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OPTIMAL IMPACT ANGLE CONTROL GUIDANCE LAWS AGAINST A MANEUVERING TARGET

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ABSTRACT. Optimal impact angle control guidance law and its variants for intercepting a maneuvering target are introduced in this paper. The linear quadratic(LQ) optimal control theory is reviewed first to setup framework of guidance law derivation, called the sweep method. As an example, the inversely weighted time-to-go energy optimal control problem to obtain the optimal impact angle control guidance law for a fixed target is solved via the sweep method. Since this optimal guidance law is not applicable for a moving target due to the angle mismatch at the impact instant, the law is modified to three different biased proportional navigation(PN) laws: the flight path angle control law, the line-of-sight(LOS) angle control law, and the relative flight path angle control law. Effectiveness of the guidance laws are verified via numerical simulations.

1. INTRODUCTION

Impact angle control in a guidance problem has drawn a lot of attentions due to its wide applications in unmanned aerial vehicles(UAVs) and missiles. Tactics of missiles such as approaching the most vulnerable side of a ship or a tank and maximizing radar cross section(RCS) of an air target can easily be achieved by impact angle control laws. Vertical attack of a surface-to-surface missile from air to a ground target is very important to maximize the warhead effect as well as to minimize miss distance due to navigation error inherently embedded in the vertical channel of the inertial navigation system. The impact angle control laws make it possible to easier path planning of unmanned aerial vehicles(UAVs) by freely designating the passing angles of waypoints.

Kim and Grider[1] proposed an optimal guidance law to vertically guide a re-entry vehicle to a designated ground point and this is the first attempt to control flight path angle of an aerial vehicle. The rendezvous problem is a sort of impact angle control problem and Bryson and Ho[2] showed it could be solved by the optimal control theory, where the velocity component normal to the specified rendezvous course was to be nullified by the proposed guidance law. In the previous works of this author in [3, 4], optimal guidance laws with impact angle constraints have been derived based on the linear quadratic(LQ) optimal control theory. Ohlmeyer[5] also proposed an optimal impact angle control law called the generalized vector

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explicit guidance(GENEX). Here, the same cost function as that of [4] is minimized and zero effort miss(ZEM) and zero effort terminal velocity are considered as the terminal constraints. As an application of the optimal impact angle control laws for UAVs, the real-time energy optimal path planning method has been proposed by the author in [6].

LQ optimal control theory deals with the optimal control problem of minimizing a LQ cost function with linear system and linear constraints. The sweep method[2], the solution approach to Euler-Lagrange necessary conditions for LQ optimal control, provides a design framework for optimal state feedback control which includes backward integration of differential Riccati equations. Actually, the theory of LQ regulator design is an extended version of the sweep method for infinite time horizon control problems. The sweep method is well posed for deriving new guidance laws. In this paper, hence, the sweep method is systematically reviewed and the optimal guidance law proposed in [4] is derived via the sweep method as an example.

One of the problems of impact angle control laws previously studied is that they cannot be directly applied to intercept a maneuvering target. Most impact angle control laws include the flight path angle and the line-of-sight(LOS) angle(or relative position of the missile to the target) as the state variables. For a fixed target, the flight path angle and the LOS angle are the same at the impact instant. However, if the target is moving or maneuvering, both angles are not coincident with each other at the impact instant because the flight path angle is defined without consideration of target's motion while the LOS angle is defined relative to the target position. To apply the impact angle control laws to intercept a maneuvering target, they should be modified to be a biased proportional navigation(PN) which consists of the conventional PN term and the impact angle control term. In this paper, three variants of optimal impact angle control law is proposed in the sense of the flight path angle control, the LOS angle control, and the relative flight path angle control against a maneuvering target.

This paper is organized as follows. In section 2, the sweep method based on the LQ optimal control theory is reviewed. Optimal impact angle control guidance law studied in [4] is derived again based on the sweep method and the variants are introduced in Section 3. Numerical examples for understanding characteristics of the guidance laws are provided in Section 4. Finally, the concluding remarks are in Section 5.

2. Review of LQ Optimal Control - The Sweep Method

In this section, the sweep method[7-9] to provide a framework of derivation of guidance laws as the solution of LQ optimal problem is reviewed.

2.1. **Nonlinear optimal control problem.** Consider the problem of finding an optimal control, a continuously differentiable function $u : [t_0, t_f] \to \mathrm{R}$, which minimizes the cost function for a fixed t_f

$$J = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x, u, t) dt. \tag{2.1}$$

subject to the differential equations

$$\dot{x} = f(x, u, t), \quad x(t_0) = x_0 \tag{2.2}$$

and the terminal constraints

$$\psi(x(t_f), t_f) = 0. \tag{2.3}$$

Here, $x \in \mathrm{R}^n$ and $u \in \mathrm{R}^r$ denote the state vector and the control vector, respectively. And, f is the n real functions; φ and L are scalar functions; ψ is the p real functions where $p < n - 1$. Each of functions has continuous partial derivatives in x and u.

2.2. **Euler-Lagrange necessary conditions.** Hamiltonian to augment the nonlinear system equations to the integral cost with the Lagrange multipliers $\lambda \in \mathrm{R}^n$ to augment the system constraint to the integral cost is defined by

$$\mathcal{H}(x, u, \lambda, t) = L(x, u, t) + \lambda^T f(x, u, t). \tag{2.4}$$

And the endpoint equation to augment the terminal constraints to the terminal cost with the multipliers $\nu \in \mathrm{R}^p$is defined by

$$G = \varphi(x(t),\ t) + \nu^T \psi(x(t),\ t). \tag{2.5}$$

If u^* minimizes (2.1) subject to (2.2) and (2.3), u^* should satisfy the control equation

$$\frac{\partial \mathcal{H}}{\partial u}(x^*, u^*, \lambda, t) = 0 \tag{2.6}$$

where x^* is the optimal state vector. The co-state equations to define the dynamics of the Lagrange multiplier is

$$-\dot{\lambda}^T = \frac{\partial \mathcal{H}}{\partial x}(x^*, u^*, \lambda, t), \quad \lambda^T(t_f) = \frac{\partial \varphi}{\partial x}(x^*(t_f), t_f) + \nu^T \frac{\partial \psi}{\partial x}(x^*(t_f), t_f). \tag{2.7}$$

As (2.2) and (2.7) are introduced, these necessary conditions called the Euler-Lagrange equations are “weak” in the sense that there is no bound on control input and form a two-point boundary value problem.

