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A TUTORIAL ON LINEAR QUADRATIC OPTIMAL GUIDANCE FOR MISSILE APPLICATIONS

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ABSTRACT. In this tutorial the theoretical background of LQ optimal guidance is reviewed, starting from calculus of variations. LQ optimal control is then introduced and applied to missile guidance to obtain the basic form of LQ optimal guidance laws. Extension of LQ optimal guidance methodology for handling weighted cost function, dynamic lag associated with the missile dynamics and the autopilot, constrained impact angle, and constrained impact time is also described with a brief discussion on the asymptotic properties of the optimal guidance laws. Furthermore, an introduction to polynomial guidance and generalized impact-angle-control guidance, which are closed related with LQ optimal guidance, is provided to demonstrate the current status of missile guidance techniques.

1. INTRODUCTION

The purpose of this tutorial is to provide fundamentals of guidance laws based on linear quadratic (LQ) optimal control theory that have been developed for missile applications since the 1960s. The scope of the tutorial is restricted to the works done by the author and his ex-students, and other previous works on guidance laws are not considered unless necessary. Technical details of the topics discussed here can be found in the accompanying papers published in this special issue of the journal.

Loosely speaking, guidance laws are algorithms calculating the control commands that the vehicle follows to achieve the mission objectives. In missile applications, the set of the objectives depends on the purpose of the missile system although the fundamental objective is to hit the target with zero miss distance. For example, an air-to-air missile may be designed to just hit a highly-maneuvering agile aircraft while an anti-tank missile needs to hit a slowly moving tank with a certain range of impact angles. Furthermore, the means to acquire the target information and to produce forces and moments required for vehicle control can limit the range of applicable guidance algorithms. With an infrared seeker, the missile cannot measure the target range (the distance from the missile to the target) so that some LQ optimal guidance laws cannot be applied unless the target range is estimated by using a tracking filter. Due to the diversity of missile seekers, actuators, operational environments, target types, and mission objectives, various guidance algorithms have been studied to accommodate the specific needs

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of each missile system. To prevent the readers from being distracted, the discussion of this tutorial is focused on the terminal homing phase, for which feedback guidance laws based on LQ optimal control theory have been successfully applied. Mid-course guidance required when the target is beyond the operational range of the on-board seeker will not be much discussed here since it needs different mathematical approaches such as trajectory optimization for which LQ optimal guidance is not useful.

This tutorial starts with a brief introduction of optimal control theory and LQ optimal control given in Section 2. LQ optimal guidance, which is a direct application of LQ optimal control to missile guidance is also described. Section 3 treats various topics associated with practical applications, showing how LQ optimal guidance can meet specific needs and provide practical solutions. Guidance laws associated with weighted cost function, dynamic lag, impact angle control, and impact time control are discussed one by one. Asymptotic properties of the guidance gain are also explained to demonstrate the importance of accurate time-to-go estimation for integrated guidance. Section 4 is about polynomial guidance and generalized impact-angle-control guidance, which are closely related with LQ optimal guidance. The motivation of polynomial guidance and the analysis results on generalized impact-angle-control guidance are briefly discussed. Section 5 provides concluding remarks.

2. Optimal Homing Guidance

In this section, the fundamentals of LQ optimal guidance are provided for the readers who are not familiar with optimal control theory. LQ optimal guidance is a direct application of LQ optimal control which has a solution in the feedback form. Although the calculation of the feedback gains needs numerical integration for general cases, analytic solutions have been obtained for many cases of practical importance.

2.1. **Optimal Control Theory.** In general, optimal guidance laws are referred to as guidance laws derived from optimal control theory, which is founded on calculus of variations. Johann Bernoulli is known to be the first mathematician to consider the brachistochrone curve problem, a simple problem of calculus of variations, but Euler and Lagrange contributed extensively to lay the foundations of calculus of variations. The necessary condition for optimality, known as the Euler-Lagrange equation, is expressed as

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{2.1}$$

where $L(t, x, \dot{x})$ is the functional to be optimized.

