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THE ELECTRON FRACTION AND THE FERMI ENERGY OF RELATIVISTIC ELECTRONS IN A NEUTRON STAR

Zhi Fu Gao[1,2], X. D. Li[3], N. Wang[1], and Q. H. Peng[3]
[1]Xinjiang Astronomical Observatory, CAS, 150, Science 1-Street, Urumqi, Xinjiang, 830011, China
[2]Key Laboratory of Radio Astronomy, CAS, Nanjing, Jiangshu, 210008, China
[3]Shchool of Astronomy and Space Science, Nanjing University, 22, Hankou Road, Nanjing, Jiangshu, 210093, China

E-mail: zhifugao@xao.ac.cn; na.wang@xao.ac.cn
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ABSTRACT

We first deduce a uniform formula forthe Fermi energy of degenerate and relativistic electrons in the weak-magnetic field approximation. Then we obtain an expression of the special solution for the electron Fermi energy through this formula, and express the electron Fermi energy as a function of electron fraction and matter density. Our method is universally suitable for relativistic electron- matter regions in neutron stars in the weak-magnetic field approximation.

Key words: neutron star: weak-magnetic field approximation: generate and relativistic electrons: electron Fermi energy

1. INTRODUCTION

Pulsars are commonly recognized as magnetized neutron stars (NSs), but sometimes have been argued to be quark stars (e.g., Xu 2007; Lai et al. 2013). For completely degenerate ($T \to 0$, i.e., $\mu/kT \to \infty$, and relativistic electrons in equilibrium, the distribution function $f(E_e)$ can be expressed as

$$f(E_e) = \frac{1}{Exp[(E_e - \mu_e)/kT] + 1} \quad , \tag{1}$$

where the sign + refers to Fermi-Dirac statistics, k represents Boltzmann's constant; and μ_e is the electron chemical potential, also called the electron Fermi energy, $E_{\rm F}(e)$); when $E_e \leq E_{\rm F}(e), f(E_e) = 1$; when $E_e > E_{\rm F}(e)$, $f(E_e) = 0$. The electron fraction is the average electron number per baryon, and defined as $Y_e = n_e/n_B$, where n_e and n_B are electron number density and baryon number density, respectively. The electron fraction and the Fermi energy of relativistic electrons in NSs are two important physical parameters directly influencing weak-interaction processes including MURCA reactions, electron capture (e.g., Gao et al., 2011a, 2011b; Liu, 2013; Wang et al., 2012) and so forth. This influence will change the intrinsic equations of states, interior structure and heat evolution of a NS, and even affect large scale properties of the star. The effects of a superhigh magnetic field $B^* \gg 1$ ($B^* = B/B_{\rm cr}$ and $B_{\rm cr} = 4.414 \times 10^{13}$ G is the electron critical field) on the equilibrium composition of a NS have been shown in detail in our previous studies (e.g., Gao et al., 2012a, 2012b, 2013, 2014). It is well known that determining the Fermi energy of electrons in different matter-density regions in a NS is very complicated. Here we will present a simple and reliable way to get values for relativistic electrons in a NS in the "weak-magnetic field approximation" $B^* \ll 1$.

The remainder of this paper is organized as follows: in Section 2, we will deduce a general formula for the electron Fermi energy, $E_F(e)$, which is suitable for relativistic matter regions ($\rho \geq 10^7$ g cm^{-3}) in the "weak-magnetic field approximation", and will present a special solution of $E_F(e)$; in Section 3, we will simulate Y_e and $E_F(e)$ in the interior of a NS, and finally we give a summary in Section 5.

2. DEDUCTIONS OF FORMULAE

According to statistical physics, the microscopic state number in a 6-dimension phase-space element $dxdydzdp_xdp_ydp_z$ is $dxdydzdp_xdp_ydp_z/h^3$, where h is Plank's constant. The electron Fermi energy $E_{\rm F}(e)$ has the simple form

$$E_{\rm F}^2(e) = p_{\rm F}^2(e)c^2 + m_e^2c^4 \quad , \tag{2}$$

with $p_{\rm F}(e)$ being the electron Fermi momentum. In the "weak-magnetic field approximation", the microscopic state number of electrons, $N_{\rm pha}$, can be calculated by

$$N_{pha} = n_e = \frac{g}{h^3}\int_0^{p_{\rm F}(e)} 4\pi p^2 dp = \frac{8\pi}{3h^3}p_{\rm F}^3(e). \tag{3}$$