2.3. **LQ optimal control problems.** Now consider an LQ optimal control problem: For a given t_f, find $u(t)$ which minimizes

$$J = \frac{1}{2} x(t_f)^T S_f x(t_f) + \int_{t_0}^{t_f} \left(\frac{1}{2} x^T Q x + u^T C x + \frac{1}{2} u^T R u \right) dt \tag{2.8}$$

subject to

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \tag{2.9}$$

with

$$Dx(t_f) = E \tag{2.10}$$

where $S_f \geq 0$, $Q \geq 0$, $C \geq 0$, and $R > 0$ are weighting matrices with proper dimensions.

2.4. **Euler-Lagrange equations.** By the definition, Hamiltonian of the above optimal control problem is given by

$$\mathcal{H} = \frac{1}{2}x^T Qx + u^T Cx + \frac{1}{2}u^T Qu + \lambda^T (Ax + Bu) \tag{2.11}$$

Let u^* be the optimal control. Then, from Pontryagin's minimum principle,

$$\dot{\lambda} = -\frac{\partial \mathcal{H}^T}{\partial x} = -A^T\lambda - Qx - C^T u^*, \quad \lambda(t_f) = S_f x(t_f) + D^T \nu \tag{2.12}$$

and

$$\frac{\partial \mathcal{H}^T}{\partial u} = 0 = Cx + Ru^* + B^T\lambda \tag{2.13}$$

Substituting (2.11) into (2.13), we have

$$u^* = -R^{-1}Cx - R^{-1}B^T\lambda \tag{2.14}$$

Also, substitute (2.11) into (2.12) and combine with (2.9) to obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{2.15}$$

with

$$x(t_0) = x_0 \text{ and } \lambda(t_f) = S_f x(t_f) + D^T\nu \tag{2.16}$$

where

$$H = \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^T \\ -Q + C^T R^{-1} C & -(A - BR^{-1}C)^T \end{bmatrix}. \tag{2.17}$$

Note that H is not the Hamiltonian. There is a state transition matrix Φ_H for H such that

$$\dot{\Phi}_H(t, t_0) = H(t)\Phi_H(t, t_0), \quad \Phi_H(t_0, t_0) = I \tag{2.18}$$

Since $\Phi_H(t_f, t_0) = \Phi_H(t_f, t)\Phi_H(t, t_0)$,

$$\dot{\Phi}_H(t_f, t_0) = 0 = \dot{\Phi}_H(t_f, t)\Phi_H(t, t_0) + \Phi_H(t_f, t)\dot{\Phi}_H(t, t_0) \tag{2.19}$$

Substituting (2.18) into (2.19) and rearranging, we have a backward differential equation

$$\dot{\Phi}_H(t_f, t) = -\Phi_H(t_f, t)H(t), \quad \Phi_H(t_f, t_f) = I \tag{2.20}$$

It is easy to show that Φ_H has the symplectic property so that

$$\Phi_H(t, t_0) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \Phi_H^T(t, t_0) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{2.21}$$

Let Φ_H be denoted by

$$\Phi_H(t, t_0) \triangleq \begin{bmatrix} \phi_{11}(t, t_0) & \phi_{12}(t, t_0) \\ \phi_{21}(t, t_0) & \phi_{22}(t, t_0) \end{bmatrix}. \tag{2.22}$$

Now we define

$$\bar{\Phi}_H(t,t_0) \triangleq \begin{bmatrix} \bar{\phi}_{11}(t,t_0) & \bar{\phi}_{12}(t,t_0) \\ \bar{\phi}_{21}(t,t_0) & \bar{\phi}_{22}(t,t_0) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \Phi_H(t,t_0) \\ = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) - S_f\phi_{11}(t,t_0) & \phi_{22}(t,t_0) - S_f\phi_{12}(t,t_0) \end{bmatrix} \tag{2.23}$$

Since

$$\bar{\Phi}_H(t,t_0) \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \bar{\Phi}_H^T(t,t_0) = \begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & -S_f^T \\ 0 & I \end{bmatrix} \\ = \begin{bmatrix} 0 & I \\ -I & S_f^T - S_f \end{bmatrix}, \tag{2.24}$$

$\bar{\Phi}_H$ is also symplectic if S_f is symmetric. Thus,

$$\begin{aligned} \bar{\phi}_{11}(t,t_0)\bar{\phi}_{12}^T(t,t_0) &= \bar{\phi}_{12}(t,t_0)\bar{\phi}_{11}^T(t,t_0) \\ \bar{\phi}_{21}(t,t_0)\bar{\phi}_{22}^T(t,t_0) &= \bar{\phi}_{22}(t,t_0)\bar{\phi}_{21}^T(t,t_0) \\ \bar{\phi}_{11}(t,t_0)\bar{\phi}_{22}^T(t,t_0) - \bar{\phi}_{12}(t,t_0)\bar{\phi}_{21}^T(t,t_0) &= I \end{aligned} \tag{2.25}$$

Moreover,

$$\dot{\bar{\Phi}}_H(t_f,t) = \begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \dot{\Phi}_H(t_f,t) = -\begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \Phi_H(t_f,t)H(t) \\ = -\bar{\Phi}_H(t_f,t)H(t) \tag{2.26}$$

with

$$\bar{\Phi}_H(t_f,t_f) = \begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \Phi_H(t_f,t_f) = \begin{bmatrix} I & 0 \\ -S_f & I \end{bmatrix} \tag{2.27}$$

2.5. **Determination of ν , $\lambda(t_0)$, and $x(t_f)$.** The solution of (2.15) is given by

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \Phi_H(t,t_0) \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t,t_0)x_0 + \phi_{12}(t,t_0)\lambda(t_0) \\ \phi_{21}(t,t_0)x_0 + \phi_{22}(t,t_0)\lambda(t_0) \end{bmatrix}$$

Then

$$x(t_f) = \phi_{11}(t_f,t_0)x_0 + \phi_{12}(t_f,t_0)\lambda(t_0), \tag{2.28}$$

$$\lambda(t_f) = \phi_{21}(t_f,t_0)x_0 + \phi_{22}(t_f,t_0)\lambda(t_0). \tag{2.29}$$

For the time being, we use ϕ_{ij} instead of $\phi_{ij}(t_f,t_0)$ for brevity. Using (2.16) and (2.28), we have

$$\begin{aligned} \lambda(t_f) &= S_f x(t_f) + D^T\nu \\ &= S_f[\phi_{11}x_0 + \phi_{12}\lambda(t_0)] + D^T\nu \end{aligned} \tag{2.30}$$

