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Characterization of Additive (m, n)-Semihyperrings

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Abstract. We say that (R, f, g) is an additive (m, n)-semihyperring if R is a non-empty set, f is an m-ary associative hyperoperation, g is an n-ary associative operation and g is distributive with respect to f. In this paper, we describe a number of characterizations of additive (m, n)-semihyperrings which generalize well-known results. Also, we consider distinguished elements, hyperideals, Rees factors and regular relations. Later, we give a natural method to derive the quotient (m, n)-semihyperring.

1. Introduction

Canonical hypergroups [24] is a special class of Marty's hypergroup [22]. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation, then we say that R is an additive hyperring. A special case of this type is the hyperring introduced by Krasner [16]. According to [7], an *additive semihyperring* is a system consisting of a set S together with a binary hyperoperation on S called *hypersum* and a binary operation *multiplication* (denoted in the usual manner) such that (1) S together with hypersum $+$, is a (commutative) semihypergroup, (2) S together with multiplication $\cdot$ is a semigroup, (3) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$, for all

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$a, b, c \in S$.

The idea of investigations of n-ary algebras, i.e., sets with one n-ary operation, seems to be going back to Kasner's lecture [15] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. But the first paper concerning the theory of n-ary groups was written (under inspiration of Emmy Noether) by Dörnte in 1928 (see [12]). Since then many papers concerning various n-ary algebras have appeared in the literature, for example see [5, 25, 26]. The concept of n-ary hypergroup is defined by Davvaz and Vougiouklis in [9], which is a generalization of the concept of hypergroup in the sense of Marty and a generalization of n-ary group, too. Then this concept was studied by Anvariyeh, Davvaz, Dudek, Leoreanu-Fotea Mirvakili, Vougiouklis, and others, for example see [1, 10, 11, 14, 18, 19, 20, 21]. The concept of n-ary algebraic hyperstructures constitute a generalization of well-known algebraic hyperstructures (semihypergroup, hypergroup, hyperring and so on).

Let S be a set. A map f from $S \times \ldots \times S$ to $\wp^*(S)$, the non-empty subsets of S, where S appears n times, is called an *n-ary hyperoperation*. If f is an n-ary hyperoperation defined on S, then (S, f) is called an *n-ary hypergroupoid*. We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. In this convention

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$$

will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \ldots = y_j = y$ the last expression will be written in the form $f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^n)$. Also, for non-empty subsets $A_1, \ldots, A_n$ of S we define $f(A_1^n) = f(A_1, \ldots, A_n) = \cup\{f(x_1^n) \mid x_i \in A_i,\ i = 1, \ldots n\}$. An n-ary hyperoperation f is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for every $i, j \in \{1, \ldots, n\}$ and all $x_1, x_2, \ldots, x_{2n-1} \in S$. An n-ary hypergroupoid with the associative hyperoperation is called an *n-ary semihypergroup*. An n-ary semihypergroup (S, f) is called *n-ary hypergroup* if for every $x_1^n \in S$ and $i = \{1, \ldots, n\}$ we have $f(x_1^{i-1}, S, x_{i+1}^n) = S$. An n-ary hypergroupoid (S, f) is *commutative* if for all $\sigma \in S_n$ and for every $a_1^n \in S$ we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in S$ we denote $a_{\sigma(1)}^{\sigma(n)}$ as the $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$. An element e of S is called a *neutral element (scalar neutral element)* if $x \in f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e})$$(x = f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}))$, for all $x \in S$ and all $1 \leq i \leq n$. An n-ary semihypergroup (S, f) is *i-cancellative*, if for every $a_2, \ldots, a_n \in S$, $f(a_2^i, x, a_{i+1}^n) = f(a_2^i, y, a_{i+1}^n)$ implies $x = y$, for all $x, y \in S$. If this implication is valid for all $i = 1, 2, \ldots, n$, then we say that (S, f) is *cancellative*. If for some $a_2, \ldots, a_n \in S$, $f(a_2^i, x, a_{i+1}^n) = f(a_2^i, y, a_{i+1}^n)$ implies $x = y$, for all $x, y \in S$ then the elements $a_2, \ldots, a_n$ are called *cancellable*.

In some papers several authors generalize the study of ordinary rings to the case where the ring operations are respectively m-ary and n-ary. (m,n)-rings were studied by Crombez [2], Crombez and Timm [3], Dudek [13] and Lee [17].

Now, in this paper we study a generalization of additive semihyperrings and a generalization of (m,n)-semirings.

Definition 1. An *additive* (m,n)*-semihyperring* is an algebraic hyperstructure (R,f,g), which satisfies the following axioms:

(1) (R,f) is an m-ary semihypergroup,

(2) (R,g) is an n-ary semigroup,

(3) the n-ary operation g is distributive with respect to the m-ary hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R,\ 1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

