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ERROR ESTIMATES FOR FULLY DISCRETE MIXED DISCONTINUOUS GALERKIN APPROXIMATIONS FOR PARABOLIC PROBLEMS

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Abstract. In this paper, we introduce fully discrete mixed discontinuous Galerkin approximations for parabolic problems. And we analyze the error estimates in $l^\infty(L^2)$ norm for the primary variable and the error estimates in the energy norm for the primary variable and the flux variable.

1. Introduction

To approximate the solution of elliptic or parabolic problems, many authors [1, 6, 20] introduced discontinuous Galerkin methods with interior penalties which generalized the Nitsche method in [11]. Because of its advantages such as the mesh adaptivity and the local mass conservativeness, the discontinuous Galerkin methods which have now a lot of forms and names are widely used for many partial differential problems. We refer to [2, 3] and the literatures cited therein, for more details.

Riviere and Wheeler [19] introduced semidiscrete and fully discrete locally conservative discontinuous Galerkin methods for nonlinear parabolic equations and they obtained optimal error estimates in $L^2(H^1)$ and suboptimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and obtained optimal error estimates in $\ell^2(H^1)$ and suboptimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations. Ohm et. al [12, 13] obtained optimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and optimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations, which improved the results of Riviere and Wheeler [19]. And Ohm et. al [14] introduced fully discrete discontinuous Galerkin method for nonlinear parabolic equations based on Crank-Nicolson method for time stepping and obtained optimal error estimates in $\ell^\infty(L^2)$ for both spatial and temporal directions.

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To approximate both primary variable and its flux variable simultaneously, Raviart and Thomas [18] and Nedelec [10] introduced mixed finite element methods with the inf-sup conditions. These mixed finite element methods were widely used for elliptic or parabolic problems [5, 7, 9]. And Pani [16] introduced H^1-Galerkin mixed finite element method without inf-sup conditions for parabolic problems. Applications of these H^1-Galerkin mixed finite element method were given in [8] for the Sobolev equation and in [17] for parabolic integro-differential equations.

Chen [3] introduced a family of mixed discontinuous finite element methods for second-order elliptic equations. Chen and Chen [4] developed a theory for stability and convergence for mixed discontinuous finite element methods in a general form for second-order partial differential problems. Ohm et. al [15] introduced a semidiscrete mixed discontinuous Galerkin method with an interior penalty to approximate the solution of parabolic problems and obtained optimal error estimates in $L^\infty(L^2)$ for the primary variable u, optimal error estimates in $L^2(L^2)$ for u_t and suboptimal error estimates in $L^\infty(\boldsymbol{L}^2)$ for the flux variable $\boldsymbol{\sigma}$.

In this paper, we consider fully discrete mixed discontinuous Galerkin approximations for parabolic problems and obtain error estimates for the primary variable and its flux variable. In section 2, we introduce a model problem, finite element spaces, and the mixed formulation with an interior penalty for the model problem. In section 3, we state some projections with approximation properties which will be used later. And in section 4, we obtain the error estimates in $l^\infty(L^2)$ norm for the primary variable and the error estimates in the energy norm for the primary variable and the flux variable.

2. Model problems and finite element spaces

We consider the following parabolic problem

$$u_t - \nabla \cdot (a\nabla u) = f, \quad \text{in} \ \ \Omega \times (0,T], \tag{2.1}$$

$$u = g_D, \quad \text{on} \ \ \Gamma_D \times (0,T], \tag{2.2}$$

$$a\nabla u \cdot \boldsymbol{n} = g_N, \quad \text{on} \ \ \Gamma_N \times (0,T], \tag{2.3}$$

$$u(x,0) = u_0(x), \quad \text{in} \ \ \Omega. \tag{2.4}$$

Here $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, is an open bounded convex domain with the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \phi$ and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. We assume that a is a symmetric, positive definite bounded tensor and the given functions $f \in L^2(\Omega)$, $u^0 \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$.