Compare (2.30) with (2.29) to obtain

$$(\phi_{22} - S_f\phi_{12})\lambda(t_0) = -(\phi_{21} - S_f\phi_{11})x_0 + D^T\nu \tag{2.31}$$

or

$$\bar{\phi}_{22}\lambda(t_0) = -\bar{\phi}_{21}x_0 + D^T\nu. \tag{2.32}$$

Assume that $\bar{\phi}_{22}$ is invertible. Then

$$\lambda(t_0) = -\bar{\phi}_{22}^{-1}\bar{\phi}_{21}x_0 + \bar{\phi}_{22}^{-1}D^T\nu \tag{2.33}$$

Substituting (2.33) into (2.28) and considering $\phi_{11} = \bar{\phi}_{11}$, $\phi_{12} = \bar{\phi}_{12}$, we have

$$\begin{aligned} x(t_f) &= (\bar{\phi}_{11} - \bar{\phi}_{12}\bar{\phi}_{22}^{-1}\bar{\phi}_{21})x_0 + \bar{\phi}_{12}\bar{\phi}_{22}^{-1}D^T\nu \\ &= \bar{\phi}_{22}^{-T}x_0 + \bar{\phi}_{12}\bar{\phi}_{22}^{-1}D^T\nu \end{aligned} \tag{2.34}$$

where the second equality in (2.34) is obtained from a symplectic property given by

$$\bar{\phi}_{22}^{-T} = \bar{\phi}_{11} - \bar{\phi}_{12}\bar{\phi}_{22}^{-1}\bar{\phi}_{21}. \tag{2.35}$$

Replacing $x(t_f)$ in (2.10) by (2.34) gives

$$E = D\bar{\phi}_{12}\bar{\phi}_{22}^{-1}D^T\nu + D\bar{\phi}_{22}^{-T}x_0. \tag{2.36}$$

Eqs. (2.33) and (2.36) can be written as

$$\begin{bmatrix} \lambda(t_0) \\ E \end{bmatrix} = \begin{bmatrix} -\bar{\phi}_{22}^{-1}(t_f,t_0)\bar{\phi}_{21}(t_f,t_0) & \bar{\phi}_{22}^{-1}(t_f,t_0)D^T \\ [\bar{\phi}_{22}^{-1}(t_f,t_0)D^T]^T & D\bar{\phi}_{12}(t_f,t_0)\bar{\phi}_{22}^{-1}(t_f,t_0)D^T \end{bmatrix} \begin{bmatrix} x_0 \\ \nu \end{bmatrix}. \tag{2.37}$$

In order to obtain $\lambda(t_0)$ and ν, we have to know all the terms of the state transitions matrix.

2.6. **Determination of differential equations.** Now define

$$\begin{aligned} S(t_f,t) &= -\bar{\phi}_{22}^{-1}(t_f,t)\bar{\phi}_{21}(t_f,t) \\ F(t_f,t) &= \bar{\phi}_{22}^{-1}(t_f,t)D^T \\ G(t_f,t) &= D\bar{\phi}_{12}(t_f,t)\bar{\phi}_{22}^{-1}(t_f,t)D^T. \end{aligned} \tag{2.38}$$

Expand (2.26) with (2.27) to obtain

$$\begin{aligned} &\dot{\bar{\phi}}_{11}(t_f,t) = -\bar{\phi}_{11}(t_f,t)(A - BR^{-1}C) - \bar{\phi}_{12}(t_f,t)(-Q + C^TR^{-1}C), \quad \bar{\phi}_{11}(t_f,t_f) = I \\ &\dot{\bar{\phi}}_{12}(t_f,t) = \bar{\phi}_{11}(t_f,t)BR^{-1}B^T + \bar{\phi}_{12}(t_f,t)(A - BR^{-1}C)^T, \quad \bar{\phi}_{12}(t_f,t_f) = 0 \\ &\dot{\bar{\phi}}_{21}(t_f,t) = -\bar{\phi}_{21}(t_f,t)(A - BR^{-1}C) - \bar{\phi}_{22}(t_f,t)(-Q + C^TR^{-1}C), \quad \bar{\phi}_{21}(t_f,t_f) = -S_f \\ &\dot{\bar{\phi}}_{22}(t_f,t) = \bar{\phi}_{21}(t_f,t)BR^{-1}B^T + \bar{\phi}_{22}(t_f,t)(A - BR^{-1}C)^T, \quad \bar{\phi}_{22}(t_f,t_f) = I. \end{aligned} \tag{2.39}$$

Differentiating $\bar{\phi}_{22}(t_f,t)\bar{\phi}_{22}^{-1}(t_f,t) = I$ and using $\dot{\bar{\phi}}_{22}(t_f,t)$ in (2.39), we have

$$\dot{\bar{\phi}}_{22}^{-1}(t_f,t) = -\left[(A - BR^{-1}C)^T - S(t_f,t)BR^{-1}B^T\right]\bar{\phi}_{22}^{-1}(t_f,t) \tag{2.40}$$

and

$$\begin{aligned}\frac{d}{dt}\left[\bar{\phi}_{12}(t_f,t)\bar{\phi}_{22}^{-1}(t_f,t)\right] &= \dot{\bar{\phi}}_{12}\bar{\phi}_{22}^{-1} + \bar{\phi}_{12}\dot{\bar{\phi}}_{22}^{-1} \\ &= \left[\bar{\phi}_{11}BR^{-1}B^T + \bar{\phi}_{12}(A - BR^{-1}C)^T\right]\bar{\phi}_{22}^{-1} \\ &\quad - \bar{\phi}_{12}\left[(A - BR^{-1}C)^T - S(t_f,t)BR^{-1}B^T\right]\bar{\phi}_{22}^{-1} \\ &= \left[\bar{\phi}_{11} - \bar{\phi}_{12}\bar{\phi}_{22}^{-1}\bar{\phi}_{21}\right]BR^{-1}B^T\bar{\phi}_{22}^{-1} = \bar{\phi}_{22}^{-T}BR^{-1}B^T\bar{\phi}_{22}^{-1}\end{aligned} \tag{2.41}$$