For the convenience of calculations, we introduce a dimensionless momentum of electrons, $x_e = p_{\rm F}(e)/m_ec$. According to Pauli's exclusion principle, electron num-

ber density n_e is equal to its energy state density,

$$n_e = N_{pha} = \frac{1}{3\pi^2\lambda_e^3} x_e^3 \ , \quad (4)$$

where $\lambda_e/m_e c$ is the Compton wavelength of an electron. The average mass of a baryon, m_B is defined as

$$m_B \equiv \frac{1}{n}\sum_i n_i m_i = \frac{\sum_i n_i m_i}{\sum_i n_i A_i}, \quad (5)$$

with $A_{\rm i}$ the baryon number of species i. In the interior of a NS, the relation of $m_B \equiv m_u \equiv 1.6606\times10^{-24}$ g always holds, where m_u is the mass of an atom. Thus, the matter density, ρ, can be expressed as:

$$\rho = n_B m_B = \frac{n_e m_B}{Y_e} \ , \quad (6)$$

Combining Eq.(4) and Eq.(6), we get

$$x_e = (\frac{3\pi^2\lambda_e^3}{m_u} Y_e \rho)^{\frac{1}{3}} \ . \quad (7)$$

Inserting $\lambda_e = 3.8614 \times 10^{-11}$ cm and $m_B \equiv 1.6606 \times 10^{-24}$ g into Eq.(7), we get

$$x_e = 1.0088 \times 10^{-2} (Y_e \rho)^{\frac{1}{3}} \ , \quad (8)$$

where ρ in units of g cm^3. Utilizing the relation of $\mu_e = \frac{m_B}{m_u Y_e} = \frac{1}{Y_e}$, Eq.(6) is rewritten as

$$\rho = \mu_e m_u n_e = 0.97395 \times 10^6 \frac{x_e^3}{Y_e}, \quad (9)$$

in units of g cm^{-3}, where μ_e is the average molecular weight of electrons. Combining Eq.(2) with Eqs.(8) and (9), we obtain a general formula for the electron Fermi energy, $E_F(e)$,

$$E_{\rm F}(e) = [1 + 1.018 \times 10^{-4} (\rho Y_e)^{\frac{2}{3}}]^{\frac{1}{2}} \times 0.511 \text{ MeV}. \quad (10)$$

The formula above is be approximately suitable for relativistic matter regions ($\rho \geq 10^7$ g cm^{-3}) over the whole interior of a NS in the "weak-magnetic field approximation". In the case of 0.5 $\rho_0 \leq \rho \leq 2\rho_0$, electrons are relativistic, and neutrons and protons are non-relativistic. Shapiro & Teukolsky (1983) gave an expressions for Y_e and $E_F(e)$ as follows:

$$Y_e \approx \frac{n_e}{n_n} \approx 0.005647 \times (\frac{\rho}{\rho_0}). \quad (11)$$

$$E_{\rm F}(e) = 60 \times (\frac{\rho}{\rho_0})^{2/3} \quad \text{(MeV)}. \quad (12)$$

Now, we give a simple proof for Eq.(12) using Eq.(10). Inserting Eq.(11) into Eq.(10) yields

$$\begin{aligned} E_{\rm F}(e) &= [1 + 1.018 \times 10^{-4} (\rho \times 0.005647 \\ &\times (\frac{\rho}{\rho_0}))^{2/3}]^{1/2} \times 0.511 \ \text{MeV} \\ &= [1 + 1.018 \times 10^{-4} (2.8 \times 10^{14} \times 0.005647 \\ &\times (\frac{\rho}{\rho_0}))^{4/3}]^{1/2} \times 0.511 \ \text{MeV} \\ &= [1 + 1.018 \times 10^{-4} (1.58116 \times 10^{12})^{2/3}]^{1/2} \\ &\times (\frac{\rho}{\rho_0})^{2/3} \times 0.511 \text{ MeV} = 60 \times (\frac{\rho}{\rho_0})^{2/3} \text{ MeV} \end{aligned} \quad (13)$$

Table 1
Values of Y_e, ρ_m, and $E_{\rm F}(e)$ in BPS model below neutron drip.