By introducing $\boldsymbol{\sigma} = a(x)\nabla u$ in (2.1), we get the following mixed form

$$u_t - \nabla \cdot \sigma = f, \quad \text{in } \Omega \times (0,T], \tag{2.5}$$

$$\boldsymbol{\sigma} = a\nabla u, \quad \text{in } \Omega \times (0,T], \tag{2.6}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = g_N, \quad \text{on } \Gamma_N \times (0,T], \tag{2.7}$$

$$u = g_D, \quad \text{on } \Gamma_D \times (0,T], \tag{2.8}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega. \tag{2.9}$$

To introduce the mixed discontinuous Galerkin finite element approximations of (2.5)-(2.9), let $\{T_h\}$ be a sequence of finite element partitions of Ω for $h > 0$, and each subdomain $T \in T_h$ have a Lipschitz boundary. We allow the property that a vertex of one element can lie on the edge or face of another element for the given two adjacent elements in T_h. For a given T_h, let $\mathcal{E}_h^I$ be the set of all interior boundaries e of T_h, $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$ the sets of boundaries e on Γ_D and Γ_N, respectively, $\mathcal{E}_h^B = \mathcal{E}_h^D \cup \mathcal{E}_h^N$ the set of the boundaries e on $\partial\Omega$, $\mathcal{E}_h^{ID} = \mathcal{E}_h^I \cup \mathcal{E}_h^D$, and $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$. For an $e \in \mathcal{E}_h^B$, we denote $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$ and for an $e \in \mathcal{E}_h^I$ with $e = T_1 \cap T_2$ and T_1, $T_2 \in T_h$, we associate the direction of $\boldsymbol{n}$ with the definition of jump across e.

For $l \geq 0$, we define

$$H^\ell(T_h) = \{v \in L^2(\Omega) : v|_T \in H^\ell(T),\ T \in T_h\}$$

together with its norm

$$\|v\|_{H^\ell(T_h)} = \Big(\sum_{T\in T_h} \|v\|^2_{H^\ell(T)}\Big)^{1/2}.$$

For $v \in H^\ell(T_h)$ with $l > \frac{1}{2}$, we define the jump of v across $e \in \mathcal{E}_h^I$ by

$$[v] = v|_{T_2\cap e} - v|_{T_1\cap e}$$

and the jump of v across $e \in \mathcal{E}_h^B$ by

$$[v] = \begin{cases} v, & e \in \mathcal{E}_h^D, \\ 0, & e \in \mathcal{E}_h^N. \end{cases}$$

And we define the average of v on $e \in \mathcal{E}_h^I$ by

$$\{v\} = \frac{1}{2}\big(v|_{T_1\cap e} + v|_{T_2\cap e}\big)$$

and the average of v on $e \in \mathcal{E}_h^B$ by

$$\{v\} = v|_e.$$

Let $V = H^1(T_h)$, $\boldsymbol{W} = \{\boldsymbol{w} \in (H^1(T_h))^d \mid \nabla \cdot \boldsymbol{w} \in L^2\}$, and $V_h \subset V$ and $\boldsymbol{W}_h \subset \boldsymbol{W}$ the finite element subspaces, respectively. Then they are defined locally on each element $T \in T_h$, so that $\boldsymbol{W}_h(T) = \boldsymbol{W}_h|_T$ and $V_h(T) = V_h|_T$. Neither continuity constraint nor boundary values are imposed on $\boldsymbol{W}_h \times V_h$.

Now the corresponding mixed formulation with an interior penalty for (2.1)-(2.4) can be defined as follows: find $u \in V$ and $\boldsymbol{\sigma} \in \boldsymbol{W}$ such that

$$(u_t, v) + \sum_{T \in T_h} (\boldsymbol{\sigma}, \nabla v)_T - \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\sigma} \cdot \boldsymbol{n}\}, [v])_e + J(u, v) = (f, v) + \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e + \sum_{e \in \mathcal{E}_h^D} h^{-1}(g_D, v)_e, \quad \forall\, v \in V, \tag{2.10}$$

$$(\alpha(x)\boldsymbol{\sigma}, \boldsymbol{\tau}) - \sum_{T \in T_h} (\nabla u, \boldsymbol{\tau})_T + \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\tau} \cdot n\}, [u])_e = \sum_{e \in \mathcal{E}_h^D} (g_D, \boldsymbol{\tau} \cdot n)_e, \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{W}, \tag{2.11}$$

where $\alpha(x) = a(x)^{-1}$, $J(u, v) = \sum_{e \in \mathcal{E}_h^{ID}} h_e^{-1} \int_e [u][v] dx$, $h_e = |e|$, $(\ ,\ )$ denote an L^2 inner product on Ω, $(\ ,\ )_T$ an L^2 inner product on T, and $(\ ,\ )_e$ an L^2 inner product on e. Notice that the solution u and $\boldsymbol{\sigma}$ of (2.5)-(2.9) satisfies the system (2.10)-(2.11).