By differentiating (2.38) to have

$$\begin{aligned}\dot{S}(t_f,t) &= -\dot{\bar{\phi}}_{22}^{-1}\bar{\phi}_{21} - \bar{\phi}_{22}^{-1}\dot{\bar{\phi}}_{21} \\ &= \left[S(t_f,t)BR^{-1}B^T - (A - BR^{-1}C)^T\right]S(t_f,t) \\ &\quad - S(t_f,t)(A - BR^{-1}C) + (C^TR^{-1}C - Q) \\ \dot{F}(t_f,t) &= \dot{\bar{\phi}}_{22}^{-1}D^T = -\left[(A - BR^{-1}C)^T - S(t_f,t)BR^{-1}B^T\right]F(t_f,t) \\ \dot{G}(t_f,t) &= D\frac{d}{dt}\left[\bar{\phi}_{12}(t_f,t)\bar{\phi}_{22}^{-1}(t_f,t)\right]D^T = D\bar{\phi}_{22}^{-T}BR^{-1}B^T\bar{\phi}_{22}^{-1}D^T \\ &= F^T(t_f,t)BR^{-1}B^TF(t_f,t).\end{aligned} \tag{2.42}$$

The boundary conditions of (2.42) are given by

$$\begin{aligned}S(t_f,t_f) &= -\bar{\phi}_{22}^{-1}(t_f,t_f)\bar{\phi}_{21}(t_f,t_f) = S_f \\ F(t_f,t_f) &= \bar{\phi}_{22}^{-1}(t_f,t_f)D^T = D^T \\ G(t_f,t_f) &= D\bar{\phi}_{12}(t_f,t_f)\bar{\phi}_{22}^{-1}(t_f,t_f)D^T = 0\end{aligned} \tag{2.43}$$

2.7. **Determination of control equation.** Now (2.37) can be rewritten as

$$\begin{bmatrix}\lambda(t_0) \\ E\end{bmatrix} = \begin{bmatrix}S(t_f,t_0) & F(t_f,t_0) \\ F^T(t_f,t_0) & G(t_f,t_0)\end{bmatrix}\begin{bmatrix}x_0 \\ \nu\end{bmatrix} \tag{2.44}$$

Assuming that G is invertible,

$$\nu = G^{-1}(t_f,t_0)[E - F^T(t_f,t_0)x_0] \tag{2.45}$$

Thus,

$$\begin{aligned}\lambda(t_0) &= S(t_f,t_0)x_0 + F(t_f,t_0)\nu \\ &= \bar{S}(t_f,t_0)x_0 + F(t_f,t_0)G^{-1}(t_f,t_0)E\end{aligned} \tag{2.46}$$

where

$$\bar{S}(t_f,t_0) = S(t_f,t_0) - F(t_f,t_0)G^{-1}(t_f,t_0)F^T(t_f,t_0) \tag{2.47}$$

Since (2.46) is valid for any $t_0 < t_f$ for which $G(t_f,t_0)$ is invertible, we can write

$$\lambda(t) = \bar{S}(t_f,t)x(t) + F(t_f,t)G^{-1}(t_f,t)E \tag{2.48}$$

where

$$\bar{S}(t_f,t) = S(t_f,t) - F(t_f,t)G^{-1}(t_f,t)F^T(t_f,t) \tag{2.49}$$

Substituting (2.48) into (2.14), we have

$$\begin{aligned} u^* &= -R^{-1}Cx - R^{-1}B^T\left[\bar{S}x + FG^{-1}E\right] \\ &= -R^{-1}(C + B^T\bar{S})x - R^{-1}B^TFG^{-1}E \end{aligned} \tag{2.50}$$

The above result can be summarized as follows.

2.8. **Summary of the sweep method.** Consider an LQ optimal control problem: Find $u(t)$ which minimizes (2.8) subject to (2.9) with the terminal constraints of (2.10). Then, the optimal control u^* is given by

$$u^* = -R^{-1}(C + B^T\bar{S})x - R^{-1}B^TFG^{-1}E \tag{2.51}$$

where

$$\begin{aligned} &\bar{S} = S - FG^{-1}F^T \\ &\dot{S} = -(A - BR^{-1}C)^TS - S(A - BR^{-1}C) - (Q - C^TR^{-1}C) \\ &\qquad + SBR^{-1}B^TS, \;\; S(t_f, t_f) = S_f \\ &\dot{F} = -[(A - BR^{-1}C)^T - SBR^{-1}B^T]F, \;\; F(t_f, t_f) = D^T \\ &\dot{G} = F^TBR^{-1}B^TF, \;\; G(t_f, t_f) = 0 \end{aligned} \tag{2.52}$$

2.9. **The simplified sweep method - Terminal constraint only.** Most widely used LQ optimal control formulation in guidance problems involves the minimization of control effort with terminal constraints. In this case, the generalized optimal control problem is greatly simplified: Find $u(t)$ which minimizes

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^TRu\,dt \tag{2.53}$$

subject to $\dot{x} = Ax + Bu$, $x(t_0) = x_0$ with $Dx(t_f) = E$.

Then, the solution to this LQ optimal control problem is given by

$$u^* = R^{-1}B^TFG^{-1}(F^Tx - E) \tag{2.54}$$

where

$$\dot{F} = -A^TF, \;\; F(t_f, t_f) = D^T \tag{2.55}$$

$$\dot{G} = F^TBR^{-1}B^TF, \;\; G(t_f, t_f) = 0 \tag{2.56}$$

3. Impact Angle Control Law and Its Variants

In this section, the optimal impact angle control guidance law in [4] is briefly introduced to show how to derive a guidance law via the sweep method explained in the previous section. Consider the planar homing engagement geometry as shown in Fig. 1 where M and T represent the missile and the target, respectively. The target is assumed to be fixed. The relative distance and the LOS angle of the target with respect to the missile are represented by R and σ, respectively. The speed and flight path angle of the missile are respectively denoted by V_M

and γ_M. The acceleration command u is normal to the missile velocity. Let γ_f be the desired impact angle of the missile to the target. Under the assumption that V_M is constant, we have

$$\dot{\gamma}_M = \frac{u}{V_M}. \tag{3.1}$$

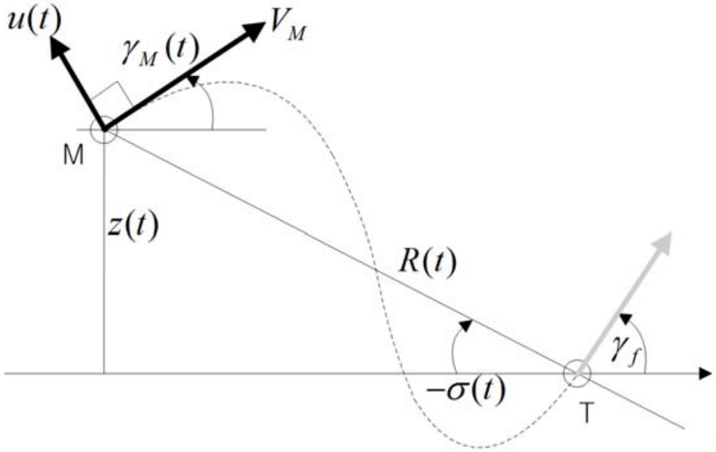


FIGURE 1. The planar homing engagement geometry

Note that z represents the lateral distance from the reference line. Hence,

$$\dot{z} = V_M \sin \gamma_M. \tag{3.2}$$

Assuming that γ_M is small during the flight, we obtain the linearized equations of motion as follows

