(z-z_2)^{n(q_2+k)}\cdots(z-z_t)^{n(q_t+k)}}, \end{aligned} \tag{2.5}$$

where

$$p_1(z)=(z-z_0)^p(z-z_1)^{(n-1)q_1+nk}(z-z_2)^{(n-1)q_2+nk}\cdots(z-z_t)^{(n-1)q_t+nk}$$

and

$$p_2(z)=(z-z_0)^{n(p-k)}g^n(z).$$

It follows that $\deg(p_1(z))=(n-1)q+nkt$ and $\deg(p_2(z))=n(p-k)+n\deg(g(z))$. We can claim that

$$\deg(p_1(z))\neq\deg(p_2(z)). \tag{2.6}$$

If $p\leq q+k$, then $\deg(p_1(z))-\deg(p_2(z))\geq(n-1)(q-p)+nk\geq 1$, which implies that $\deg(p_1(z))\neq\deg(p_2(z))$.

If $p>q+k$, then $\deg(g(z))=kt$. Hence $\deg(p_2(z))-\deg(p_1(z))=(n-1)(p-q)-nk\geq 1$, which also yields that $\deg(p_1(z))\neq\deg(p_2(z))$, as claimed.

Since $n(p-k)>p$ for $n\geq k+1$ and $p\geq k+2$. It follows from (2.5) that

$$\begin{aligned} D(f) &= f+a\left(f^{(k)}\right)^n \\ &= \frac{A(z-z_0)^pg_1(z)}{(z-z_1)^{n(q_1+k)}(z-z_2)^{n(q_2+k)}\cdots(z-z_t)^{n(q_t+k)}}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} g_1(z) &= (z-z_1)^{(n-1)q_1+nk}(z-z_2)^{(n-1)q_2+nk}\cdots(z-z_t)^{(n-1)q_t+nk} \\ &\quad +aA^{n-1}(z-z_0)^{n(p-k)-p}g^n(z). \end{aligned}$$

By the assumption that $f+a(f^{(k)})^n$ has exactly one zero z_0 with multiply l, we deduce from (2.5) that

$$\begin{aligned} D(f) &= f+a\left(f^{(k)}\right)^n \\ &= \frac{C(z-z_0)^l}{(z-z_1)^{n(q_1+k)}(z-z_2)^{n(q_2+k)}\cdots(z-z_t)^{n(q_t+k)}}. \end{aligned} \tag{2.8}$$

Hence (2.7) and (2.8) mean that

$$C(z-z_0)^l \equiv A(z-c_0)^p g_1(z), \tag{2.9}$$

where C is a non-zero constant.

Case 1.2.1. $l>p$. It follows from (2.9) that g_1 has a zero z_0 and then $(z-z_1)^{(n-1)q_1+nk}(z-z_2)^{(n-1)q_2+nk}\cdots(z-z_t)^{(n-1)q_t+nk}$ has a zero z_0, which leads to a contradiction.

Case 1.2.2. $l=p$. In view of (2.9), one has that

$$h_1(z)+h_2(z)\equiv \frac{C}{A}, \tag{2.10}$$

where

$$h_1(z)=(z-z_1)^{(n-1)q_1+nk}(z-z_2)^{(n-1)q_2+nk}\cdots(z-z_t)^{(n-1)q_t+nk}$$

and

$$h_2(z)=aA^{n-1}(z-z_0)^{n(p-k)-p}g^n(z).$$

We easily obtain from (2.10) that $\deg(h_1)=\deg(h_2)$. On the other hand, (2.6) and the definitions of $p_1(z), p_2(z), g_1, h_1$ and h_2 yield that $\deg(h_1)\neq\deg(h_2)$. We thus have a contradiction.

Case 2. Let $D(f)$ has no zeros. it is easily obtained that f is not a polynomial, otherwise $D(f)$ becomes a polynomial of degree at least 4. Hence f is a non-polynomial rational function. Now putting $l=0$ in (2.8) and proceeding as case 1.2, we thus arrive at a contradiction.