We define bilinear forms A, B, and C as follows: for any $\boldsymbol{\tau}, \boldsymbol{r} \in \boldsymbol{W}$ and $u, v \in V$

$$A(\boldsymbol{\tau}, \boldsymbol{r}) = (\alpha \boldsymbol{\tau}, \boldsymbol{r}), \tag{2.12}$$

$$B(\boldsymbol{\tau}, v) = \sum_{T \in T_h} (\boldsymbol{\tau}, \nabla v)_T - \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\tau} \cdot \boldsymbol{n}\}, [v])_e, \tag{2.13}$$

$$C(u, v) = J(u, v) + \lambda(u, v), \tag{2.14}$$

where λ is a positive real number. And also we define linear functionals F, G_D^1, G_D^2, and G_N as follows: for any $\boldsymbol{\tau} \in \boldsymbol{W}$ and $v \in V$

$$F(v) = (f, v), \tag{2.15}$$

$$G_D^1(\boldsymbol{\tau}) = \sum_{e \in \mathcal{E}_h^D} (g_D, \boldsymbol{\tau} \cdot \boldsymbol{n})_e, \tag{2.16}$$

$$G_D^2(v) = \sum_{e \in \mathcal{E}_h^D} h^{-1}(g_D, v)_e, \tag{2.17}$$

$$G_N(v) = \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e. \tag{2.18}$$

Then, by using the bilinear forms (2.12)-(2.14) and linear functionals (2.15)-(2.18), the system (2.10)-(2.11) can be rewritten into the system

$$(u_t, v) + B(\boldsymbol{\sigma}, v) + C(u, v) - \lambda(u, v) = F(v) + G_N(v) + G_D^2(v), \quad \forall v \in V \tag{2.19}$$

$$A(\boldsymbol{\sigma}, \boldsymbol{\tau}) - B(\boldsymbol{\tau}, u) = G_D^1(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{W}. \tag{2.20}$$

Define the following broken norms on V and $\boldsymbol{W}$: for any $v \in V$ and $\boldsymbol{\tau} \in \boldsymbol{W}$

$$\|v\|_S^2 = \|v\|_1^2 + J(v,v), \tag{2.21}$$

$$\|\boldsymbol{\tau}\|_{\boldsymbol{W}}^2 = \|\boldsymbol{\tau}\|^2 + \sum_{T \in T_h} h_T^2 \|\nabla \cdot \boldsymbol{\tau}\|_T^2, \tag{2.22}$$

$$\|v\|_C^2 = J(v,v) + \lambda \|v\|^2, \tag{2.23}$$

$$\|\boldsymbol{\tau}\|_A^2 = A(\boldsymbol{\tau},\boldsymbol{\tau}), \tag{2.24}$$

where $\| \, \|_1$ denotes H^1 norm on $\boldsymbol{W}$ and $\| \, \|$ denotes L^2 norm on V or $\boldsymbol{W}$.

3. Auxiliary projections and some estimates

For a given $(u, \boldsymbol{\sigma}) \in V \times \boldsymbol{W}$, we can define $(\tilde{u}, \tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ such that

$$B(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}, v) + C(u - \widetilde{u}, v) = 0, \qquad \forall v \in V_h, \tag{3.1}$$

$$A(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) - B(\boldsymbol{\tau}, u - \widetilde{u}) = 0, \qquad \forall \boldsymbol{\tau} \in \boldsymbol{W}_h. \tag{3.2}$$

The unique existence of $(\tilde{u}, \tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ follows from the following Lemmas 3.1 and 3.2 whose proofs can be found in [15].