$$\dot{z} \approx V_M \gamma_M \tag{3.3}$$

$$V_M \dot{\gamma}_M = a_M. \tag{3.4}$$

Consider a following optimal control problem: Find the optimal control u that minimizes the performance cost

$$J = \frac{1}{2}\int_0^{t_f} \frac{u^2(t)}{t_{go}^m} dt, \ \ m \geq 0, \ \ t_{go} = t_f - t, \tag{3.5}$$

subject to the kinematics constraints given by (3.3) and (3.4). And the terminal constraints are

$$z\left(t_f\right) = 0 \tag{3.6}$$

$$\gamma_M(t_f) = \gamma_f. \tag{3.7}$$

Note that t_f in (3.5) denotes the final time when the missile reaches to the target. Now we introduce the following state vector to solve the above LQ optimal control problem

$$x = \begin{bmatrix} z \\ v \end{bmatrix} \triangleq \begin{bmatrix} z \\ V_M \gamma_M \end{bmatrix}. \tag{3.8}$$

Then, the matrices and weightings in (2.54)-(2.56) are given by

$$\dot{x} = Ax + Bu \text{ where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$Dx(t_f) = E \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; E = \begin{bmatrix} 0 \\ v_f \end{bmatrix} \triangleq \begin{bmatrix} 0 \\ V_M \gamma_f \end{bmatrix}, \tag{3.9}$$

and

$$R = \frac{1}{(t_f - t)^m}, \;\; m \geq 0. \tag{3.10}$$

From D in (2.55), we know that the size of F is 2×2. Define $F \triangleq [f_{ij}]\,, \; i, j = 1, 2$, then

$$\dot{F} = -A^T F, \;\; F(t_f) = D^T \Rightarrow \begin{cases} \dot{f}_{11} = 0, \;\; f_{11} = 1 \;\; (f_{11}(t_f) = 1) \\ \dot{f}_{12} = 0, \; f_{12} = 0 \;\; (f_{12}(t_f) = 0) \\ \dot{f}_{21} = -f_{11} = -1, \; f_{21} = t_f - t \; (f_{21}(t_f) = 0) \\ \dot{f}_{22} = -f_{12} = 0, \; f_{22} = 1 \;\; (f_{22}(t_f) = 1) \end{cases} .$$
$$\Rightarrow \;\; F = \begin{bmatrix} 1 & 0 \\ t_{go} & 1 \end{bmatrix}, \;\; \text{where } t_{go} \triangleq t_f - t \tag{3.11}$$

The right hand side of (2.56) is calculated as

$$F^T B R^{-1} B^T F = \begin{bmatrix} 1 & t_{go} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} t_{go}^m \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_{go} & 1 \end{bmatrix} = \begin{bmatrix} t_{go} \\ 1 \end{bmatrix} t_{go}^m \begin{bmatrix} t_{go} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} t_{go}^{m+2} & t_{go}^{m+1} \\ t_{go}^{m+1} & t_{go}^m \end{bmatrix} \tag{3.12}$$

Define $G \triangleq [g_{ij}], \; i, j = 1, 2$, and solve (2.56) to obtain

$$G = \begin{bmatrix} -\frac{1}{m+3} t_{go}^{m+3} & -\frac{1}{m+2} t_{go}^{m+2} \\ -\frac{1}{m+2} t_{go}^{m+2} & -\frac{1}{m+1} t_{go}^{m+1} \end{bmatrix} \tag{3.13}$$

or

$$G^{-1} = -\frac{(m+2)}{t_{go}^{m+3}} \begin{bmatrix} (m+3)(m+2) & -(m+3)(m+1) t_{go} \\ -(m+3)(m+1) t_{go} & (m+2)(m+1) t_{go}^2 \end{bmatrix} \tag{3.14}$$

Finally, from (2.54) the optimal guidance law is derive as

$$\begin{aligned} u^* &= R^{-1}B^TFG^{-1}\left[F^Tx - E\right] \\ &= -\frac{(m+2)}{t_{go}^3}\begin{bmatrix} t_{go} & 1 \end{bmatrix}\begin{bmatrix} (m+3)(m+2) & -(m+3)(m+1)t_{go} \\ -(m+3)(m+1)t_{go} & (m+2)(m+1)t_{go}^2 \end{bmatrix} \\ &\quad \times \left\{\begin{bmatrix} 1 & t_{go} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ v_f \end{bmatrix}\right\} \\ &= -\frac{(m+2)}{t_{go}^3}\begin{bmatrix} (m+3)t_{go} & -(m+1)t_{go}^2 \end{bmatrix}\begin{bmatrix} z + t_{go}v \\ v - v_f \end{bmatrix} \\ &= -\frac{1}{t_{go}^2}\left[(m+2)(m+3)z + 2(m+2)t_{go}v + (m+1)(m+2)t_{go}v_f\right] \end{aligned} \tag{3.15}$$

or

$$u^* = -\frac{V_M}{t_{go}}\left[(m+2)(m+3)\frac{z}{t_{go}V_M} + 2(m+2)\gamma_M + (m+1)(m+2)\gamma_f\right] \tag{3.16}$$

The time-to-go is simply calculated by range over range rate, i.e., $t_{go} = R/\dot{R}$.

The optimal impact angle control law given in (3.16) is derived under the approximations of small flight path angle and small LOS angle. The first term in the bracket can be easily changed to the LOS angle and by doing so the effect of the small angle approximations are reduced so that the capture region of the law can be greatly expanded. The performance of the guidance law is varied according to which of the reference line to define the angles is selected and the detailed topics are found in [10].