Nuclei	Y_e	$\rho_m^\dagger$ (g cm^{-3})	$\frac{\Delta\rho}{\rho}$ (%)	$E_F(e)$ (MeV)	$E_F^*(e)$ (MeV)
$^{56}_{26}Fe$	0.4643	8.1×10^6	2.9	0.95	0.96
$^{62}_{28}Ni$	0.4516	2.7×10^8	3.1	2.60	2.61
$^{64}_{28}Ni$	0.4375	1.2×10^9	7.9	4.20	4.19
$^{84}_{34}Se$	0.4048	8.2×10^9	3.5	7.70	7.71
$^{82}_{32}Ge$	0.3902	2.2×10^{10}	3.8	10.60	10.57
$^{80}_{30}Zn$	0.3750	5.91×10^{10}	4.1	13.60	13.52
$^{78}_{28}Ni$	0.3590	8.21×10^{10}	4.6	20.0	19.90
$^{76}_{26}Fe$	0.3421	1.8×10^{11}	2.2	20.20	20.37
$^{124}_{42}Mo$	0.3387	1.9×10^{11}	3.1	20.50	20.67
$^{122}_{40}Zr$	0.3279	2.7×10^{11}	3.3	22.90	22.98
$^{120}_{38}Sr$	0.3167	3.7×10^{11}	3.5	25.2	25.23
$^{118}_{36}Kr$	0.3051	4.3×10^{11}	− − −	26.20	26.20

Apart from column 6, all the data in this Table are cited from Baym, Pethick & Sutherland, 1971.
* Calculated by our Fermi energy special solution of Eq.(13).
† The maximum equilibrium density at which the nuclide is present.

The above proof indicates that Eq.(10) is a good approximate expression. If we insert Eq.(11) into Eq.(12), we can derive a special solution of the Fermi energy of relativistic electrons in the "weak-magnetic field approximation", a simple deduction is given as follows:

$$\begin{aligned} E_{\rm F}(e) &= 60 \times (\frac{\rho}{\rho_0})^{1/3} (\frac{\rho}{\rho_0})^{1/3} \\ &= 60 \times (\frac{\rho}{\rho_0})^{1/3} (\frac{\frac{\rho_0 Y_e}{0.005647}}{\rho_0})^{1/3} \\ &= 60 \times (\frac{\rho}{\rho_0})^{1/3} (\frac{Y_e}{0.005647})^{1/3} \quad \text{(MeV)}. \end{aligned} \quad (14)$$

It is worth emphasizing that, unlike the general expression of $E_F(e)$ (see Eq.(10)), there could be several special solutions of Fermi energy of relativistic electrons in the "weak-magnetic field approximation" because there are several (or more) known and accepted values of $E_F(e)$ (e.g., Baym, Pethick & Sutherland, 1971; Baym, Bethe & Pethick, 1971; Canuto, 1974; Haensel, Potekhin & Yakovlev, 2007). In this paper, we introduce a special solution of $E_F(e)$ for the convenience of calculations and numerical simulations in the later subsections.

2.1. Applications of Special Solution

Considering shell effects on the binding energy of a given nucleus, Salpeter (1961) first calculated the composition and EOS in the region of $10^7 - 3.4\times10^{11}$ g cm^{-3} . By introducing the lattice energy, Baym, Pethick and Sutherland (hereafter BPS) improved on Salpeter's treatment, and described the nuclear composition and EOS for catalyzed matter in complete thermodynamic equilibrium below the neutron drop $\rho_d \sim 4.4\times10^{11}$ g cm^{-3}. According to the BPS model, the total matter energy density ε and the total matter pressure P are given by

$$\varepsilon = \varepsilon_N + \varepsilon_e + \varepsilon_L,$$

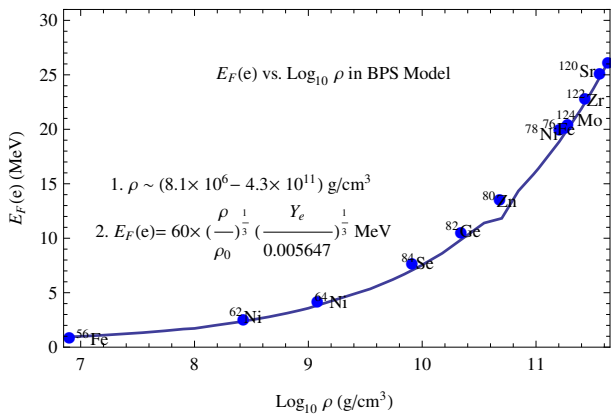

Figure 1. Numerically simulating Y_e in BPS model.