Throughout this paper, every (m,n)-semihyperring is an additive (m,n)-semihyperring. If f is an m-ary operation then (R,f,g) is called an (m,n)-semiring. An additive (m,n)-semihyperring is called an *additive* (m,n)*-hyperring* if (R,f) is an m-ary hypergroup. Let (R,f,g) be an (m,n)-semihyperring such that (R,f) has a neutral (scalar neutral) element 0, then 0 is called a *zero (scalar zero)* element if $g(x_1^{i-1}, 0, x_{i+1}^n) = 0$, for every $x_1^n \in R$. A special subclass of additive (m,n)-hyperrings is the Krasner (m,n)-hyperring. We recall the following definition from [23]. A *Krasner* (m,n)*-hyperring* is an additive (m,n)-hyperring such that (R,f) is a canonical m-ary hypergroup and relating to the n-ary multiplication, (R,g) is an n-ary semigroup having zero element 0. In an additive (m,n)-semihyperring (R,f,g), fixing elements a_2^{m-1} and b_2^{n-1} we obtain a hyperoperation $\oplus$ and an operation $\odot$ as follows: $x \oplus y = f(x, a_2^{m-1}, y)$ and $x \odot y = f(x, b_2^{n-1}, y)$. Choosing different elements a_2^{m-1} and b_2^{n-1}, we obtain different binary relations. Obviously, $(R,\oplus,\odot)$ is an additive semihyperring. Such obtained additive semihyperrings are called *retracts* of (R,f,g). Let $(R,\oplus,\odot)$ be an additive semihyperring. Let f be an m-ary hyperoperation and g be an n-ary operation on R as follows: $f(x_1^m) = x_1 \oplus \ldots \oplus x_m$ and $g(y_1^n) = y_1 \odot \ldots \odot y_n$, for all $x_1^m, y_1^n \in R$. Then, (R,f,g) is an (m,n)-semihyperring.

Example 1. Let N be the set of all positive integers. We define an m-ary hyperoperation and an n-ary multiplication on N in the following way:

$$f(x_1, \ldots, x_m) = \bigcup_{i=1}^m \{x_i\} \quad \text{and} \quad g(x_1, \ldots, x_n) = \prod_{i=1}^n x_i,$$

Then, (N,f,g) is an (m,n)-semihyperring. It has not zero element.

Example 2. Let $(R,+,\cdot)$ be a semiring. We define an m-ary hyperoperation and an n-ary multiplication on R in the following way:

(1) $f(x_1, \ldots, x_m) =< x_1, \ldots, x_m >$, the ideal generated by $x_1, \ldots, x_n$,

(2) $g(x_1^n) = x_1 \cdot \ldots \cdot x_n$.

Then, (R, f, g) is an (m, n)-semihyperring. If R has a zero element 0, then 0 is a zero element of (R, f, g).

Example 3. Let I be the real interval $[0, 1]$ and for every $x, y \in I$, set $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. On I we define

(1) $f(x_1, \ldots, x_m) = \{t \in I \mid x_1 \wedge \ldots \wedge x_m \leq t \leq x_1 \vee \ldots \vee x_m\}$,

(2) $g(x_1^n) = x_1 \wedge \ldots \wedge x_n$.

Then, (I, f, g) is an (m, n)-semihyperring.

Example 4.([6]) If $(L, \wedge, \vee)$ is a relatively complemented distributive lattice and if $\oplus$ and g are defined as:

(1) $a \oplus b = \{c \in L \mid a \wedge c = b \wedge c = a \wedge b,\ a, b \in L\}$,

(2) $g(a, b, c) = a \vee b \vee c$.

Then, $(L, \oplus, g)$ is a $(2, 3)$-semihyperring.

Example 5. Let $(R, +, \cdot)$ be a semihyperring and $b \in Z(R)$, this means for every $x \in R$, $x \cdot b = b \cdot x$. Now, we set $g(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n \cdot b$. Then, $(R, +, g)$ is a $(2, n)$-semihyperring.

Example 6.([6]) Let $R = Z_2 \times Z_3$. We define a hyperoperation $+$ on R as follows:

$$(a, b) + (c, d) = \begin{cases} (0, Z_3) & \text{if } a + c = 0 \\ (1, Z_3) & \text{if } a + c = 1 \\ (Z_2, Z_3) & \text{if } a + c = 2 \end{cases}$$

and define a ternary multiplication $g((x_1, y_1), (x_2, y_2), (x_3, y_3)) = (x, y)$ such that $x \equiv x_1 x_2 x_3 (mod\ 2)$ and $y \equiv y_1 - y_2 + y_3 (mod\ 3)$. Then, $(R, +, g)$ is a $(2, 3)$-semihyperring.

Example 7. Let $(G, \circ)$ be an abelian group. We define an m-ary hyperoperation f and $(2n-1)$-ary multiplication g on G in the following way:

$$f(x_1, \ldots x_m) = \bigcup_{i=1}^{m} \{x_i\}, \text{ for all } x_1^m \in R,$$

$$g(x_1^{2n-1}) = y_1 \circ y_2 \circ \ldots \circ y_{2n-1}, \text{ where } y_i = \begin{cases} x_i & \text{if } i \text{ is odd} \\ x_i^{-1} & \text{if } i \text{ is even.} \end{cases}$$

Then, (G, f, g) is an $(m, 2n-1)$-semihyperring.

Example 8.([6]) Let $G = (Z_{16}, +, \cdot)$ and $R = 2Z_{16}$. We define a binary hyperoperation and a ternary multiplication on R in the following way:

$$x \oplus y = \{x, y\} \quad \text{and} \quad g(x, y, z) = x \cdot y \cdot z + 4.$$

Then, g is associative, since for every $x_1^5 \in R$, we have

$$g(g(x_1^3), x_4^5) = g(x_1, g(x_2^4), x_5) = g(x_1^2, g(x_3^5)) = 4.$$

It is not difficult to see that $(R, \oplus, g)$ is a $(2, 3)$-semihyperring.

