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Analysis, Control, and Synchronization of a 3-D Novel Jerk Chaotic System with Two Quadratic Nonlinearities

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Abstract. In this research work, a seven-term 3-D novel jerk chaotic system with two quadratic nonlinearities has been proposed. The basic qualitative properties of the novel jerk chaotic system have been described in detail. Next, an adaptive backstepping controller is designed to stabilize the novel jerk chaotic system with two unknown parameters. Moreover, an adaptive backstepping controller is designed to achieve complete chaos synchronization of the identical novel jerk chaotic systems with two unknown parameters. MATLAB simulations have been shown in detail to illustrate all the main results developed for the 3-D novel jerk chaotic system.

1. Introduction

Chaos theory describes the qualitative study of unstable aperiodic behaviour in deterministic nonlinear dynamical systems. A chaotic system is mathematically defined as a dynamical system with at least one positive Lyapunov exponent. In simple language, a chaotic system is a dynamical system, which is very sensitive to small changes in the initial conditions. Interest in nonlinear dynamics and in particular chaotic dynamics has grown rapidly since 1963, when Lorenz published his numerical work on a simplified model of convection and discussed its implications for weather prediction [5].

Nonlinear dynamics occurs widely in engineering, physics, biology and many other scientific disciplines [14]. Poincaré was the first to observe the possibility of *chaos*, in which a deterministic system exhibits aperiodic behaviour that depends on the initial conditions, thereby rendering long-term prediction impossible, since then it has received much attention [27, 16].

Chaos has developed over time. For example, Ruelle and Takens [34] proposed a

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theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors. Later, May [22] found examples of chaos in iterated mappings arising in population biology. Feigenbaum [9] discovered that there are certain universal laws governing the transition from regular to chaotic behaviours. That is, completely different systems can go chaotic in the same way, thus, linking chaos and phase transitions.

One of the hallmarks of nonlinear dynamics is the concept of equilibrium, which helps in characterizing a system's behaviour - especially its long-term motion. There are numerous types of equilibrium behaviour that can occur in continuous dynamical systems, but such long-time behaviours are restricted by the number of degrees-of-freedom (that is, by the dimensionality) of the system. In order words, one ignores the transient behaviour of a dynamical system and only considers the limiting behaviour as $t \to \infty$.

Chaos is a kind of motion, which is erratic, but not simply quasiperiodic with large number of periods [2]. Chaotic behaviour has been observed in driven acoustic systems, resonantly forced surface water, irradiated superconducting Josephson junction, ac-driven diode circuits, driven piezoelectric resonators, periodically forced neural oscillators, Ratchets, periodically modulated Josephson junction, the rigid body, gyroscopes, etc. For the motion of a system to be chaotic, the system variables should contain nonlinear terms and it must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions.

The study of chaos in the last decades had a tremendous impact on the foundations of science and engineering and one of the most recent exciting developments in this regard is the discovery of chaos synchronization, whose possibility was first reported by Fujisaka and Yamada [10] and later by Pecora and Carroll [24].

Different types of synchronization such as complete synchronization [24], anti-synchronization [45, 55, 46], hybrid synchronization [49, 15], lag synchronization [31], phase synchronization [31, 32], anti-phase synchronization [4], generalized synchronization [35], projective synchronization [21], generalized projective synchronization [36, 37, 65], etc. have been studied in the chaos literature.

Since the discovery of chaos synchronization, different approaches have been proposed to achieve it, such as PC method [24], active control method [1, 71, 12, 66], adaptive control method [47, 48, 38, 67], backstepping control method [53, 28, 29, 52, 30, 68], sliding mode control method [73, 51, 56, 69, 23, 63], etc.

The first famous chaotic system was accidentally discovered by Lorenz, when he was deriving a mathematical model for atmospheric convection [19]. Subsequently, Rössler discovered a chaotic system in 1976 [33], which is algebraically simpler than the Lorenz system.

Some well-known 3-D chaotic systems are Arneodo system [3], Sprott systems [39], Chen system [8], Lü-Chen system [20], Liu system [18], Cai system [6], T-system [54], etc. Many new chaotic systems have been also discovered like Li system [17], Sundarapandian system [44], Vaidyanathan systems [58, 59, 60, 61, 62, 70], Vaidyanathan-Madhavan system [64], Sundarapandian-Pehlivan system [50], Pehlivan system [25], Jafari system [13], Pham system [26], etc.

In the recent decades, there is some good interest in finding novel chaotic systems, which can be expressed by an explicit third order differential equation describing the time evolution of the single scalar variable x given by

$$\dddot{x} = j(x, \dot{x}, \ddot{x}) \tag{1.1}$$

The differential equation (1.1) is called "jerk system" because the third order time derivative in mechanical systems is called *jerk*. Sprott's work [39] on jerk systems inspired Gottlieb [11] to pose the question of finding the simplest jerk function that generates chaos. This question was successfully answered by Sprott [40], who proposed a jerk function containing just three terms with a quadratic nonlinearity:

$$j(x, \dot{x}, \ddot{x}) = -A\ddot{x} + \dot{x}^2 - x \quad \text{(with } A = 2.017\text{)} \tag{1.2}$$

Sprott showed that the jerk system with the jerk function (1.2) is chaotic with the Lyapunov exponents $L_1 = 0.0550, L_2 = 0$ and $L_3 = -2.0720$, and corresponding to Kaplan-Yorke dimension of $D_{KY} = 2.0265$.