$$P = P_e + P_L = P_e + \frac{1}{3}\varepsilon_L, \tag{15}$$

respectively, where ε_N is the mass-energy of nuclei in a unit volume, ε_e the free electron energy including the rest mass of electrons in a unit volume, ε_L is the *bcc* Coulomb lattice energy in a unit volume, P_e the electron pressure, and P_L the lattice pressure, respectively. The value of electron Fermi energy $E_F(e)$ is obtained by solving the following differential equation:

$$E_{\rm F}(e) = \mu_e = \frac{\partial \varepsilon_e}{\partial n_e} = \frac{\partial}{\partial n_e}(n_e E_e) \quad , \tag{16}$$

where E_e is the free energy of an electron. For the specific equilibrium nuclei (A, Z) in BPS model, the values of quantities Y_e, ρ_m, and $E_{\rm F}(e)$ are tabulated in Table 1.

Note that, for a given nucleus with proton number Z and nucleon number A, the relation of $Y_e = Y_p = Z/A$ always approximately holds in the BPS model, where Y_p is the proton fraction. From Table 1, it is obvious that the electron fraction Y_e decreases with matter density ρ, and the decrease in Y_e is caused by an increase in the ratio of Z/A in nuclei in the BPS model. In order to compare the method of calculating $E_F(e)$ in the BPS model with our Fermi energy solution, we add column 6 in Table 1. Our calculation results are almost consistent with those of the BPS model, which can be seen by comparing the data of column 5 with the data of column 6 in the Table. Based on this, we obtain an analytical expression for the electron fraction Y_e and matter density ρ in the BPS model by numerically simulation, as shown in Figure 1.

$$Y_{\rm e} = 0.464 - 2.46\times 10^{-6} e^{Log_{10}\rho} + 9.2\times 10^{-12} e^{2Log_{10}\rho}. \tag{17}$$

In Figure 1, circles denote the values of Y_e and ρ of each individual nuclide in BPS model. Inserting Eq.(16) into the special solution of Eq.(13), we can calculate the value of $E_F(e)$ for any given matter density in the BPS model. Combining Table 1 with Eq.(13) and Eq.(16), we plot the diagram of $E_F(e)$ as a continuous function of matter density ρ in the BPS model, as shown in Figure 2. In Figure 2, circles denote the values of Y_e and ρ

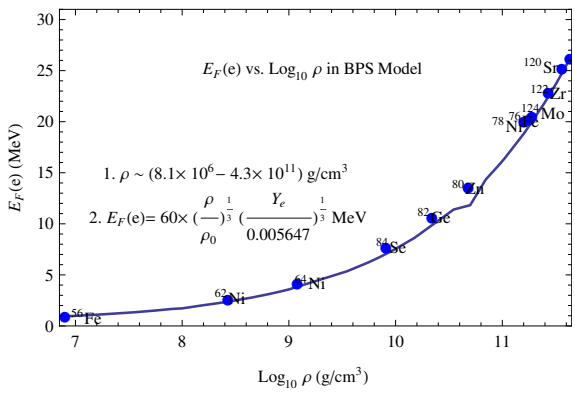

Figure 2. Numerically simulating $E_F(e)$ in BPS model.

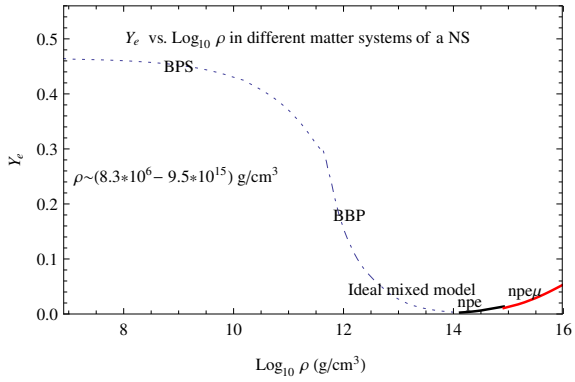

Figure 3. Numerically simulating the relation of Y_e and ρ for relativistic matter regions in the whole interior of a NS in the "weak-magnetic field approximation". Here "BBP"–(Baym, Bethe & Pethick 1971); "ST-83"–Shapiro & Teukolsky 1983.

of each individual nuclide in BPS model, circles denote the values of $E_F(e)$ and ρ of each individual nuclide in the BPS model. From Figure 2, it is obvious that the electron Fermi energy $E_F(e)$ decreases with matter density ρ, and the increase in $E_F(e)$ is caused by an increase in n_e in the BPS model.