The proof is complete. □

Lemma 6. *Let n,k be two positive integers such that $n\geq k+1$, and let f be a non-constant rational function with following properties :*

(1) *Zeros of f have multiplicity at least $k+2$ and poles (if exists) of f are of multiplicity at least 2.*

(2) *Zeros of* $f^{(k)}$ *are not the* b *points of* f, *where* b *is a non-zero constant. then* $D(f)-b$ *has at least two distinct zeros.*

Proof. We consider the following cases.

Case 1. $D(f)-b$ has exactly one zero z_0 (say) with multiplicity l.

Case 1.1. f is a non-constant polynomial. It is easily obtained that $D(f)-b$ has zeros since all zeros of f have multiplicity at least $k+2$. Suppose that $D(f)-b$ has exactly one zero z_0 with multiplicity l, then $D(f)-b$ has the form $D(f)-b = A(z-z_0)^l$, where A is a non-zero constant, l is a positive integer. Obviously, $l \geq k+2$ since f has only zeros with multiplicity at least $k+2$. So

$$(D(f))^{(k)} = f^{(k)} + a\left[\left(f^{(k)}\right)^n\right]^{(k)} = Al(l-1)\cdots(l-k+1)(z-z_0)^{l-k}.$$

On the other hand, the simple calculation implies that

$$(D(f))^{(k)} = f^{(k)} + a[(f^{(k)})^n]^{(k)} = f^{(k)}\left[1 + Q\left(f^{(k)}\right)\right],$$

where $Q\left(f^{(k)}\right)$ is given in Case 1.1 of lemma 5. This shows that z_0 is only zero of $(D(f))^{(k)}$. Note that a zero of $f^{(k)}$ is also a zero of $(D(f))^{(k)} = f^{(k)}\left[1+Q\left(f^{(k)}\right)\right]$ and $f^{(k)}$ is a nonconstant function. We thus conclude that z_0 is a zero of $f^{(k)}$, which yields that $f(z_0) = b$. This is a contradiction that our assumptions. Thus $D(f)-b$ has at least two distinct zeros.

Case 1.2. f is a nonconstant rational function which is not a polynomial. By the hypothesis, we may put

$$f(z) = \frac{A(z-\alpha_1)^{p_1}(z-\alpha_2)^{p_2}\cdots(z-\alpha_s)^{p_s}}{(z-z_1)^{q_1}(z-z_2)^{q_2}\cdots(z-z_t)^{q_t}}, \tag{2.11}$$

where A is a non-zero constant, $p_i(i=1,2,\ldots,s), q_j(j=1,2,\ldots,t)$ are positive integers such that $p_i \geq k+2(i=1,2,\ldots,s)$ and $q_j \geq 2(j=1,2,\ldots,t)$.

For brevity, we denote

$$p = \sum_{i=1}^{s} p_i \geq (k+2)s, \quad q = \sum_{j=1}^{t} q_j \geq 2t.$$

From (2.11), it follows that

$$f^{(k)}(z) = \frac{A(z-\alpha_1)^{p_1-k}(z-\alpha_2)^{p_2-k}\cdots(z-\alpha_s)^{p_s-k}g_1(z)}{(z-z_1)^{q_1+k}(z-z_2)^{q_2+k}\cdots(z-z_t)^{q_t+k}}, \tag{2.12}$$

where $g_1(z)$ is a polynomial and $c_{kt-1},\ldots,c_1,c_0$ are constants. Then (2.11) and (2.12) imply that $(f)_\infty = p-q$ and $(f^{(k)})_\infty = p-ks+\deg(g(z))-q-kt$, By Lemma 3, it is easy to see that

$$\deg(g_1(z)) \leq k(s+t-1). \tag{2.13}$$

From (2.11) and (2.12), then

$$(2.14)\qquad \begin{aligned} D(f) &= f + a\left(f^{(k)}\right)^n \\ &= \frac{(z-\alpha_1)^{p_1}(z-\alpha_2)^{p_2}\cdots(z-\alpha_s)^{p_s}g(z)}{(z-z_1)^{n(q_1+k)}(z-z_2)^{n(q_2+k)}\cdots(z-z_t)^{n(q_t+k)}}, \end{aligned}$$