Motivated by the research on jerk systems in the literature like the Sprott system with the jerk function (1.2), this research work announces a 3-D novel jerk chaotic system with two quadratic nonlinearities. The work on jerk functions and jerk chaotic systems are related to mechanical systems and the research on the control, chaos and synchronization of jerk systems have important literature on mechanical systems involving jerk functions.

First, we detail the fundamental qualitative properties of the novel jerk chaotic system. We show that the novel chaotic system is dissipative and derive the Lyapunov exponents and Kaplan-Yorke dimension of the novel jerk chaotic system.

Next, this paper derives an adaptive backstepping control law that stabilizes the novel jerk chaotic system about its unique equilibrium point at the origin, when the system parameters are unknown. The backstepping control method is a recursive procedure that links the choice of a Lyapunov function with the design of a controller and guarantees global asymptotic stability of strict feedback systems [7, 72, 57].

This paper also derives an adaptive backstepping control law that achieves global chaos synchronization of the identical 3-D novel jerk chaotic systems with unknown parameters. All the main adaptive results in this paper are proved using Lyapunov stability theory. MATLAB simulations are depicted to illustrate the phase portraits of the novel jerk chaotic system, dynamics of the Lyapunov exponents, adaptive stabilization and synchronization results for the novel jerk chaotic system.

2. A 3-D Novel Jerk Chaotic System

Recently, there is some interest in finding chaotic jerk functions having the special form

$$\dddot{x} + A\ddot{x} + \dot{x} = G(x), \tag{2.1}$$

where G is a nonlinear function having some special properties [41].

Such systems are called as *chaotic memory oscillators* in the literature. In [43], Sprott has made an exhaustive study on autonomous dissipative chaotic systems. Especially, Sprott has listed a set of 16 chaotic memory oscillators (Table 3.3, p. 74, [43]), named as $MO_0, MO_1, \ldots, MO_{15}$ with details of their Lyapunov exponents.

In this work, we propose a new jerk system, which is given in a system form as

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & ax_1 - bx_2 - x_3 - x_1^2 - x_2^2 \end{array} \right. \tag{2.2}$$

where a and b are positive parameters.

In this paper, we shall show that the system (2.2) is *chaotic* when the parameters a and b take the values

$$a = 7.5, \quad b = 4 \tag{2.3}$$

The system (2.2) with parameter values fixed at $a = 7.5$ and $b = 4$ will be chaotic for all initial conditions. In fact, the Lyapunov exponents of the system (2.2) are obtained as

$$L_1 = 0.12476, \quad L_2 = 0, \quad L_3 = -1.12451 \tag{2.4}$$

The presence of a positive Lyapunov exponent, L_1, shows that the system (2.2) is chaotic and the motion is dissipative since $L_1 + L_2 + L_3 < 0$.

Thus, for all initial conditions, the system (2.2) is chaotic with a strange chaotic attractor.

For numerical simulations, we take the initial conditions of the system (2.2) as

$$x_1(0) = 0.2, \quad x_2(0) = 0.6, \quad x_3(0) = 0.4 \tag{2.5}$$

The initial conditions in (2.5) have been chosen arbitrarily for the sake of simulations. For other initial conditions in $\mathbf{R}^3$ also, the system (2.2) is chaotic with a similar strange attractor.

Figure 1 depicts the chaotic attractor of the novel jerk system (2.2) in 3-D view, while in Figures 2-4, the 2-D projection of the strange chaotic attractor of the novel jerk chaotic system (2.2) on $(x_1, x_2), (x_2, x_3)$ and (x_3, x_1) planes, is shown, respectively.

3. Analysis of the 3-D Novel Jerk System

In this section, we describe the fundamental properties of the 3-D novel jerk chaotic system described by (2.2).

3.1 Dissipativity

In vector notation, the new jerk system (2.2) can be expressed as

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix}, \tag{3.1}$$

where

$$\begin{cases} f_1(x_1, x_2, x_3) &= x_2 \\ f_2(x_1, x_2, x_3) &= x_3 \\ f_3(x_1, x_2, x_3) &= ax_1 - bx_2 - x_3 - x_1^2 - x_2^2 \end{cases} \tag{3.2}$$

Let Ω be any region in $\mathbf{R}^3$ with a smooth boundary and also, $\Omega(t) = \Phi_t(\Omega)$, where Φ_t is the flow of f. Furthermore, let $V(t)$ denote the volume of $\Omega(t)$.

By Liouville's theorem, we know that

$$\dot{V}(t) = \int\limits_{\Omega(t)} (\nabla \cdot f)\, dx_1\, dx_2\, dx_3 \tag{3.3}$$

The divergence of the novel jerk system (3.1) is found as:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -1 < 0 \tag{3.4}$$

Inserting the value of $\nabla \cdot f$ from (3.4) into (3.3), we get

$$\dot{V}(t) = \int\limits_{\Omega(t)} (-1)\, dx_1\, dx_2\, dx_3 = -V(t) \tag{3.5}$$

Integrating the first order linear differential equation (3.5), we get

$$V(t) = \exp(-t)V(0) \tag{3.6}$$

Thus, $V(t) \to 0$ exponentially as $t \to \infty$. This shows that the novel 3-D jerk chaotic system (2.2) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the novel jerk chaotic system (2.2) settles onto a strange attractor of the system.