Lemma 3.1. *For any $u, v \in V$ and any $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{W}$, the followings hold:*
(1) $A(\boldsymbol{\sigma},\boldsymbol{\tau}) \le K\|\boldsymbol{\sigma}\|_A \|\boldsymbol{\tau}\|_A$, $\quad A(\boldsymbol{\sigma},\boldsymbol{\tau}) \le K\|\boldsymbol{\sigma}\|_{\boldsymbol{W}} \|\boldsymbol{\tau}\|_{\boldsymbol{W}}$;
(2) $B(\boldsymbol{\sigma}, v) \le K\|\boldsymbol{\sigma}\|_{\boldsymbol{W}} \|v\|_S$;
(3) $C(u, v) \le K\|u\|_S \|v\|_S$.

Lemma 3.2. *For any $v \in V$ and any $\boldsymbol{\tau} \in \boldsymbol{W}_h$, the followings hold:*
(1) $A(\boldsymbol{\tau},\boldsymbol{\tau}) \ge K\|\boldsymbol{\tau}\|_{\boldsymbol{W}}^2$;
(2) $C(v,v) \ge K\|v\|_C^2$.

Lemma 3.3. *For any $(u, \boldsymbol{\sigma}), (u_t, \boldsymbol{\sigma}_t) \in V \times \boldsymbol{W}$, we have*

$$\|u - \widetilde{u}\|_C + \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A \le Kh^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}),$$

$$\|u_t - \widetilde{u}_t\|_C + \|\boldsymbol{\sigma}_t - \widetilde{\boldsymbol{\sigma}}_t\|_A \le Kh^k\big(\|u_t\|_{k+1} + \|\boldsymbol{\sigma}_t\|_{k+1}\big),$$

$$\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_{\boldsymbol{W}} \le Kh^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}),$$

$$\|\boldsymbol{\sigma}_t - \widetilde{\boldsymbol{\sigma}}_t\|_{\boldsymbol{W}} \le Kh^k\big(\|u_t\|_{k+1} + \|\boldsymbol{\sigma}_t\|_{k+1}\big),$$

where $(\tilde{u}, \tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ is given in (3.1) and (3.2).

Proof. The proofs of these results can be found in Lemma 3.1 - Lemma 3.3 of [15]. □

Lemma 3.4. *For any $(u,\boldsymbol{\sigma}), (u_t,\boldsymbol{\sigma}_t) \in V \times \boldsymbol{W}$, we have*

$$\|u^n - \widetilde{u}^n\| \le Kh^{k+1}(\|u^n\|_{k+1} + \|\boldsymbol{\sigma}^n\|_{k+1}),$$
$$\|u_t^n - \widetilde{u}_t^n\| \le Kh^{k+1}(\|u_t^n\|_{k+1} + \|\boldsymbol{\sigma}_t^n\|_{k+1}),$$
$$\|u_{tt}^n - \widetilde{u}_{tt}^n\| \le Kh^{k+1}(\|u_{tt}^n\|_{k+1} + \|\boldsymbol{\sigma}_{tt}^n\|_{k+1}),$$

where $(\tilde{u},\tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ are given in (3.1) and (3.2).

Proof. The proofs of first two results can be found in Lemma 3.4 of [15]. The proof of the last result is very similar to ones of first two results. □

4. Error Estimates

In this section, we want to construct the fully discrete mixed discontinuous Galerkin approximations of (2.1)-(2.4). For the given positive integer N, let $\triangle t = \frac{T}{N}$, $t^j = j\triangle t$ for $j = 0, 1, 2, \cdots, N$. Then the fully discrete mixed discontinuous Galerkin approximations $\{U^j\}_{j=0}^N \subset V_h, \{\boldsymbol{\Sigma}^j\}_{j=1}^N \subset \boldsymbol{W}_h$ of (2.1)-(2.4) are defined as follows: for $j = 1, 2, \cdots, N$

$$(4.1)\qquad \Big(\frac{U^j - U^{j-1}}{\triangle t}, v_h\Big) + B(\boldsymbol{\Sigma}^j, v_h) + C(U^j, v_h) - \lambda(U^j, v_h) = F(v_h) + G_N(v_h) + G_D^2(v_h), \quad \forall\, v_h \in V_h,$$

$$(4.2)\qquad A(\boldsymbol{\Sigma}^j, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, U^j) = G_D^1(\boldsymbol{\tau}_h), \quad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{W}_h$$

and $U^0(x)$ is an appropriate projection of $u_0(x)$.