where $g(z) = g_1^n(z)$ is a polynomial and

$$(2.15)\ \deg(g(z)) \leq \max\{(n-1)q + nkt, (n-1)p - nks + n\deg(g_1(z))\}.$$

Noting that $D(f) - b$ has exactly one zero z_0 with multiplicity l, from (2.14) we have

$$(2.16)\qquad \begin{aligned} D(f) &= f + a\left(f^{(k)}\right)^n \\ &= b + \frac{B(z-z_0)^l}{(z-z_1)^{n(q_1+k)}(z-z_2)^{n(q_2+k)}\cdots(z-z_t)^{n(q_t+k)}}, \end{aligned}$$

where B is a non-zero number.

It follows from (2.14) and (2.16) that

$$(2.17)\ (D(f))' = \frac{(z-\alpha_1)^{p_1-1}(z-\alpha_2)^{p_2-1}\cdots(z-\alpha_s)^{p_s-1}h_1(z)}{(z-z_1)^{n(q_1+k)+1}(z-z_2)^{n(q_2+k)+1}\cdots(z-z_t)^{n(q_t+k)+1}}$$

and

$$(2.18)\ (D(f))' = \frac{(z-z_0)^{l-1}h_2(z)}{(z-z_1)^{n(q_1+k)+1}(z-z_2)^{n(q_2+k)+1}\cdots(z-z_t)^{n(q_t+k)+1}},$$

where $h_1(z), h_2(z)$ are polynomials. Both (2.14) and (2.17) imply that $(D(f))_\infty = p + \deg(g(z)) - nq - nkt$ and $((D(f))')_\infty = p - s + \deg(h_1(z)) - nq - nkt - t$. Lemma 3 tells us that $((D(f))')_\infty \leq (D(f))_\infty - 1$, then

$$(2.19)\quad \deg(h_1(z)) \leq s + t - 1 + \deg(g(z)) = s + t - 1 + n\deg(g_1(z)).$$

Similarly(2.16) and (2.18) yields that

$$(2.20)\qquad \deg(h_2(z)) \leq t.$$

Since $\alpha_i \neq z_0$ for $i = 1, 2, \ldots, s$, it follows from (2.17) and (2.18) that $p - s \leq \deg(h_2(z)) \leq t$, which implies that $p \leq s + t \leq \frac{p}{k+2} + \frac{q}{2}$. One can deduce that

$$(2.21)\qquad p < q.$$

Blow we divided into the following two cases again.

Case 1.2.1. $l \neq nq + nkt$. It follows from (2.14) that

$$(2.22)\qquad nq + nkt \leq p + \deg(g(z)).$$

If $\deg(g(z)) \leq (n-1)q + nkt$, we thus from (2.22) obtain that $nq + nkt \leq p + (n-1)q + nkt$, which implies that $q \leq p < q$ by (2.21). This is impossible.

If $\deg(g(z)) \leq (n-1)p - nks + n\deg(g_1(z))$, since $\deg(g_1(z)) \leq k(s+t-1)$, hence $nq + nkt \leq (n-1)p - nks + +nk(s+t-1)$, we have $q \leq p - 1 < q - 1$ by (2.21). We thus arrive at a contradiction.

Case 1.2.2. $l = nq + nkt$. It is obtained from (2.17) and (2.18) that

$$l - 1 \leq \deg(h_1(z)) \leq s + t - 1 + \deg(g(z)). \tag{2.23}$$

If $\deg(g(z)) \leq (n-1)q+nkt$, we thus from (2.23) obtain that $l \leq s+t+\deg(g(z))$, which implies that $nq+nkt \leq s+t+(n-1)q+nkt$. We have $q \leq s+t \leq \frac{p}{k+2} + \frac{q}{2} \leq \frac{q}{k+2} + \frac{q}{2} < q$ by (2.21). This is impossible.