3.2 Equilibrium Points

The equilibrium points of the 3-D novel jerk chaotic system (2.2) are obtained by solving the equations

$$\left.\begin{aligned} f_1(x_1, x_2, x_3) &= x_2 &= 0 \\ f_2(x_1, x_2, x_3) &= x_3 &= 0 \\ f_3(x_1, x_2, x_3) &= ax_1 - bx_2 - x_3 - x_1^2 - x_2^2 &= 0 \end{aligned}\right\} \tag{3.7}$$

We take the parameter values as in the chaotic case, *viz.* $a = 7.5$ and $b = 4$.

Thus, the equilibrium points of the system (2.2) are characterized by the equations

$$ax_1 - x_1^2 = 0, \quad x_2 = 0, \quad x_3 = 0 \tag{3.8}$$

Solving the system (3.8), we get the equilibrium points of the system (2.2) as

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 7.5 \\ 0 \\ 0 \end{bmatrix} \tag{3.9}$$

To test the stability type of the equilibrium points E_0 and E_1, we calculate the Jacobian matrix of the novel jerk chaotic system (2.2) at any point $\mathbf{x} = \mathbf{x}^\star$:

$$J(\mathbf{x}^\star) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7.5 - 2x_1^\star & -4 - 2x_2^\star & -1 \end{bmatrix} \tag{3.10}$$

We note that

$$J_0 \triangleq J(E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7.5 & -4 & -1 \end{bmatrix} \tag{3.11}$$

which has the eigenvalues

$$\lambda_1 = 1.1555, \quad \lambda_{2,3} = -1.0778 \pm 2.3085i \tag{3.12}$$

This shows that the equilibrium point E_0 is a saddle-focus point.

Next, we note that

$$J_1 \triangleq J(E_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7.5 & -4 & -1 \end{bmatrix} \tag{3.13}$$

which has the eigenvalues

$$\lambda_{1,2} = 0.2737 \pm 2.1845i, \quad \lambda_3 = -1.5474 \tag{3.14}$$

This shows that the equilibrium point E_1 is also a saddle-focus point.

Hence, the novel jerk chaotic system (2.2) has two equilibrium points E_0 and E_1 defined by (3.9), which are saddle-foci. Thus, the equilibrium points E_0 and E_1 are unstable equilibrium points.

3.3 Lyapunov Exponents and Kaplan-Yorke Dimension

For the parameter values defined in (2.3), the Lyapunov exponents of the 3-D novel jerk system (2.2) are numerically obtained using MATLAB as

$$L_1 = 0.12476, \quad L_2 = 0, \quad L_3 = -1.12451 \tag{3.15}$$

Thus, the maximal Lyapunov exponent (MLE) of the novel jerk system (2.2) is positive, which means that the system has a chaotic behavior.

Since $L_1 + L_2 + L_3 = -0.99975 < 0$, it follows that the novel jerk chaotic system (2.2) is dissipative.

Also, the Kaplan-Yorke dimension of the novel jerk chaotic system (2.2) is obtained as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.11095 \tag{3.16}$$

The importance of calculating Kaplan-Yorke dimension, D_{KY}, can be explained as follows. D_{KY} represents an upper bound for the information dimension of the system [42].

Figure 5 depicts the dynamics of the Lyapunov exponents of the novel jerk chaotic system (2.2).

4. Adaptive Backstepping Control of the 3-D Novel Jerk Chaotic System with Unknown Parameters

In Section 3.2, we showed that the novel jerk chaotic system has two equilibrium points E_0 and E_1, which are unstable. Thus, the novel jerk chaotic system is an unstable chaotic system.

In this section, we use backstepping control method to derive an adaptive feedback control law for globally stabilizing the 3-D novel jerk chaotic system with unknown parameters.

Thus, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= ax_1 - bx_2 - x_3 - x_1^2 - x_2^2 + u \end{cases} \tag{4.1}$$

where a and b are unknown constant parameters, and u is a backstepping control law to be determined using estimates $\hat{a}(t)$ and $\hat{b}(t)$ for a and b, respectively.

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \end{cases} \tag{4.2}$$

Differentiating (4.2) with respect to t, we obtain the following equations:

$$\left\{\begin{array}{lcl} \dot{e}_a(t) & = & -\dot{\hat{a}}(t) \\ \dot{e}_b(t) & = & -\dot{\hat{b}}(t) \end{array}\right. \tag{4.3}$$

Next, we shall state and prove the main result of this section.

Theorem 4.1. *The 3-D novel jerk chaotic system* (4.1)*, with unknown parameters a and b, is globally and exponentially stabilized by the adaptive feedback control law,*

$$u(t) = -(3 + \hat{a}(t))x_1 - (5 - \hat{b}(t))x_2 - 2x_3 + x_1^2 + x_2^2 - kz_3 \tag{4.4}$$

where $k > 0$ is a gain constant,

$$z_3 = 2x_1 + 2x_2 + x_3, \tag{4.5}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t)$ is given by

$$\left\{\begin{array}{lcl} \dot{\hat{a}}(t) & = & x_1 z_3 \\ \dot{\hat{b}}(t) & = & -x_2 z_3 \end{array}\right. \tag{4.6}$$

Proof. We prove this result via backstepping control method and Lyapunov stability theory.