Regular(strongly regular) relations play an important role in hyperstructure theory. Let ρ be an equivalence relation on an n-ary semihypergroup (S, f). H_ρ denotes the set of equivalence classes of ρ. We denote by $\overline{\overline{\rho}}$ the relation defined on $\mathcal{P}^*(S)$ as follows. If $A, B \in \mathcal{P}^*(S)$, then

$$A \;\overline{\overline{\rho}}\; B \Longleftrightarrow a \;\rho\; b \;\text{ for all, }\; a \in A, b \in B.$$

It follows immediately that $\overline{\overline{\rho}}$ is symmetric and transitive. In general, $\overline{\overline{\rho}}$ is not reflexive. Also, we denote by $\overline{\rho}$ the relation defined on $\mathcal{P}^*(S)$ as follows. If $A, B \in \mathcal{P}^*(S)$, then

$$A \;\overline{\rho}\; B \Longleftrightarrow \quad \begin{array}{l}\text{for all } a \in A, \text{ there exists } b \in B \text{ such that } a \;\rho\; b \text{ and} \\ \text{for all } b \in B, \text{ there exists } a \in A \text{ such that } a \;\rho\; b.\end{array}$$

Let (S, f) be an n-ary semihypergroup and ρ be an equivalence relation on S. Then, ρ is a *regular* relation if $a_i \;\rho\; b_i$ for all $1 \le i \le n$ then $f(a_1, \ldots, a_n) \;\overline{\rho}\; f(b_1, \ldots, b_n)$. Also, ρ is called a *strongly regular* relation if $a_i \;\rho\; b_i$ for all $1 \le i \le n$ then $f(a_1, \ldots, a_n) \;\overline{\overline{\rho}}\; f(b_1, \ldots, b_n)$. By a *regular(strongly regular)* relation on an (m, n)-semihyperring R we mean a regular(strongly regular) relations on (R, f) and (R, g). Mirvakili and Davvaz proved the next theorem:

Theorem 1.([23]) *Let (R, f, g) be an (m, n)-semihyperring and the relation ρ be a regular(strongly regular) relation on (R, f, g). Then, the quotient (R_ρ, f_ρ, g_ρ) is an (m, n)-semihyperring((m, n)-semiring) under $f_\rho(\rho(x_1), \ldots, \rho(x_m)) = \rho(f(x_1^m))$ and $g_\rho(\rho(y_1), \ldots, \rho(y_n)) = \rho(g(y_1^n))$, for all x_1^m and y_1^n in R.*

Theorem 2. *Let (R, f, g) and (S, f', g') be two (m, n)-semihyperrings and $\varphi : R \longrightarrow S$ be a homomorphism. Then, $ker\varphi = \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\}$ is a regular relation on R and there exists a unique one to one homomorphism ψ from $R_{ker\varphi}$ into S.*

Proof. It is straightforward. □

Corollary 3. *Let (R, f, g) be an (m, n)-semihyperring and ρ, σ be two regular relations on R with $\rho \subseteq \sigma$. Then, $\sigma_\rho = \{(\rho(a), \rho(b)) \mid (a, b) \in \sigma\}$ is a regular relation on R_ρ and $(R_\rho)_{(\sigma_\rho)} \cong R_\sigma$.*

2. Hyperideals of (m, n)-Semihyperrings

Let S be a non-empty subset of an (m, n)-semihyperring (R, f, g). If (S, f, g) is an (m, n)-semihyperring, then S is called a *sub-semihyperring* of R.

Definition 5. Let (R, f, g) be an (m, n)-semihyperring. By an (i, j)-*center* of R we mean the set

$$Z_{ij}(R) = \{a \in R \mid f(x_1^{i-1}, a, x_i^{n-1}) = f(x_1^{j-1}, a, x_j^{n-1}), \text{ for } x_1^{n-1} \in R\}.$$

The set $Z(R) = \bigcap_{i=1}^{n} Z_{ij}(R) = \bigcap_{j=1}^{n} Z_{ij}(R)$ is called the *center* of R.

Proposition 6. *Let (R, f, g) be an (m, n)-semihyperring. Then,*

(1) *For every $i, j \in \{1, \ldots, n\}$, $Z_{ij} = Z_{ji}$.*

(2) *If $a \in Z_{ij} \cap Z_{jk}$, then $a \in Z_{ik}$.*

(3) *If $Z_{ij}(R)$ is non-empty, then it is a sub-semihyperring of R.*

(4) *If $Z(R)$ is non-empty, then it is a maximal commutative sub-semihyperring of R.*

Proof. The proof is straightforward. □

Definition 7. Let I be a non-empty subset of an (m, n)-semihyperring (R, f, g) and $1 \leq i \leq n$; we call I an (i)-*hyperideal* of R if

(1) I is a sub-semihypergroup of the m-ary semihypergroup (R, f), i.e., (I, f) is an m-ary semigroup,

(2) for every $x_1^n \in R$, $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$.

Also, if for every $1 \leq i \leq n$, I is an (i)-hyperideal, then I is called a *hyperideal* of R.

If X is a subset of an (m, n)-semihyperring R, then $< X >$ is the hyperideal generated by elements of X. Let $A_1, \ldots, A_n$ be non-empty subsets of R. We set

$$\prod_{i=1}^{n} A_i = \{f_{(k)}([g(a_{i1}^{in})]_{i=1}^{i=m_k}) \mid a_{ij} \in A_j, m_k = k(m-1)+1\}.$$

Then, $\prod_{i=1}^{n} A_i$ called the *product* of A_i.