From the geometry,

$$\sigma = -\sin\frac{z}{R} = -\frac{z}{R} \approx -\frac{z}{V_M t_{go}} \tag{3.17}$$

or

$$z = -V_M t_{go}\sigma. \tag{3.18}$$

Substitute (3.18) into (3.15) to obtain the angular form of the optimal impact angle control law[4]

$$u^*(t) = \frac{V_M}{t_{go}}(N_\sigma\sigma - N_\gamma\gamma_M - N_f\gamma_f) \tag{3.19}$$

where

$$\begin{aligned} N_\sigma &= (m+2)(m+3) \\ N_\gamma &= 2(m+2) \\ N_f &= (m+1)(m+2). \end{aligned} \tag{3.20}$$

The gain sets according to m are shown in Table 1. Special concern for $m = 0$ is required, because it provides the pure energy optimality of the guidance law. For $m > 0$, the weighting in (3.5) is growing to infinity as t_{go} approaches zero. To have a finite cost, the control should be zero as t_{go} approaches zero.

TABLE 1. Examples of the gain set according to m

m	N_σ	N_γ	N_f
0	6	4	2
1	12	6	6
2	20	8	12

3.1. **Flight path angle control against a maneuvering target.** The optimal impact angle control law given in (3.19) is simple enough but it requires the measurements of the LOS angle as well as the flight path angle. The flight path angle can be typically obtained by the INS(inertial navigation system) measurement while there are no sensor systems to directly provide the LOS angle. If the target's location is known, the LOS angle can be geometrically calculated by combining with the missile's location. If the target acquisition sensors such as RF(radio frequency) or IR(infra-red) seekers, we requires a state estimator to obtain the LOS angle.

Another important weakness in application of the law given in (3.19) is that it cannot intercept a maneuvering target even for a constant speed without maneuver. For a fixed target, the LOS angle and the flight path angle are always same as the desired impact angle at the impact instant. If the target is moving or maneuvering, however, σ and γ_M are not same each other at the impact, because σ is a state variable of the missile relative to the target so that it can vary according to the target's velocity vector, while γ_M is a state variable only for the missile regardless of target's motion. It means that angle mismatch will occur at the impact instant and the guidance command eventually blows up if (3.19) is applied to a maneuvering target.

To solve this problem we change the form of the optimal impact angle control law. From (3.17) and (3.18), the LOS rate becomes.

$$\dot{\sigma} = -\frac{\dot{z}V_M t_{go} + zV_M}{V_M^2 t_{go}^2} = -\frac{V_M \gamma_M t_{go} + z}{V_M t_{go}^2} = -\frac{\gamma_M - \sigma}{t_{go}} \tag{3.21}$$

or

$$\sigma = \gamma_M + t_{go}\dot{\sigma}. \tag{3.22}$$

Substitute (3.22) into (3.19) to obtain

$$u^*(t) = N_\sigma V_M \dot{\sigma} + \frac{N_f V_M}{t_{go}}(\gamma_M - \gamma_f). \tag{3.23}$$

As introducing the LOS rate to the guidance law instead of the LOS angle, angle mismatch disappears so that it is possible to intercept the maneuvering target. Note that the above optimal impact angle control law is a biased PN where the first term in the right hand side of (3.23) is the conventional PN to intercept a target and the second term is to satisfy the desired impact angle.

3.2. **LOS angle control against a maneuvering target.** Sometimes we need to intercept the target with a desired LOS, for example, to maintaining the target within a field of view(FOV) of the seeker of the missile. It is recommended that the missile is guided to the target with a small LOS angle. The flight path angle control is not adequate for this purpose because the law given in (3.23) intercepts the target without consideration of target maneuver. From (3.22),

$$\gamma_M = \sigma - t_{go}\dot{\sigma}. \tag{3.24}$$

Substituting (3.24) into (3.19), we have

$$u^*(t) = N_\gamma V_M \dot{\sigma} + \frac{N_f V_M}{t_{go}}(\sigma - \gamma_f) \tag{3.25}$$

For the physical matching of the angle control term, the LOS angle control guidance law can be rewritten

$$u^*(t) = N_\gamma V_M \dot{\sigma} + \frac{N_f V_M}{t_{go}}(\sigma - \sigma_f) \tag{3.26}$$

Note that the guidance gain for the PN term in (3.26) is given by N_γ but different from N_σ of (3.23). Referring to Table 1, small gain is required for the LOS angle control when it compared to the flight path angle control law given in (3.23). Also, as illustrated in Fig. 2 we note that $\gamma_M(t_f) \neq \sigma(t_f) = 0$ if target maneuvering is introduced.

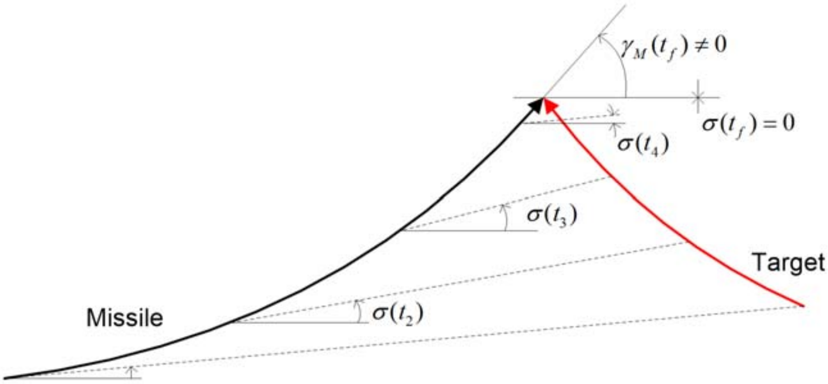


FIGURE 2. Difference between flight path angle and LOS angle at the impact instant for a maneuvering target

3.3. **Relative flight angle control against a maneuvering target.** In some applications like the interception of a ballistic target, maintaining the specified impact angle relative to the target's flight direction is important to magnify the radar cross section(RCS) and to enhance kill probability by hitting the sweet spot. If $\Delta\gamma$ denotes the relative flight path impact angles of the missile to the target's velocity, the desired terminal flight path impact angle can be given by

$$\gamma_f = \gamma_T + \Delta\gamma \tag{3.27}$$

The desired terminal flight path impact angle given in (3.27) is applied to (3.23) or

$$u^*(t) = N_\sigma V_M \dot{\sigma} + \frac{N_f V_M}{t_{go}}(\gamma_M - \gamma_T - \Delta\gamma). \tag{3.28}$$

In (3.28), the target flight path angle at impact instant γ_T should be predicted but it is not possible in real application because future target maneuver is not determined. Hence, we assume that the current target flight path angle is maintained until the impact instant.