First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2}\, z_1^2 \tag{4.7}$$

where

$$z_1 = x_1 \tag{4.8}$$

Differentiating V_1 along the dynamics (4.1), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = x_1 x_2 = -z_1^2 + z_1(x_1 + x_2) \tag{4.9}$$

Now, we define

$$z_2 = x_1 + x_2 \tag{4.10}$$

Using (4.10), we can simplify the equation (4.9) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{4.11}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}\, z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{4.12}$$

Differentiating V_2 along the dynamics (4.1), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3) \tag{4.13}$$

Now, we define

$$z_3 = 2x_1 + 2x_2 + x_3 \tag{4.14}$$

Using (4.14), we can simplify the equation (4.13) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{4.15}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b) = V_2(z_1, z_2) + \frac{1}{2}z_3^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 \tag{4.16}$$

which is a positive definite function on $\mathbf{R}^5$.

Differentiating V along the dynamics (4.1), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} \tag{4.17}$$

Eq. (4.17) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} \tag{4.18}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3 \tag{4.19}$$

A simple calculation gives

$$S = (3 + a)x_1 + (5 - b)x_2 + 2x_3 - x_1^2 - x_2^2 + u \tag{4.20}$$

Substituting the adaptive control law (4.4) into (4.20), we obtain

$$S = (a - \hat{a})x_1 - (b - \hat{b})x_2 - kz_3 \tag{4.21}$$

Using the definitions (4.3), we can simplify (4.21) as

$$S = e_a x_1 - e_b x_2 - kz_3 \tag{4.22}$$

Substituting the value of S from (4.22) into (4.18), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k)z_3^2 + e_a\left(x_1 z_3 - \dot{\hat{a}}\right) + e_b\left(-x_2 z_3 - \dot{\hat{b}}\right) \tag{4.23}$$

Substituting the update law (4.6) into (4.23), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k)z_3^2, \tag{4.24}$$

which is a negative semi-definite function on $\mathbf{R}^5$.

From (4.24), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) \end{bmatrix} \in \mathbf{L}_\infty \tag{4.25}$$

Also, it follows from (4.24) that

$$\dot{V} \le -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2 \tag{4.26}$$

That is,

$$\|\mathbf{z}\|^2 \le -\dot{V} \tag{4.27}$$

Integrating the inequality (4.27) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \le V(0) - V(t) \tag{4.28}$$

From (4.28), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$.

From Eq. (4.1), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$.

Thus, using Barbalat's lemma, we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathbf{R}^3$.

Hence, it is immediate that $\mathbf{x}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{x}(0) \in \mathbf{R}^3$.

This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (4.1) and (4.6), when the adaptive control law (4.4) is applied.

The parameter values of the novel jerk chaotic system (4.1) are taken as $a = 7.5$ and $b = 4$, and the positive gain constant as $k = 10$.

Furthermore, as initial conditions of the novel jerk chaotic system (4.1), we take $x_1(0) = 3.2$, $x_2(0) = -4.5$ and $x_3(0) = 7.1$. Also, as initial conditions of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$, we take $\hat{a}(0) = 12.7$ and $\hat{b}(0) = 17.5$.

In Figure 6, the exponential convergence of the controlled states $x_1(t), x_2(t), x_3(t)$ is depicted, when the adaptive control law (4.4) and (4.6) are implemented.

5. Adaptive Backstepping Synchronization of the Identical 3-D Novel Jerk Chaotic Systems with Unknown Parameters

In this section, we use backstepping control method to derive an adaptive control law for globally and exponentially synchronizing the identical 3-D novel jerk chaotic systems with unknown parameters.

As the master system, we consider the 3-D novel jerk chaotic system given by

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & ax_1 - bx_2 - x_3 - x_1^2 - x_2^2\end{array}\right. \tag{5.1}$$

where x_1, x_2, x_3 are the states of the system, and a and b are unknown constant parameters.

As the slave system, we consider the controlled 3-D novel jerk chaotic system given by

$$\left\{\begin{array}{lcl}\dot{y}_1 & = & y_2 \\ \dot{y}_2 & = & y_3 \\ \dot{y}_3 & = & ay_1 - by_2 - y_3 - y_1^2 - y_2^2 + u\end{array}\right. \tag{5.2}$$

where y_1, y_2, y_3 are the states of the system, and u is a backstepping control to be determined using estimates $\hat{a}(t)$ and $\hat{b}(t)$ for a and b, respectively.