Lemma 8. *Let R be an (m, n)-semihyperring. Then,*

(1) *If $I_1, \ldots, I_m$ are hyperideals of R, then $f(I_1^m)$ is a hyperideal of R.*

(2) *If $I_1, \ldots, I_m$ are subsets of R and there exists $1 \leq j \leq n$ such that I_j is a hyperideal of R and R is commutative, then $\prod_{i=1}^{n} I_i$ is a hyperideal of R.*

(3) *If $I_1, \ldots, I_n$ are hyperideals of R and $\bigcap_{i=1}^{n} I_i \neq \emptyset$, then $\bigcap_{i=1}^{n} I_i$ is a hyperideal of R and $< \prod_{i=1}^{n} I_i > \subseteq \bigcap_{i=1}^{n} I_i$.*

(4) *If I is a hyperideal of R and $a_2^n \in I$, then $f(I, a_2^n) = I$.*

Proof. The proof is similar to the proof of Lemma 3.4 in [23]. □

An element $\omega \in R$ is called *(i,j)-distinguished element* of the (m,n)-semihyperring R if it satisfies $f(x_1^i, \omega, x_{i+1}^m) = \omega$ and $g(y_1^j, \omega, y_{j+1}^n) = \omega$, for all $x_1^m, y_1^n \in R$, where $1 \le i \le m$ and $1 \le j \le n$. An element $\omega \in R$ is called *distinguished element* of the (m,n)-semihyperring R if it is an (i,j)-distinguished for all $1 \le i \le m$ and $1 \le j \le n$. Every (m,n)-semihyperring can not contain two different distinguished elements. We shall always call "ω" the distinguished element of every (m,n)-semihyperring.

Theorem 9. *Let R be an (m,n)-semihyperring and $\omega \in R$. Then, the following conditions are equivalent:*

(1) *ω is a distinguished element of R.*

(2) *ω is a $(1,1)$-distinguished element and an (n,n)-distinguished element of R.*

(3) *ω is a $(1,n)$-distinguished element and an $(n,1)$-distinguished element of R.*

(4) *for some $1 < i < n$, ω is an $(i,1)$-distinguished element and an (i,n)-distinguished element of R.*

(5) *for some $1 < j < n$, ω is a $(1,j)$-distinguished element and an (n,j)-distinguished element of R.*

(6) *for some $1 < i,j < n$, ω is an (i,j)-distinguished element of R.*

Proof. (1) → (2) It is clear by using the definition.

(2) → (3) It is straightforward.

(3) → (4) We have $f(\omega, x_2^m) = \omega$, $g(y_1^{n-1}, \omega) = \omega$, $f(x_1^{m-1}, \omega) = \omega$ and $g(\omega, y_2^n) = \omega$, for every $x_1^m, y_1^n \in R$. Now, we have

$$\begin{aligned} f(x_1^i, \omega, x_{i+1}^m) &= f(x_1^i, f(x_{i+1}^{m-1}, \omega, \ldots, \omega), x_{i+1}^m) \\ &= f(f(x_1^{m-1}, \omega), \omega, \ldots, \omega, x_{i+1}^m) \\ &= f(\omega, \omega, \ldots, \omega, x_{i+1}^m) \\ &= \omega. \end{aligned}$$

(4) → (5) Similar to the proof of (3) → (4), we obtain $g(x_1^j, \omega, x_{j+1}^m) = \omega$. Now, we have

$$\begin{aligned} f(\omega, x_2^m) &= f(f(\overset{(m)}{\omega}), x_2^m) \\ &= f(\overset{(m-i)}{\omega}, f(\overset{(i)}{\omega}, x_2^{m-i+1}), x_{m-i+2}^m) \\ &= f(\overset{(m-i)}{\omega}, \omega, x_{m-i+2}^m) \\ &\cdots \\ &= f(\overset{(m)}{\omega}) = \omega. \end{aligned}$$

and

$$\begin{aligned} f(x_1^{m-1},\omega) &= f(x_1^{m-1}, f(\overset{(m)}{\omega})) \\ &= f(x_1^{m-i}, f(x_{m-i+1}^{m-1}, \overset{(m-i+1)}{\omega}), \overset{(i-1)}{\omega}) \\ &= f(x_1^{m-i}, \omega, \overset{(i-1)}{\omega}) \\ &\ldots \\ &= f(\overset{(m)}{\omega}) = \omega. \end{aligned}$$

(5) → (6) The proof is similar to the proof of (3) → (4).

(6) → (1) Let $1 < i < m$ and $f(x_1^{i-1},\omega,x_{i+1}^m)=\omega$ for every $x_1^m \in R$. Now, for every $x_1^m \in R$ we have

$$\begin{aligned} f(x_1^{i-2},\omega,x_i^m) &= f(x_1^{i-2}, f(\overset{(m)}{\omega}), x_{i+1}^m) \\ &= f(x_1^{i-2},\omega, f(\overset{(m-1)}{\omega}, x_i), x_{i+1}^m) \\ &= f(x_1^{i-2},\omega,\omega,x_{i+1}^m) \\ &= \omega. \end{aligned}$$

In the similar way, we obtain $f(x_1^i,\omega,x_{i+2}^m)=\omega$, for every $x_1^m \in R$. Also, in the similar way, for the m-ary operation g, we have $g(x_1^j,\omega,x_{j+2}^n)=\omega$ and $g(x_1^{i-1},\omega,x_i^n)=\omega$. Hence, ω is an (h,k)-distinguished element when $h=i-1,i,i+1$ and $k=i-1,i,i+1$.