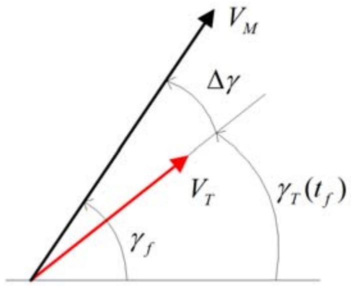
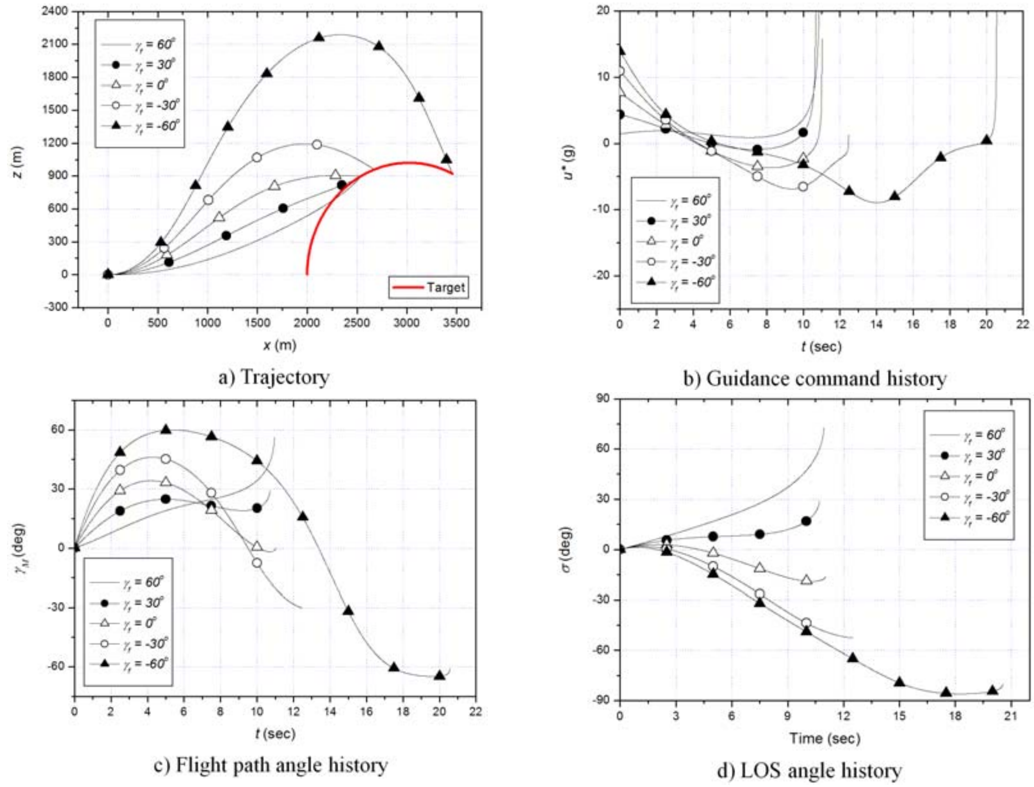


FIGURE 3. Definition of the relative flight path impact angle

4. Numerical Examples

In this section, the performance of the proposed guidance laws for intercepting a maneuvering target is verified via numerical simulations in the planar engagement scenarios. The speeds of the missile and the target are 250m/s and 100m/s, respectively, and remain constant during the engagement. The target laterally maneuvers with 1g. The guidance gains are set by $m = 0$ in Table 1 so that the guidance laws have pure energy optimality.

Figure 4 shows the simulation results under the application of the flight path angle control law given in (3.23) with various flight path impact angles. As shown in Fig. 4(c), all the flight path angle constraints are satisfied with slight impact angle errors due to the target maneuver. These errors cause that the guidance commands blow up as shown in Fig. 4(b). For $\gamma_f = -60^o$, the missile detours far away compared to other impact angle cases.



a) Trajectory

b) Guidance command history

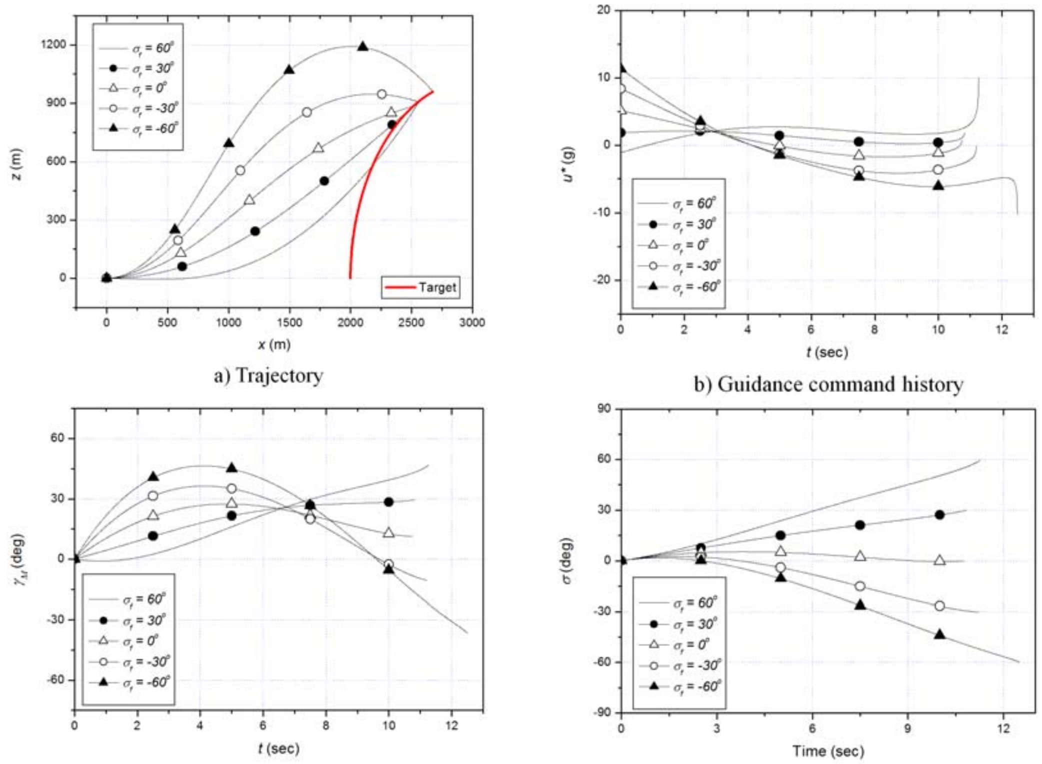
c) Flight path angle history

d) LOS angle history

FIGURE 4. Simulation results for the flight path angle control law

From Fig. 5, we observe that the simulation results for the LOS angle control law in (3.26)for various LOS impact angles. As shown in Fig. 5(d), the LOS impact angle constraints are satisfied with no errors and command divergence is not observed from Fig. 5(b). The variation range of flight path angle at the impact instant is from -40 degrees to 45 degrees as shown in