Theorem 4.1. *If $(u,\boldsymbol{\sigma}) \in V \times \boldsymbol{W}$ is the solution of* (2.5)-(2.9) *and $\{U^j\}_{j=0}^N \subset V_h$ and $\{\boldsymbol{\Sigma}^j\}_{j=1}^N \subset \boldsymbol{W}_h$ are the solution of* (4.1)-(4.2), *then*

$$\|u^j - U^j\|^2 \le K\big(h^{2(k+1)} + (\triangle t)^2\big), \quad j = 0, 1, 2, \cdots, N,$$

and

$$\triangle t \sum_{j=1}^{N} \big(|\!|\!| u^j - U^j |\!|\!|_C^2 + \|\boldsymbol{\sigma}^j - \boldsymbol{\Sigma}^j\|_A^2\big) \le K\big(h^{2k} + (\triangle t)^2\big).$$

Proof. It is clear that

$$\|\widetilde{u}^0 - U^0\| \le \|\widetilde{u}^0 - u^0\| + \|u^0 - U^0\| \le Kh^{k+1}\big(\|u^0\|_{k+1} + \|\boldsymbol{\sigma}^0\|_{k+1}\big).$$

From (4.1)-(4.2) and the fact that the solution u and $\boldsymbol{\sigma}$ of (2.5)-(2.9) satisfies the system (2.19)-(2.20), we obtain the error equation for j

$$(4.3)\qquad (u_t^j - \partial_t U^j, v_h) + B(\boldsymbol{\sigma}^j - \boldsymbol{\Sigma}^j, v_h) + C(u^j - U^j, v_h) = \lambda(u^j - U^j, v_h), \quad \forall\, v_h \in V_h,$$

$$(4.4)\qquad A(\boldsymbol{\sigma}^j - \boldsymbol{\Sigma}^j, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, u^j - U^j) = 0, \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{W}_h,$$

where $\partial_t U^j = \frac{U^j - U^{j-1}}{\triangle t}$. Then we have

$$
\begin{aligned}
(4.5)\qquad & (\partial_t \widetilde{u}^j - \partial_t U^j, v_h) + B(\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j, v_h) + C(\widetilde{u}^j - U^j, v_h) \\
& = (\partial_t \widetilde{u}^j - \widetilde{u}_t^j, v_h) + (\widetilde{u}_t^j - u_t^j, v_h) + B(\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\sigma}^j, v_h) \\
& \qquad + C(\widetilde{u}^j - u^j, v_h) + \lambda(u^j - U^j, v_h), \\
& = (\partial_t \widetilde{u}^j - \widetilde{u}_t^j, v_h) + (\widetilde{u}_t^j - u_t^j, v_h) + \lambda(u^j - U^j, v_h), \ \ \forall\, v_h \in V_h,
\end{aligned}
$$

$$
\begin{aligned}
(4.6)\qquad & A(\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, \widetilde{u}^j - U^j) \\
& = A(\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\sigma}^j, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, \widetilde{u}^j - u^j) = 0, \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{W}_h.
\end{aligned}
$$

From (4.5)-(4.6) with $v_h = \widetilde{u}^j - U^j$ and $\boldsymbol{\tau}_h = \widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j$, we get

$$
\begin{aligned}
& (\partial_t \widetilde{u}^j - \partial_t U^j, \widetilde{u}^j - U^j) + \|\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j\|_A^2 + \|\!|\widetilde{u}^j - U^j|\!\|_C^2 \\
& = (\partial_t \widetilde{u}^j - \widetilde{u}_t^j, \widetilde{u}^j - U^j) + (\widetilde{u}_t^j - u_t^j, \widetilde{u}^j - U^j) + \lambda(u^j - U^j, \widetilde{u}^j - U^j).
\end{aligned}
$$