We define the synchronization errors between the states of the master system (5.1) and the slave system (5.2) as

$$\left\{\begin{array}{lcl}e_1 & = & y_1 - x_1 \\ e_2 & = & y_2 - x_2 \\ e_3 & = & y_3 - x_3\end{array}\right. \tag{5.3}$$

Then the error dynamics is easily obtained as

$$\left\{\begin{array}{lcl}\dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & e_3 \\ \dot{e}_3 & = & ae_1 - be_2 - e_3 - y_1^2 - y_2^2 + x_1^2 + x_2^2 + u\end{array}\right. \tag{5.4}$$

The parameter estimation errors are defined as:

$$\left\{\begin{array}{lcl}e_a(t) & = & a - \hat{a}(t) \\ e_b(t) & = & b - \hat{b}(t)\end{array}\right. \tag{5.5}$$

Differentiating (5.5) with respect to t, we obtain the following equations:

$$\left\{\begin{array}{lcl}\dot{e}_a(t) & = & -\dot{\hat{a}}(t) \\ \dot{e}_b(t) & = & -\dot{\hat{b}}(t)\end{array}\right. \tag{5.6}$$

Next, we shall state and prove the main result of this section.

Theorem 5.1. *The identical 3-D novel jerk chaotic systems* (5.1) *and* (5.2) *with unknown parameters a and b are globally and exponentially synchronized by the adaptive control law*

$$u(t) = -(3 + \hat{a}(t))e_1 - (5 - \hat{b}(t))e_2 - 2e_3 + y_1^2 + y_2^2 - x_1^2 - x_2^2 - kz_3 \tag{5.7}$$

where $k > 0$ is a gain constant,

$$z_3 = 2e_1 + 2e_2 + e_3, \tag{5.8}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= e_1 z_3 \\ \dot{\hat{b}}(t) &= -e_2 z_3 \end{cases} \tag{5.9}$$

Proof. We prove this result via backstepping control method and Lyapunov stability theory.

First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2}\, z_1^2 \tag{5.10}$$

where

$$z_1 = e_1 \tag{5.11}$$

Differentiating V_1 along the error dynamics (5.4), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = e_1 e_2 = -z_1^2 + z_1(e_1 + e_2) \tag{5.12}$$

Now, we define

$$z_2 = e_1 + e_2 \tag{5.13}$$

Using (5.13), we can simplify the equation (5.12) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{5.14}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}\, z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{5.15}$$

Differentiating V_2 along the error dynamics (5.4), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3) \tag{5.16}$$

Now, we define

$$z_3 = 2e_1 + 2e_2 + e_3 \tag{5.17}$$

Using (5.17), we can simplify the equation (5.16) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{5.18}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b) = V_2(z_1, z_2) + \frac{1}{2}z_3^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 \tag{5.19}$$

which is a positive definite function on $\mathbf{R}^5$.

Differentiating V along the error dynamics (5.4), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} \tag{5.20}$$

Eq. (5.20) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} \tag{5.21}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 \tag{5.22}$$

A simple calculation gives

$$S = (3 + a)e_1 + (5 - b)e_2 + 2e_3 - y_1^2 - y_2^2 + x_1^2 + x_2^2 + u \tag{5.23}$$

Substituting the adaptive control law (5.7) into (4.20), we obtain

$$S = (a - \hat{a}(t))e_1 - (b - \hat{b}(t))e_2 - kz_3 \tag{5.24}$$

Using the definitions (5.6), we can simplify (5.24) as

$$S = e_a e_1 - e_b e_2 - kz_3 \tag{5.25}$$

Substituting the value of S from (5.25) into (5.21), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k)z_3^2 + e_a\left(z_3 e_1 - \dot{\hat{a}}\right) + e_b\left(-z_3 e_2 - \dot{\hat{b}}\right) \tag{5.26}$$

Substituting the update law (5.9) into (5.26), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k)z_3^2, \tag{5.27}$$

which is a negative semi-definite function on $\mathbf{R}^5$.

From (5.27), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) \end{bmatrix} \in \mathbf{L}_\infty \tag{5.28}$$

Also, it follows from (5.27) that

$$\dot{V} \le -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2 \tag{5.29}$$

That is,

$$\|\mathbf{z}\|^2 \leq -\dot{V} \tag{5.30}$$

Integrating the inequality (5.30) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \leq V(0) - V(t) \tag{5.31}$$

From (5.31), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$.

From Eq. (5.4), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$.

Thus, using Barbalat's lemma, we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathbf{R}^3$.

Hence, it is immediate that $\mathbf{e}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{e}(0) \in \mathbf{R}^3$.

This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (5.1) and (5.2).

The parameter values of the novel jerk chaotic systems are taken as $a = 7.5$ and $b = 4$, and the positive gain constant as $k = 10$.

Furthermore, as initial conditions of the master chaotic system (5.1), we take $x_1(0) = 2.5$, $x_2(0) = 1.8$ and $x_3(0) = -1.7$. As initial conditions of the slave chaotic system (5.2), we take $y_1(0) = 4.3$, $y_2(0) = -2.8$ and $y_3(0) = 6.2$. Also, as initial conditions of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$, we take $\hat{a}(0) = 12.5$ and $\hat{b}(0) = 3.2$.

In Figures 7-9, the complete synchronization of the identical 3-D jerk chaotic systems (5.1) and (5.2) is shown, when the adaptive control law and the parameter update law are impelemented.

Also, in Figure 10, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$, is shown.