If we repeat the above process we obtain ω is an (h,k)-distinguished element for every $h\in\{1,\ldots,m\}$ and $k\in\{1,\ldots,n\}$. □

Definition 10. Let (R,f,g) be an (m,n)-semihyperring and I be a subset of R. We say that I is an (i,j)-*hyperideal* of R, where $1\le i,j\le n$, if it satisfies:

(1) $f(x_1^{i-1},I,x_{i+1}^m)\subseteq I$, for all $x_1^n\in R$,

(2) $g(y_1^{j-1},I,y_{j+1}^n)\subseteq I$, for all $y_1^n\in R$.

If I is an (i,j)-hyperideal of R, for every $1\le i,j\le n$, then we say that I is a *2-hyperideal* of R. Indeed, a 2-hyperideal is a hyperideal of the m-ary semihypergroup (R,f) and the n-ary semigroup (R,g).

Lemma 11. *Let R be an (m,n)-semihyperring and I_1^k be 2-hyperideals of R .*

(1) *If $\bigcap_{i=1}^k I_i\neq\emptyset$ then $\bigcap_{i=1}^k I_i$ is a 2-hyperideal of the (m,n)-semihyperring R.*

(2) *$\bigcup_{i=1}^k I_i$ is a 2-hyperideal of the (m,n)-semihyperring R.*

Proof. The proof is straightforward. □

We say that I is an (i,j)-*distinguished hyperideal*, where $1\le i,j\le n$, if

(1) $f(x_1^{i-1},I,x_{i+1}^m)=I$, for all $x_1^n\in R$,

(2) $g(y_1^{j-1}, I, y_{j+1}^n) = I$, for all $y_1^n \in R$.

Let I be an (i,j)-distinguished hyperideal of R, for every $1 \leq i,j \leq n$. Then, we say that I is a *distinguished hyperideal* of R. If I and J are two distinguished hyperideals, then it is clear that $I = J$.

Theorem 12. *Let R be an (m,n)-semihyperring and I be a non-empty subset of R. Then, the following conditions are equivalent:*

(1) *I is a distinguished hyperideal of R.*

(2) *I is a $(1,1)$-distinguished hyperideal and an (n,n)-distinguished hyperideal of R.*

(3) *I is a $(1,n)$-distinguished hyperideal and an $(n,1)$-distinguished hyperideal of R.*

(4) *for some $1 < i < n$, I is an $(i,1)$-distinguished hyperideal and an (i,n)-distinguished hyperideal of R.*

(5) *for some $1 < j < n$, I is a $(1,j)$-distinguished hyperideal and an (n,j)-distinguished hyperideal of R.*

(7) *for some $1 < i,j < n$, I is an (i,j)-distinguished hyperideal of R.*

Proof. The proof is similar to the proof of Theorem 9. □

A 2-hyperideal I of an (m,n)-semihyperring (R,f,g) generates the following binary relation (*Rees relation*) on R: $a\rho_I b$ if and only if $a = b$ or ($a \in I$ and $b \in I$).

Lemma 13. *Rees relation on an (m,n)-semihyperring (R,f,g) is a strongly regular relation.*

Proof. Let $a, b, x_1^n \in R$, $1 \leq i \leq n$ and $a\rho_I b$. If $a = b$, then $\rho(a) = \rho(b)$, and if $a, b \in I$, then $\rho(a) = \rho(b)$. Since $\rho(x_j) = x_j$ or $\rho(x_j) = I$, so

$$f(\rho(x_1), \ldots, \rho(x_{i-1}), \rho(a), \rho(x_{i+1}), \ldots, \rho(x_n))$$

and

$$f(\rho(x_1), \ldots, \rho(x_{i-1}), \rho(b), \rho(x_{i+1}), \ldots, \rho(x_n))$$

are same set and both are singleton or both are subsets of I. Since I is a 2-hyperideal, so

$$f(\rho(x_1), \ldots, \rho(x_{i-1}), \rho(a), \rho(x_{i+1}), \ldots, \rho(x_n))$$

$$\overline{\overline{\rho_I}} f(\rho(x_1), \ldots, \rho(x_{i-1}), \rho(b), \rho(x_{i+1}), \ldots, \rho(x_n))$$

Therefore, ρ_I is a strongly regular relation. □

For every $x \in I$, we have $\rho_I(x) = I$ and for every $x \in R - I$ we have $\rho_I(x) = \{x\}$. Now, we set $R_{\rho_I} = R/I = \{\rho(x) \mid x \in R\} = \{I\} \cup \{\{x\} \mid x \in R - I\}$. Then, we define

(1) $F(\rho_I(x_1), \ldots, \rho_I(x_m)) = \rho_I(f(x_1^m))$,

(2) $G(\rho_I(y_1), \ldots, \rho_I(y_n)) = \rho_I(g(y_1^n))$.

Lemma 14. *$(R/I, F, G)$ is an (m, n)-semihyperring and I is the distinguished element of R/I.*

Proof. The proof is straightforward. □

The (m, n)-semihyperring $(R/I, F, G)$ is called the *Rees factor (m, n)-semihyperring of R modulus I*.