If $\deg(g(z)) \leq (n-1)p - nks + n\deg(g_1(z)) \leq (n-1)p - nks + nk(s+t-1)$. Since $\deg(g_1(z)) \leq k(s+t-1)$ and $l = nq+nkt$, hence (2.23) implies that $nq+nkt \leq (n-1)p - nks + +nk(s+t-1)$, we have $q \leq p - 1 < q - 1$ by (2.21). This is a contradiction.

Case 2. Let $D(f) - b$ has no zero. Then f can not be a polynomial, otherwise $D(f)$ becomes a polynomial of degree at least 4. Hence f is non polynomial rational function. Now putting $l = 0$ in (2.16) and proceeding as case 1.2 of Lemma 6, we have a contradiction. The proof of Lemma 6 is completed. □

3. Proof of Theorem

Suppose that F is not normal in D, then there exists at least one point z_0 such that F is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$ and $D = \triangle$. We shall consider two cases.

Case 1. $b = 0$. Recall that all zeros of f have multiplicity at least $k+2$, then, by Lemma 1, there exist a sequence points $z_j \to 0 (j \to \infty)$; a sequence of positive numbers ρ_j, $\rho_j \to 0$ and a sequence of functions f_j, $f_j \in F$ such that $g_j(\xi) = \rho_j^{-\frac{nk}{n-1}} f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to spherical metric on $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function and all zeros of $g(\xi)$ have multiplicity at least $k+2$.

From the above, we obtain

$$g_j^{(k)} = \rho_j^{-\frac{k}{n-1}} f_j^{(k)} \to g^{(k)}, \tag{3.1}$$

and

$$\rho_j^{-\frac{nk}{n-1}} \left[f_j + a \left(f_j^{(k)} \right)^n \right] = g_j + a \left(g_j^{(k)} \right)^n \to g + a \left(g^{(k)} \right)^n$$

also locally uniformly with respect to the spherical metric, that is,

$$\rho_j^{-\frac{nk}{n-1}} \left[D(f_j) \right] = D(g_j) \to D(g) \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

If $D(g) = g + a(g^{(k)})^n \equiv 0$, then g clearly has no poles and is not any polynomial with order at least $k+2$, so g is a transcendental entire function. By Lemma 2, g is of exponential type. Noting that $g \neq 0$, then g has the form $g(\xi) = e^{c\xi+d}$, where $c(c \neq 0), d$ are two constants. Hence

$$e^{c\xi+d} + a\left[e^{c\xi+d}\right]^n = e^{c\xi+d}\left[1 + ac^{nk}e^{(n-1)(c\xi+d)}\right] \equiv 0.$$

So we get $e^{(n-1)(c\xi+d)} \equiv -\frac{1}{ac^{nk}}$. This is a contradiction since $n \geq k+1 \geq 2$.

Since g is a non-constant meromorphic function, by Lemmas 4 and 5, we deduce that $D(g) = g + a(g^{(k)})^n$ has at least two distinct zeros.

On the other hand, we conclude that $D(g) = g + a(g^{(k)})^n$ has just a unique zero.

Suppose that there exist two distinct zeros ξ_0 and ξ_0^* and choose $\delta(\delta > 0)$ small enough such that $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$.

From (3.1) and (3.2), by Hurwitz′s theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$D(f_j(z_j + \rho_j\xi_j)) = f_j(z_j + \rho_j\xi_j) + a\left[f_j^{(k)}(z_j + \rho_j\xi_j)\right]^n = 0,$$
$$D(f_j(z_j + \rho_j\xi_j^*)) = f_j(z_j + \rho_j\xi_j^*) + a\left[f_j^{(k)}(z_j + \rho_j\xi_j^*)\right]^n = 0.$$