6. Conclusions

In this paper, we proposed a novel seven-term jerk chaotic system with two quadratic nonlinearities. Dynamic characteristics of new system has been discovered. It is worth noting that the possibilities of control and synchronization of such system with unknown parameters are verified by constructing an adaptive backstepping controller. The main results were established using adaptive control theory and Lyapunov stability theory. MATLAB simulations were shown to demonstrate all the main results developed in this paper. It is possible to use the new jerk system in potential chaos-based applications such as secure communications, random generation, or path planning for autonomous mobile robots. It is believed that the unknown dynamical behaviors of such strange chaotic jerk systems could be further investigated in future research.

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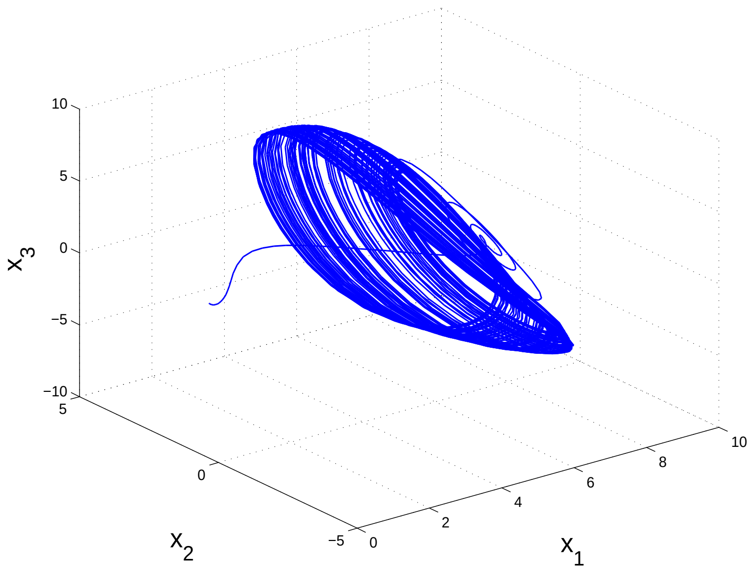