Lemma 15. *We have $R/I \cong \{\omega\} \cup (R - I)$.*

Proof. The proof is straightforward. □

Proposition 16. *Let (R, f, g) be an (m, n)-semihyperring, I be a 2-hyperideal and S be a sub-semihyperring of R. Then,*

(1) *$I \cup S$ is a subsemihyperring of R and I forms a 2-hyperideal of $I \cup S$.*

(2) *If $I \cap S \neq \emptyset$, then $I \cap S$ is a 2-hyperideal of the sub-semihyperring S.*

(3) *If $I \cap S \neq \emptyset$, then $(I \cup S)/I \cong S/(I \cap S)$.*

Proof. The proofs of (1) and (2) are straightforward. In order to prove (3), we have $(I \cup S)/I \cong ((I \cup S) - I) \cup \{\omega\} = (S - (S \cap I)) \cup \{\omega\} \cong S/(I \cap S)$. □

Proposition 17. *Let (R, f, g) be an (m, n)-semihyperring. Let I be a 2-hyperideal of R and $g : I \longrightarrow R/I$ be the natural homomorphism. Then, g induces a one-to-one correspondence which preserves inclusion, which we also call g*

$$g : K \longrightarrow K/I$$

from the set of the 2-hyperideals of R that contain I upon the set of the non-trivial 2-hyperideals of R/I. Moreover,

$$(R/I)/(K/I) \cong R/K.$$

Proof. Suppose that K is a 2-hyperideal of R such that $K \subseteq I$. Then, $g(K) = K/I$ is a 2-hyperideal of $g(R) = R/I$. Now, if J is a 2-hyperideal of R/I, then $g^{-1}(J) = K$ is a 2-hyperideal of R which contains I, so that $g(K) = J$. Therefore, g induces a mapping from the first set of the statement onto the second. Also, g induces a one to one map from the first set onto the second set, because $g(A) = g(B)$ implies $A/I = B/I$ or $A - I = B - I$, and so $A = B$. Similarly, it is easy to see that g preserves the inclusion. Finally, we have

$$\begin{aligned} (R/I)/(K/I) &\cong (R/I - K/I) \cup \{\omega\} \\ &\cong ((R - I) - (K - I)) \cup \{\omega\} \\ &\cong (R - K) \cup \{\omega\} \cong R/K. \end{aligned}$$

□

3. (m,n)-Semihyperring of Quotients

In [8], Davvaz and Salasi studied the hyperring of fractions (quotients). In [4], Darafsheh and Davvaz, defined the H_v-ring of fractions of a commutative hyperring. In [3], Crombez and Timm, proved that any commutative cancellative (n,m)-ring can be embedded into a unique (up to isomorphism) minimal (n,m)-field. Lee [17] proved (using the well-known procedure of embedding an integral domain into a field) that any commutative and cancellative (Ω,m)-ringoid A can be embedded into a quotient (Ω,m)-ringoid $Q(A)$. This extends a result of G. Crombez and J. Timm [3].

Our aim in this section is to introduce (m,n)-semihyperring of quotients.

Let (R,f,g) be a commutative (m,n)-semihyperring with at least one cancellable element respect to g and let S be the set of all cancellable elements. Consider the set $R \times S^{n-1}$ of ordered pair $(a_1,(a_2^n))$. We introduce a relation in this set by defining

$$(a_1,(a_2^n)) \sim (b_1,(b_2^n)) \Longleftrightarrow g(a_1,b_2^n) = g(b_1,a_2^n).$$

Lemma 18. *The relation $\sim$ is an equivalence relation on $R \times S^{n-1}$.*

Proof. The relation is clearly reflexive and symmetric. Now, we suppose that

$$(a_1,(a_2^n)) \sim (b_1,(b_2^n)) \text{ and } (b_1,(b_2^n)) \sim (c_1,(c_2^n)).$$

Then, $g(a_1,b_2^n) = g(b_1,a_2^n)$ and $g(b_1,c_2^n) = g(c_1,b_2^n)$. In order to prove the transitivity, we have to show that $g(a_1,c_2^n) = g(c_1,a_2^n)$. We have

$$g(g(a_1,b_2^n),c_2^n) = g(g(b_1,a_2^n),c_2^n), \qquad (*)$$
$$g(g(b_1,c_2^n),a_2^n) = g(g(c_1,b_2^n),a_2^n). \qquad (**)$$

Since g is commutative, by $(*)$ and $(**)$ we obtain $g(g(a_1,b_2^n),c_2^n) = g(g(c_1,b_2^n),a_2^n)$. Thus, $g(g(a_1,c_2^n),b_2^n) = g(g(c_1,a_2^n),b_2^n)$. Since b_2^n are cancellable elements, we have $g(a_1,c_2^n) = g(c_1,a_2^n)$ which implies that the transitivity of $\sim$. □

We now note that the equivalence class of $(a_1,(a_2^n))$ by $\frac{a_1}{[a_2^n]}$. Also, we set $\frac{a_1}{[a_{12}^{1n},a_{22}^{2n},\ldots,a_{m2}^{mn}]} := \frac{a_1}{[g(a_{12}^{m2}),g(a_{13}^{m3}),\ldots,g(a_{1n}^{mn})]}$. Let $S^{-1}R$ denote the set of these equivalence classes. We define

$$F\left(\frac{a_1}{[a_{12}^{1n}]},\ldots,\frac{a_m}{[a_{m2}^{mn}]}\right) = \{x \mid x \in \frac{f(h(a_1,a_{22}^{mn}),\ldots,h(a_m,a_{12}^{1n},\ldots,a_{(m-1)2}^{(m-1)n})}{[a_{12}^{1n},\ldots a_{m2}^{mn}]}\}$$

and

$$G\left(\frac{a_1}{[a_{12}^{1n}]},\ldots,\frac{a_m}{[a_{m2}^{mn}]}\right) = \frac{g(a_1^n)}{[a_{12}^{1n},\ldots a_{m2}^{mn}]} \quad ,$$

In the definition of F, if $l = k(m-1)+1$, then l-ary hyperoperation h given by

$$h(x_1^{k(m-1)+1}) = \underbrace{f(f(\cdots(f(f}_{k}(x_1^m),x_{m+1}^{2m-1}),\cdots),x_{(k-1)(m-1)+2}^{k(m-1)+1})$$

will be denoted by $f_{(k)}$. It is not difficult to see that F and G are well-defined.