Since

$$
\begin{aligned}
& (\partial_t \widetilde{u}^j - \partial_t U^j, \widetilde{u}^j - U^j) \\
& = \frac{1}{\triangle t}\big((\widetilde{u}^j - U^j) - (\widetilde{u}^{j-1} - U^{j-1}), \widetilde{u}^j - U^j\big) \\
& \geq \frac{1}{2\triangle t}(\|\widetilde{u}^j - U^j\|^2 - \|\widetilde{u}^{j-1} - U^{j-1}\|^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t \widetilde{u}^j - \widetilde{u}_t^j &= \frac{1}{\triangle t}\big(\widetilde{u}^j - [\widetilde{u}^j - \triangle t \widetilde{u}_t^j + \frac{1}{2}(\triangle t)^2 \widetilde{u}_{tt}^j]\big) - \widetilde{u}_t^j \\
&\cong \triangle t\, \widetilde{u}_{tt}^j,
\end{aligned}
$$

we have

$$
\begin{aligned}
(4.7)\quad & \frac{1}{2\triangle t}(\|\widetilde{u}^j - U^j\|^2 - \|\widetilde{u}^{j-1} - U^{j-1}\|^2) + \|\!|\widetilde{u}^j - U^j|\!\|_C^2 + \|\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j\|_A^2 \\
& \leq \|\partial_t \widetilde{u}^j - \widetilde{u}_t^j\|\|\widetilde{u}^j - U^j\| + \|u_t^j - \widetilde{u}_t^j\|\|\widetilde{u}^j - U^j\| \\
& \qquad\qquad + \lambda(\|u^j - \widetilde{u}^j\| + \|\widetilde{u}^j - U^j\|)\|\widetilde{u}^j - U^j\| \\
& \leq K\big(\triangle t\|\widetilde{u}_{tt}^j\| + \|u_t^j - \widetilde{u}_t^j\| + \lambda(\|u^j - \widetilde{u}^j\| + \|\widetilde{u}^j - U^j\|)\big) \times \|\widetilde{u}^j - U^j\|.
\end{aligned}
$$

And so for sufficiently small $\triangle t$, we get from (4.7)

$$
\begin{aligned}
(4.8)\quad & \|\widetilde{u}^j - U^j\|^2 - \|\widetilde{u}^{j-1} - U^{j-1}\|^2 + 2\triangle t\big(\|\!|\widetilde{u}^j - U^j|\!\|_C^2 + \|\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j\|_A^2\big) \\
& \leq K\triangle t\big[(\triangle t)^2\|\widetilde{u}_{tt}^j\|^2 + \|\widetilde{u}_t^j - u_t^j\|^2 + \lambda^2\|u^j - \widetilde{u}^j\|^2\big].
\end{aligned}
$$

Hence summing the inequality (4.8) from $j = 1$ to $j = m$, we have from Lemma 3.3

$$\begin{aligned}
&\|\widetilde{u}^m - U^m\|^2 + 2\triangle t \sum_{j=1}^{m} \big(\|\!|\widetilde{u}^j - U^j|\!\|_C^2 + \|\widetilde{\boldsymbol{\sigma}}^j - \boldsymbol{\Sigma}^j\|_A^2\big) \\
&\le \|\widetilde{u}^0 - U^0\|^2 + K\triangle t \sum_{j=1}^{m} \big[(\triangle t)^2 \|\widetilde{u}_{tt}^j\|^2 + \|\widetilde{u}_t^j - u_t^j\|^2 + \lambda^2\|u^j - \widetilde{u}^j\|^2\big] \\
&\le K\big(h^{2(k+1)} + (\triangle t)^2\big).
\end{aligned}$$

Now, by Lemma 3.3, Lemma 3.4, and the triangular inequality, we have

$$\begin{aligned}
\|u^j - U^j\|^2 &\le \ K\big(h^{2(k+1)} + (\triangle t)^2\big), \quad j = 0, 1, 2, \cdots, N, \\
\triangle t \sum_{j=1}^{N} \big(\|\!|u^j - U^j|\!\|_C^2 + \|\boldsymbol{\sigma}^j - \boldsymbol{\Sigma}^j\|_A^2\big) &\le \ K\big(h^{2k} + (\triangle t)^2\big),
\end{aligned}$$

which completes the proof. □

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