Figure 1: Strange attractor of the 3-D novel jerk system

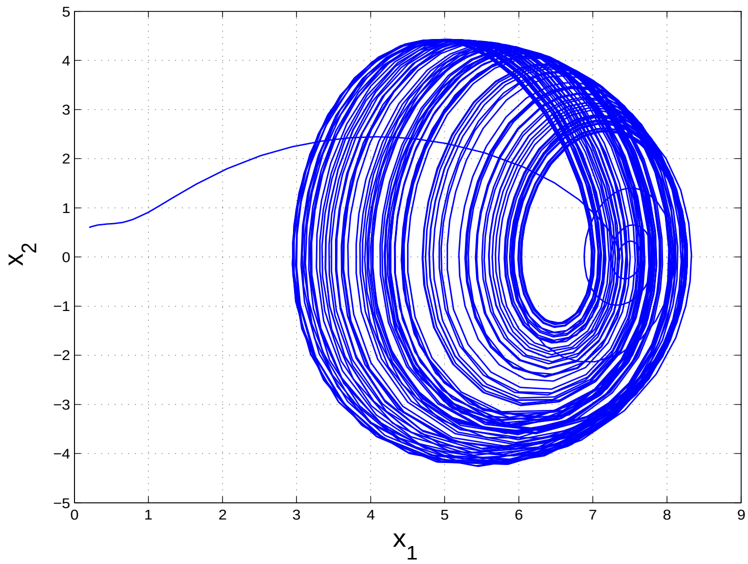

Figure 2: 2-D projection of the 3-D novel jerk system on (x_1, x_2)-plane

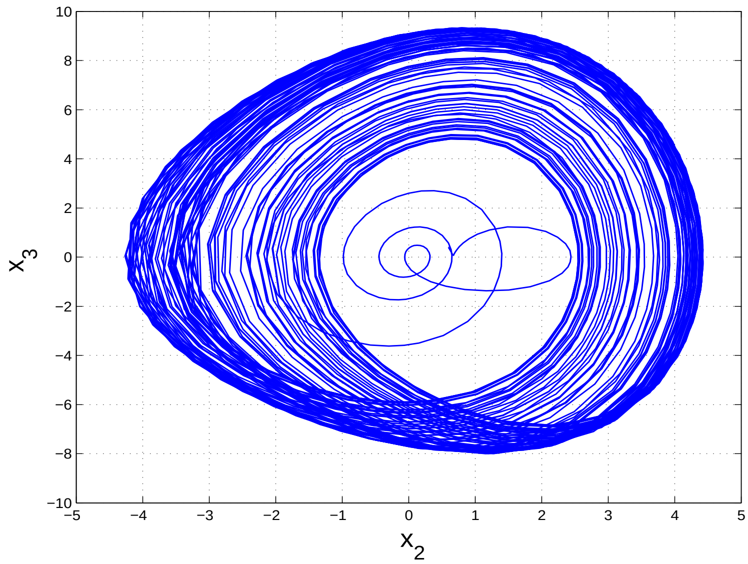

Figure 3: 2-D projection of the 3-D novel jerk system on (x_2, x_3)-plane

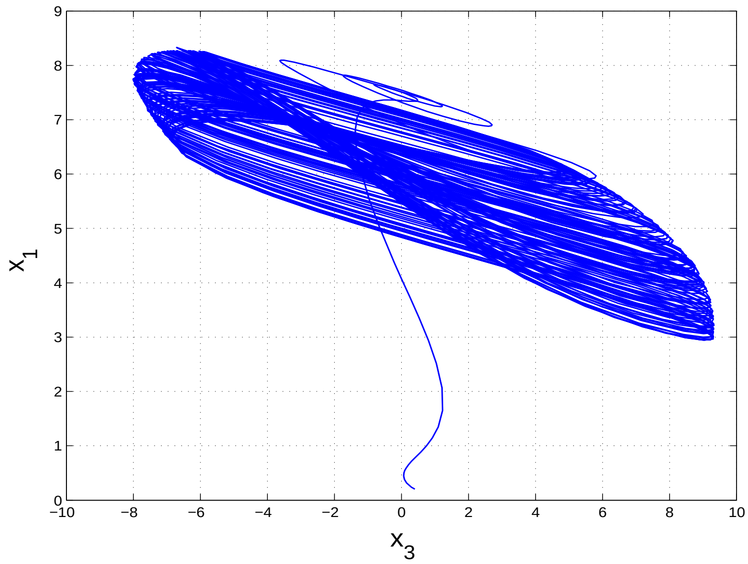

Figure 4: 2-D projection of the 3-D novel jerk system on (x_3, x_1)-plane

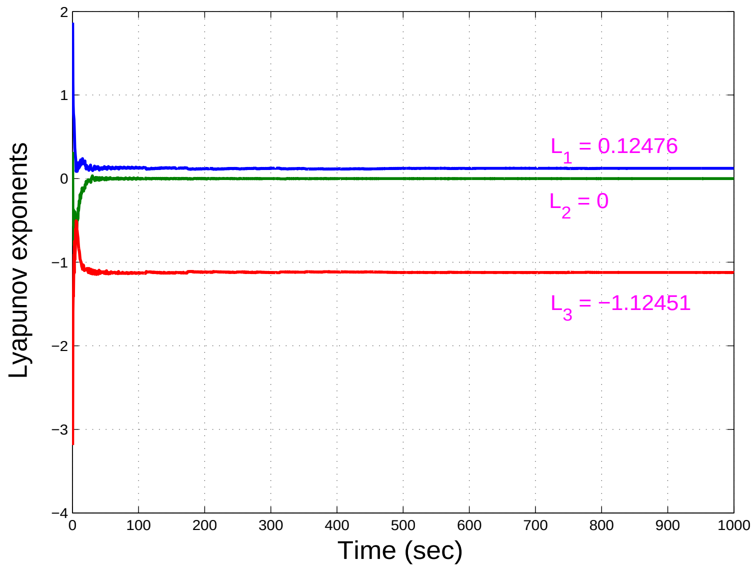

Figure 5: Dynamics of the Lyapunov exponents of the 3-D novel jerk system