Theorem 19. *If (R, f, g) is any commutative (m, n)-semihyperring with at least one cancellable element, then $(S^{-1}R, F, G)$ is an (m, n)-semihyperring of quotients for R with respect to S.*

Proof. The proof is straightforward. □

Now, we say that a commutative (m, n)-semihyperring (R, f, g) has a multiplicative identity element e, if $g(x, \overset{(n-1)}{e}) = x$ for all $x \in R$. We call (R, f, g) is a unitary commutative (m, n)-semihyperring.

Example 9. Let $R = Z_2$ and for all $x, y, z \in R$ we define a ternary hyperoperation $f(x, y, z) = R$ and a ternary operation $g(x, y, z) = x + y + z + 1$ then every element is cancellable but (R, f, g) has not a multiplicative identity element.

In this section (R, f, g, e) is a unitary commutative (m, n)-semihyperring with a multiplicative identity element e.

Lemma 20. *In every (m, n)-semihyperring (R, f, g, e) we have $S \neq \emptyset$.*

Proof. If $g(x, \overset{(n-1)}{e}) = g(y, \overset{(n-1)}{e})$ then $x = y$. So $e \in S$. □

Let (R, f, g, e) be a (m, n)-semihyperring. The map $\varphi_e : R \longrightarrow S^{-1}R$ given by $\varphi_e = \frac{a}{[\overset{(n-1)}{e}]}$ is a one to one homomorphism.

Theorem 21. *Let (R, f, g, e) and (R', f', g', e') be two (m, n)-semihyperrings. Let S be the set of all cancellable elements of R and let $\alpha : R \longrightarrow R'$ be a homomorphism of (m, n)-semihyperrings such that $\alpha(s)$ is a cancellabe element of R' for all $s \in S$ and $\varphi(e) = e'$. Then, α induces a homomorphism $\overline{\alpha} : S^{-1}R \longrightarrow \alpha(S)^{-1}R'$ such that $\overline{\alpha}\varphi_e = \varphi_{e'}\alpha$.*

Proof. We can verify that the map $\overline{\alpha} : S^{-1}R \longrightarrow \alpha(S)^{-1}R'$ given by

$$\overline{\alpha}(\frac{a_1}{[a_2, \ldots, a_n]}) = \frac{\alpha(a_1)}{[\alpha(a_2), \ldots, \alpha(a_n)]}$$

is a well-defined homomorphism of (m, n)-semihyperrings such that

$$\begin{aligned} \overline{\alpha}\varphi_e(a) &= \overline{\alpha}(\frac{a}{[\overset{(n-1)}{e}]}) \\ &= \frac{\alpha(a)}{[\overset{(n-1)}{e'}]} \\ &= \varphi_{e'}\alpha(a). \end{aligned}$$

□

Lemma 22. *Let I be a hyperideal of R, then the set*

$$S^{-1}I = \{\frac{a}{[a_2, \ldots, a_n]} \mid a \in I, a_2^n \in S\}$$

is a hyperideal of $S^{-1}R$.

Proof. The proof is straightforward. □

Lemma 23. *Let* I, J, I_1^m, J_1^n *be hyperideals of* R. *Then,*

1. $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$,
2. $S^{-1}f(I_1, \ldots, I_m) = F(S^{-1}I_1, \ldots, S^{-1}I_m)$,
3. $S^{-1}g(J_1, \ldots, J_n) = G(S^{-1}J_1, \ldots, S^{-1}J_n)$.

Proof. The proof is is straightforward. □

Theorem 24. *Let* (R, f, g, e) *be an* (m,n)*-semihyperring and* I *be a hyperideal of* R. *Then,* $S \cap I \neq \emptyset$ *if and only if* $S^{-1}I = S^{-1}R$.

Proof. If $u \in S \cap I$, then $\frac{e}{[\overset{(n-1)}{e}]} = \frac{g(\overset{(n)}{e})}{[\overset{(n-1)}{e}]} = \frac{g(e,\overset{(n-1)}{u})}{[\overset{(n-1)}{u}]} \in S^{-1}I$. Now, for every $\frac{a_1}{[a_2,{}^n]} \in S^{-1}R$ we have

$$\frac{a_1}{[a_2^n]} = \frac{g(a_1, \overset{(n-1)}{e})}{[a_2^n]} = G\left(\frac{a_1}{[a_2^n]}, \frac{e}{[\overset{(n-1)}{e}]}, \ldots, \frac{e}{[\overset{(n-1)}{e}]}\right) \in S^{-1}I$$

and this proves $S^{-1}R \subseteq S^{-1}I$.