In order to obtain the value of the Fermi energy of electrons given any density in the interior of a NS, we obtain a one to one relationship between the electron fraction and matter density at different depths in a NS by using reliable equations of state (EOSs) and numerical simulations, as shown in Figure 3.

When numerically simulating Y_e and ρ in Figure 3, we chose a BPS model for the outer crust region ($\rho \sim(8.1\times10^6 - 4.3\times10^{11})$ g cm^{-3}), the BBP (Baym, Bethe & Pethick 1971) model for the lower density region of the inner crust ($\rho \sim(4.3\times10^{11} - 4.54\times 10^{12})$ g cm^{-3}), an ideal mixed model [1] for the higher density region of the inner crust ($\rho \sim(4.54\times10^{12} - 1.32\times10^{14})$ g cm^{-3}), the ST-83 model [2] (Shapiro & Teukolsky, 1983) for the outer core region ($\rho \sim(1.32\times10^6 - 8.2\times10^{14})$ g cm^{-3}), and the "$n - p - e - \mu$" model [3] for the inner core region ($\rho \sim(8.2\times10^6 - 9.5\times10^{15})$ g cm^{-3}). In ad-

[1] The matter is composed of nuclei and neutron gas
[2] The matter is composed of neutrons, protons and electrons
[3] The matter is composed of neutrons, protons, electrons and muons

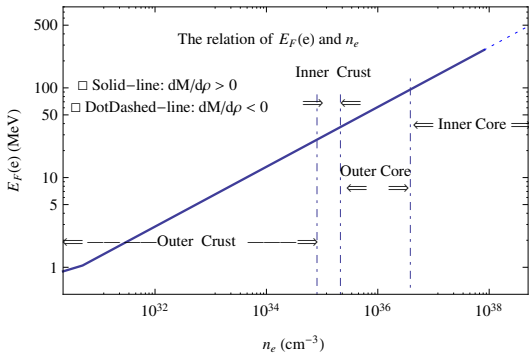

Figure 4. The relation of $E_F(e)$ and n_e for relativistic matter regions ($\rho \gg 10^7$ g cm^{-3}) in the whole interior of a NS in the "weak-magnetic field approximation".

dition, inserting Eq.(4) into Eq.(2) gives the relation of the electron Fermi energy $E_F(e)$ and electron number density n_e for relativistic matter regions over the whole interior of a NS in the "weak-magnetic field approximation",

$$\begin{aligned} E_{\rm F}(e) &= m_e c^2 (1 + x_e^2)^{1/2} \approx m_e c^2 x_e \\ &= m_e c^2 (n_e 3\pi^2 \lambda_e^3)^{1/3} \\ &= 6.121 \times 10^{-11} n_e^{1/3} \quad (\text{MeV}), \end{aligned} \tag{18}$$

where the approximate relations of $\rho \gg 10^7$ g cm^{-3} and $x_e \gg 1$ are used. We also numerically simulate the relation of $E_F(e)$ and n_e for relativistic matter regions ($\rho \gg 10^7$ g cm^{-3}) over the whole interior of a NS in the "weak-magnetic field approximation".

Since the electron Fermi energy $E_{\rm F}(e)$ increases with the electron number density n_e, and n_e increases with matter density ρ, $E_{\rm F}(e)$ also increases with matter density ρ over the whole interior of a NS. In particular, if the value of Y_e in the inner core of a young NS exceeds 1/9, the threshold Y_e of direct URCA reactions (DURCA), then DURCA would occur, and strong neutrino emissions would result in the quick cooling of a NS. Thus, the results of this paper will provide some reference values for the future study of the thermal evolution inside a NS.

3. SUMMARY

In this paper, we deduced a uniform formula for the Fermi energy of relativistic electrons in the weak-magnetic field approximation, and gave a special solution of electron Fermi energy. Compared with previous studies of the electron Fermi energy in other models, our method of calculating $E_F(e)$ is simpler more convenient, and can be universally suitable for the relativistic electron matter-region in neutron stars in the weak-magnetic field approximation.

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