Fig. 5(c). It means that the LOS angle control in this scenario is not severe as much as the previous case for flight path angle control.



a) Trajectory

b) Guidance command history

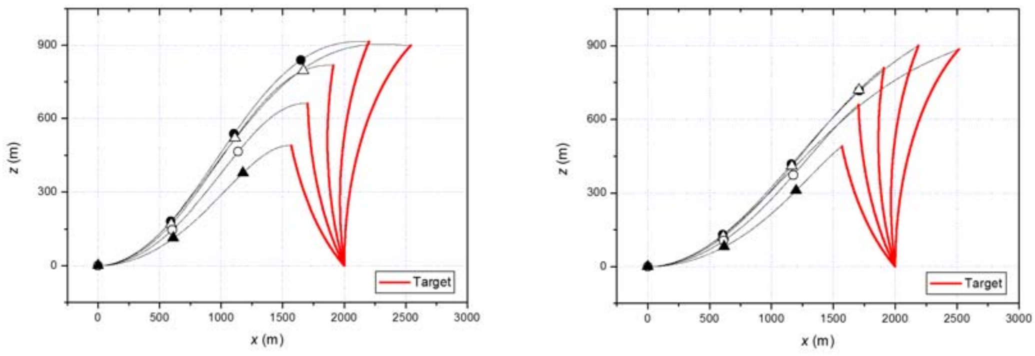
c) Flight path angle history

d) LOS angle history

FIGURE 5. Simulation results for the LOS angle control law

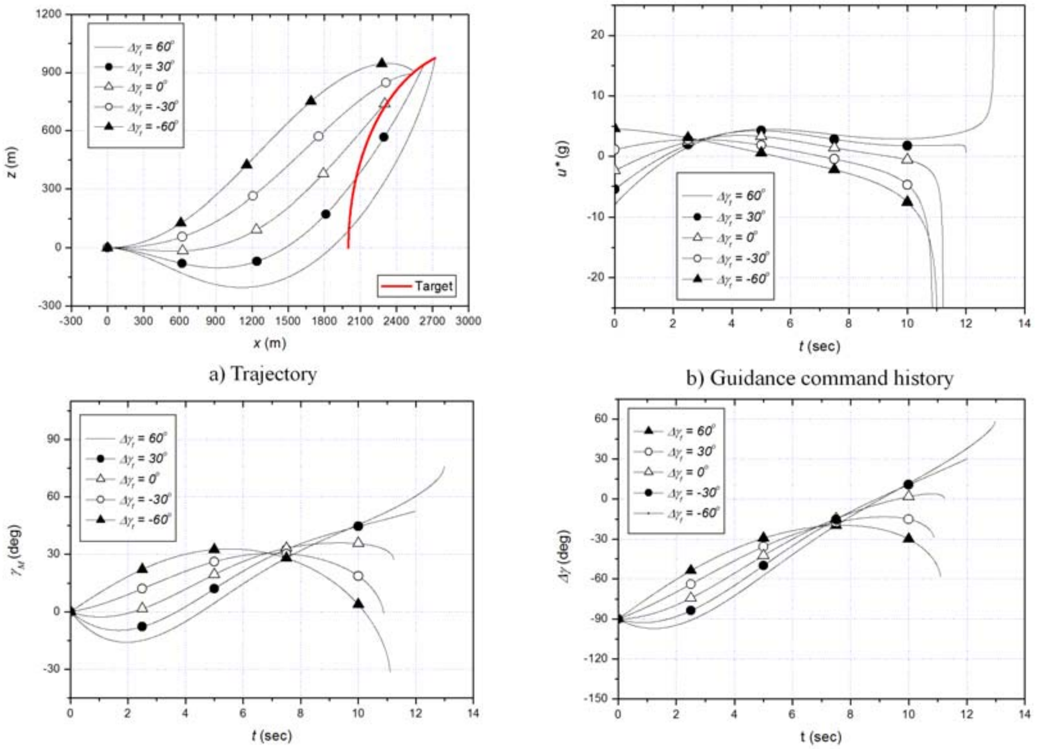
In Fig. 6, the comparison of flight path angle control of $\gamma_f = 0^o$ and LOS angle control of $\sigma_f = 0^o$ where the missile always intercepts the target with the same direction regardless of the target motion if the flight path angle control is applied. Under the application of LOS angle control, the flight path angles at the impact instant are different each other even though impact LOS angles are the same as already explained in Fig. 2.

We can see in Fig. 7 the simulation results for the relative flight path angle control for various impact angles given in (3.28). We observe from Figs. 7(a) and (d) that the guidance purpose to intercept target with the relative flight path impact angle is satisfied. Tendency of divergence in guidance command as shown in Fig. 7(b) is caused from the fact that there are slight errors in impact angles.



a) Flight path angle control with $\gamma_f = 0^o$ b) LOS angle control with $\sigma_f = 0^o$

FIGURE 6. Comparison of the flight path angle and LOS angle control laws



a) Trajectory b) Guidance command history

c) Flight path angle history d) LOS angle history

FIGURE 7. Simulation results for the relative flight path angle control law

Each guidance law has a limited capture condition even if there's no error sources such as response lag and command limit. It is thought that such a limited capture condition comes from the inadequate time-to-go calculation method. In this paper, the time-to-go is simply calculated by range over range rate since the target maneuver is introduced. It means that this method does not reflect the curved nature of flight path so that real time-to-go may be little bit different from the calculated one.

5. Conclusion

In this paper, the sweep method based on the LQ optimal control theory as a framework to derive guidance laws is reviewed in detail. The three variants of the optimal impact angle control guidance law to intercept a maneuvering target are newly introduced. The flight path angle control law is capable of intercepting a mobile target on the ground with a given flight path angle, typically vertical angle to maximize the warhead effect, regardless of target motion. The LOS rate and target range for time-to-go calculation are the only required for implementation of this law. The LOS angle control law is useful for maintaining the target in the FOV of the seeker of the missile since it can deliver a missile to an aerial target with a desired LOS angle at the impact instant. The LOS rate, LOS angle, and target range should be given as measurements for application of this law. The relative flight path angle control law has a lot of applications especially in anti-ballistic missiles, however, it requires target's flight path angle as well as the LOS rate. Not detail discussed in this paper, each of the proposed guidance laws to intercept a maneuvering target has its own limitation on capture condition which depends on the time-to-go calculation method. How to expand the capture region is remained as the further study.

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