By the hypothesis that for each pair of functions f and g in F, $D(f)$ and $D(g)$ share 0, we know that for any positive integer m

$$D(f_m(z_j + \rho_j\xi_j)) = f_m(z_j + \rho_j\xi_j) + a\left[f_m^{(k)}(z_j + \rho_j\xi_j)\right]^n = 0,$$
$$D(f_m(z_j + \rho_j\xi_j^*)) = f_m(z_j + \rho_j\xi_j^*) + a\left[f_m^{(k)}(z_j + \rho_j\xi_j^*)\right]^n = 0.$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\xi_j \to 0, z_j + \rho_j\xi_j^* \to 0$, then

$$D(f_m(0)) = f_m(0) + a\left(f_m^{(k)}\right)^n(0) = 0.$$

Since the zeros of $D(f_m) = f_m + a(f_m^{(k)})^n$ has no accumulation point, so

$$z_j + \rho_j\xi_j = 0, \quad z_j + \rho_j\xi_j^* = 0.$$

Hence

$$\xi_j = -\frac{z_j}{\rho_j}, \quad \xi_j^* = -\frac{z_j}{\rho_j},$$

which contradicts the fact that $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$. So $g + a(g^{(k)})^n$ has just a unique zero. This contradicts the fact that $D(g) = g + a(g^{(k)})^n$ has at least two distinct zeros.

Case 2. $b \neq 0$. By Lemma 1 again, there exist a sequence of points $z_j \to 0 (j \to \infty)$; a sequence of positive numbers $\rho_j, \rho_j \to 0$ and a sequence of functions $f_j, f_j \in F$ such that $g_j(\xi) = f_j(z_j + \rho_j \xi) \to g(\xi)$ locally uniformly with respect to spherical metric on $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function. Moreover g is of order of at most 2 and only zeros of g have multiplicity at least $k+2$ and poles of g have multiplicity at least 2.

From the above discussion, we have

$$D(f_j(z_j + \rho_j \xi)) - b = D(g_j(\xi)) \to D(g(\xi)) - b \quad as \quad j \to \infty \tag{3.3}$$

also locally uniformly with respect to the spherical metric.

Now we can conclude that : if $g^{(k)}(\xi_0) = 0$ then $g(\xi_0) \neq b$. Suppose that $g(\xi_0) = b$, then by Hurwitzs theorem, there exists $\xi_j \to \xi_0$ for $j \to \infty$ such that $g_j(\xi_j) = f_j(z_j + \rho_j \xi_j) = b$. It follows from $g^{(k)}(\xi_0) = 0$ that $g_j^{(k)}(\xi_j) = 0$, which implies $\rho_j^k f_j^{(k)}(z_j + \rho_j \xi_j) = 0$. We thus have $f_j^{(k)}(z_j + \rho_j \xi_j) = 0$. By condition of theorem 1, we deduce that $f_j(z_j + \rho_j \xi_j) \neq b$, which contradicts that $f_j(z_j + \rho_j \xi_j) = b$.

If $D(g(\xi)) \equiv b$. The argument in this case is completely analogous to the proof of $D(g(\xi)) = g(\xi) + a(g^{(k)})^n(\xi) \equiv 0$ and then we have a contradiction. So we omit its proof.

We derive that $D(g(\xi)) - b$ has at least two zeros by Lemmas 4 and 6 since g is a non-constant meromorphic function.

On the other hand, we can claim that $D(g(\xi)) - b$ has just a unique zero. Suppose that there exist two distinct zeros ξ_0 and ξ_0^* and choose $\delta (\delta > 0)$ small enough such that $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$.

By (3.3) and Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$D(f_j(z_j + \rho_j \xi_j)) - b = 0,$$
$$D(f_j(z_j + \rho_j \xi_j^*)) - b = 0.$$

By the hypothesis that for each pair of functions f and g in F, $D(f)$ and $D(g)$ share b, we know that for any positive integer m

$$D(f_m(z_j + \rho_j \xi_j)) - b = 0,$$
$$D(f_m(z_j + \rho_j \xi_j^*)) - b = 0.$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j \xi_j \to 0, z_j + \rho_j \xi_j^* \to 0$, then

$$D(f_m(0)) - b = 0.$$

Since the zeros of $D(f_m) - b$ has no accumulation point, so

$$z_j + \rho_j \xi_j = 0, \quad z_j + \rho_j \xi_j^* = 0.$$

Hence

$$\xi_j = -\frac{z_j}{\rho_j}, \quad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts the fact that $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$. So $a(g^{(k)})^n - b$ has just a unique zero, which contradicts that $D(g) - b$ has at least two zeros. This proves the Theorem 1. □

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