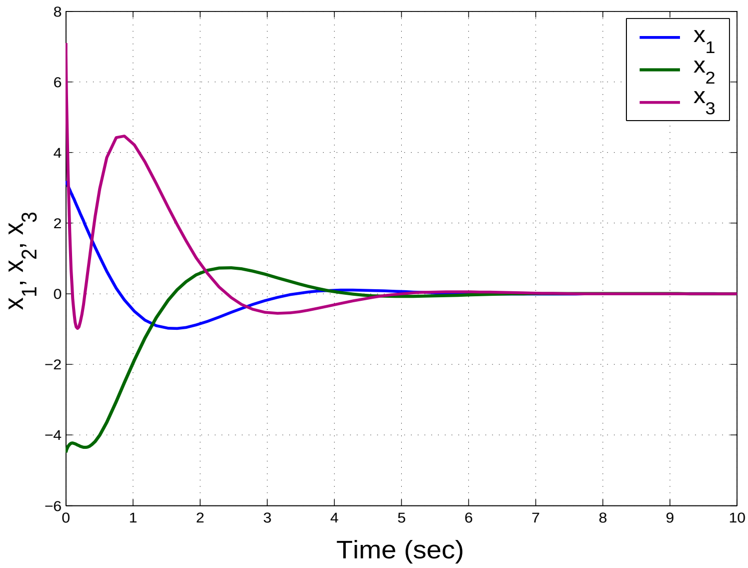

Figure 6: Time-history of the controlled states $x_1(t), x_2(t), x_3(t)$

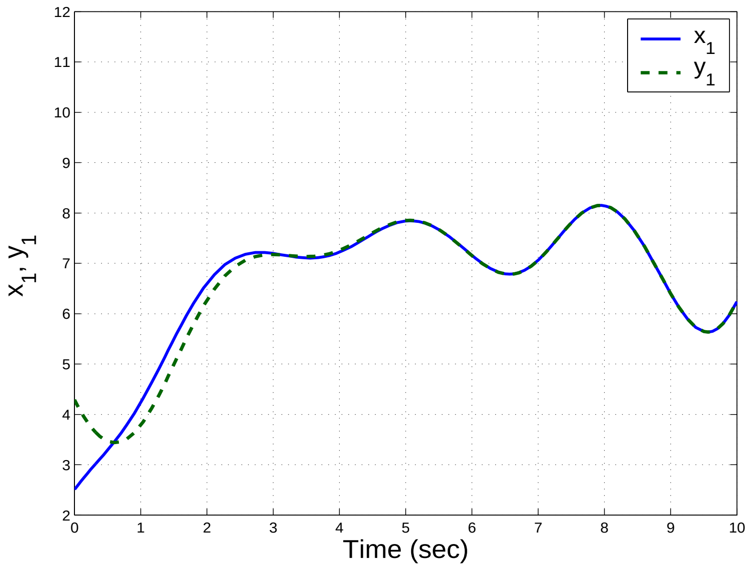

Figure 7: Synchronization of the states $x_1(t)$ and $y_1(t)$

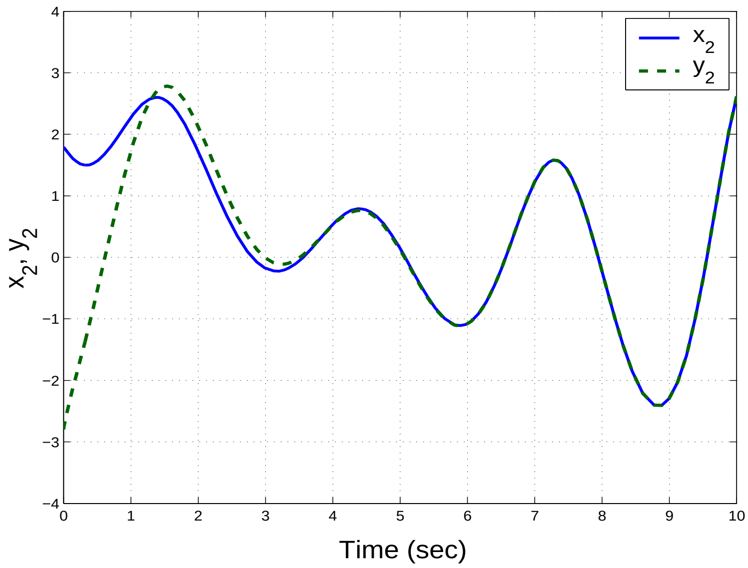

Figure 8: Synchronization of the states $x_2(t)$ and $y_2(t)$

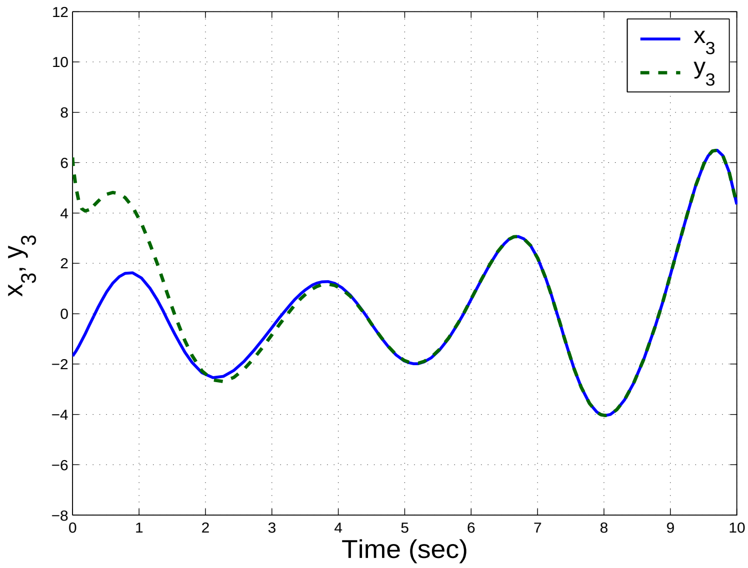

Figure 9: Synchronization of the states $x_3(t)$ and $y_3(t)$

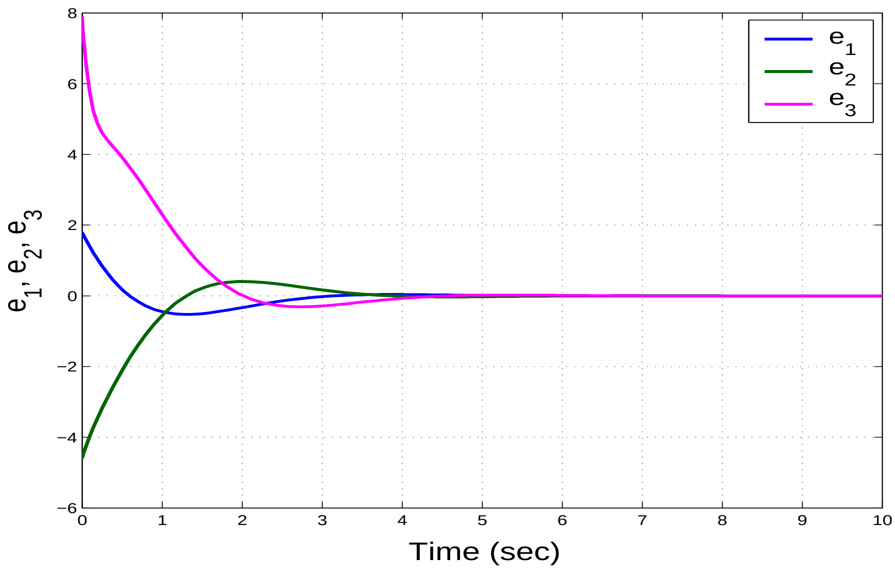

Figure 10: Time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$