Conversely, suppose that $S^{-1}I = S^{-1}R$. If we consider the natural homomorphism $\varphi_e : R \longrightarrow S^{-1}R$, then $\varphi_e(e) = \frac{e}{[\overset{(n-1)}{e}]}$. On the other hand, $\varphi_e(e) \in S^{-1}R$, consequently $\varphi_e(e) \in S^{-1}I$ and so $\varphi_S(e) = \frac{ae}{[a_2^n]}$ for some $ae \in I$ and $a_2^n \in S$. Now, we have $\frac{e}{[\overset{(n-1)}{e}]} = \frac{a}{[a_2^n]}$. Thus, $g(e, a_2^n) = g(a, \overset{(n-1)}{e}) = a \in I \cap S$. Therefore, we obtain $I \cap S \neq \emptyset$. □

Theorem 25. *Let* I *be a hyperideal of* R. *Then,*

1. $I \subseteq \varphi_e^{-1}(S^{-1}I)$,
2. *if* $I = \varphi_e^{-1}(J)$ *for some hyperideal* J *in* $S^{-1}R$, *then* $S^{-1}I = J$.

Proof. (1) If $a \in I$, then $\varphi_e(a) = \frac{a}{[\overset{(n-1)}{e}]} \in S^{-1}I$. Therefore, $I \subseteq \varphi_e^{-1}(S^{-1}I)$.

(2) Since $I = \varphi_e^{-1}(J)$, every element of $S^{-1}I$ is of the form $\frac{a}{[a_2^n]}$ with $\varphi_e(a) \in J$. Thus,

$$\begin{aligned} \frac{a}{[a_2^n]} &= \frac{g(e,a,\overset{(n-2)}{e})}{[a_2^n]} \\ &= G\left(\frac{e}{[a_2^n]}, \frac{a}{[\overset{(n-1)}{e}]}, \frac{e}{[\overset{(n-1)}{e}]}, \ldots, \frac{e}{[\overset{(n-1)}{e}]}\right) \\ &= G\left(\frac{e}{[a_2^n]}, \varphi_e(a), \varphi_e(e), \ldots, \varphi_e(e)\right) \\ &\in J. \end{aligned}$$

Therefore, $S^{-1}I \subseteq J$.

Conversely, if $\frac{a}{[a_2^n]} \in J$, then

$$\varphi_e(a) = \frac{a}{[\overset{(n-1)}{e}]} = G\left(\frac{a}{[a_2^n]}, \frac{a_2}{[\overset{(n-1)}{e}]}, \ldots, \frac{a_n}{[\overset{(n-1)}{e}]}\right) \in J,$$

whence $a \in \varphi_e^{-1}(J) = I$. Thus $\frac{a}{[a_2^n]} \in S^{-1}I$ and hence $J \subseteq S^{-1}I$. □

Now, we will prove some theorems concerning a congruence relation.

Let ρ be a congruence relation on semigroup (R, g), Then, we have

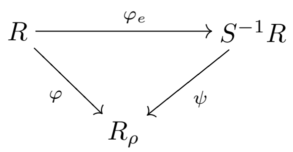
Lemma 26. *Let (R, f, g, e) be an (m, n)-semihyperring. Then, for every $a \in S$, $\rho(a)$ is cancellable in R_ρ.*

Proof. Since $e \in R$, then $\rho(e) \in R_\rho$. Now, for every $\rho(x) \in R_\rho$, we have $g_\rho(\rho(x), \rho(e), \ldots, \rho(e)) = \rho(x)$, i.e., $\rho(e)$ is a neutral element of the (m, n)-semihyperring R_ρ. On the other hand, suppose that $a_2^n \in S$ such that

$$\begin{aligned} &g_\rho(\rho(a_2), \ldots, \rho(a_i), \rho(x), \rho(a_{i+1}), \ldots, \rho(a_n)) \\ &= g_\rho(\rho(a_2), \ldots, \rho(a_i), \rho(y), \rho(a_{i+1}), \ldots, \rho(a_n)). \end{aligned}$$

Then, $\rho(g(a_2^i, x, a_{i+1}^n)) = \rho(g(a_2^i, y, a_{i+1}^n))$ or $g(a_2^i, x, a_{i+1}^n)\ \rho\ g(a_2^i, y, a_{i+1}^n)$. Since $e\rho e$ and ρ is a congruence, $g(g(a_2^i, x, a_{i+1}^n), e, \ldots, e) = g(g(a_2^i, y, a_{i+1}^n), e, \ldots, e)$. Thus, $g(a_2^i, x, a_{i+1}^n) = g(a_2^i, y, a_{i+1}^n)$ which implies that $x = y$. □

Theorem 27. *There exists a homomorphism $\psi : S^{-1}R \longrightarrow R_\rho$ such that $\psi\varphi_e = \varphi$, i.e., the following diagram is commutative.*



Proof. We define $\psi : S^{-1}R \longrightarrow R_\rho$ by setting $\psi\left(\frac{a_1}{[a_2^n]}\right) = g_\rho(\rho(a_1), \ldots, \rho(a_n)) = \rho(g(a_1^n))$. First, we show that ψ is well-defined. If $\frac{a_1}{[a_2^n]} = \frac{b_1}{[b_2^n]}$, then $g(a_1^n) = g(b_1^n)$ and so $g_\rho(\rho(a_1), \ldots, \rho(a_n)) = g_\rho(\rho(b_1), \ldots, \rho(b_n))$. Thus, ψ is well-defined. A routine calculation shows that ψ is a homomorphism. Finally, we have

$$\begin{aligned} \psi\varphi_e(a) &= \psi\left(\frac{a}{[\overset{(n-1)}{e}]}\right) = g_\rho(\rho(g(a, \overset{(n-1)}{e})), \rho(e), \ldots, \rho(e)) \\ &= \rho(g(g(a, \overset{(n-1)}{e}), \overset{(n-1)}{e})) = \rho(a) = \varphi(a). \end{aligned}$$

□

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