Given a dynamic constraint on x , for example, we have an optimal control problem formulated as follows:

$$\text{Minimize} \quad J = \phi(x_f) + \int_{t_0}^{t_f} L(x, u, t)dt \tag{2.2}$$

$$\text{subject to} \quad \dot{x} = f(t, x, u), \quad x(t_0) = x_0 \quad \text{(dynamic constraints)} \tag{2.3}$$

$$\psi(x_f) = 0 \quad \text{(terminal constraints)} \tag{2.4}$$

Let the dimensions of the state, control, and constraint vectors be n, m, p, respectively; that is,

$$x : n \times 1, \quad u : m \times 1, \quad \psi : p \times 1$$

And define augmented functions H and G as

$$H \triangleq L + \lambda^T f, \quad G \triangleq \phi + \mu^T \psi \tag{2.5}$$

where λ and μ are multipliers for the dynamics and the constraints, respectively. Then the Euler-Lagrange equation for the optimal control problem becomes

$$\text{(i)} \quad \dot{x} = H_\lambda,\ x(t_0) = x_0 \qquad \text{(n D.E's with B.C's)} \tag{2.6}$$

$$\text{(ii)} \quad \dot{\lambda} = -H_x, \quad \lambda(t_f) = G_{x_f} \qquad \text{(n D.E's with B.C's)} \tag{2.7}$$

$$\text{(iii)} \quad 0 = H_u \qquad \text{(m algebaric equations)} \tag{2.8}$$

$$\text{(iv)} \quad \psi(x_f) = 0 \qquad \text{(p algebraic equations)} \tag{2.9}$$

The optimal solution $(x^*(t),\ \lambda(t),\ u^*(t),\ \mu)$ is obtained by solving these four equations, which is a two-point boundary value problem (TPBVP). Derivation of the Euler-Lagrange equation for optimal control can be found in [1]. Note that the Euler-Lagrange equation is valid only for a weak minimum. Weierstrass was the first to provide the necessary condition for a strong minimum, which is a special case of Pontryagin's maximum(minimum) principle.

2.2. **Linear Quadratic Optimal Control.** Two-point boundary value problems associated with optimal control problems do not allow analytical solutions for most cases. However, if the system $f(t, x, u)$ is linear and $L(t, x, u)$ is a quadratic function of x and u , then the Euler-Lagrange equation reduces to a final value problem. This problem is called LQ optimal control and its solution can be obtained by integrating three differential equations backwards from the final time.

Consider an LQ optimal control problem with terminal constraints shown below:

$$\text{Minimize} \quad J = \frac{1}{2}\int_{t_0}^{t_f} (x^T A x + u^T B u) dt \quad \text{(quadratic cost)} \tag{2.10}$$

$$\text{subject to} \quad \dot{x} = Fx + Gu, \quad x(t_0) = x_0 \quad \text{(linear dynamics)} \tag{2.11}$$

$$\psi \equiv D x_f - c = 0 \quad \text{(linear terminal constraints)} \tag{2.12}$$

Then the optimal control is derived as [1]

$$u^* = -B^{-1} G^T (S - R Q^{-1} R^T) x - B^{-1} G^T R Q^{-1} c \tag{2.13}$$

where matrices $S(t), R(t), Q(t)$ satisfy the following differential equation, respectively;

$$\dot{S} = -A - F^T S - SF + SGB^{-1}G^T S, \quad S(t_f) = 0 \tag{2.14}$$

$$\dot{R} = -(F - GB^{-1}G^T S)^T R, \quad R(t_f) = D^T \tag{2.15}$$

$$\dot{Q} = RGB^{-1}G^T R, \quad Q(t_f) = 0 \tag{2.16}$$

Since the solution of feedback form is available, the LQ controllers described above can find many practical applications subject to terminal constraints.

2.3. **LQ Optimal Homing Guidance for Intercept.** The purpose of homing guidance in missile applications is to intercept (or hit) the target as accurately as possible. This requirement can be formulated as a linear terminal constraint shown in (2.12). Furthermore, the missile-target dynamics can be simplified as a linear system if the missile and the target maintain their speed constant. Finally, we can choose $A = 0$ in (2.10) by assuming that the state variables during the flight is not important. Then, (2.14) reduces to $S(t) \equiv 0$ and the optimal control $u^*(t)$ is simplified as

$$u^* = B^{-1}G^T RQ^{-1}R^T x - B^{-1}G^T RQ^{-1}c \tag{2.17}$$

where $R(t)$ satisfies

$$\dot{R} = F^T R, \quad R(t_f) = D^T \tag{2.18}$$

and $Q(t)$ is obtained from (2.16).

Now we consider a simple two-dimensional guidance geometry shown in Fig. 1 to derive the optimal guidance law as described in [1, 2]. In this figure, the Z-axis is chosen to be aligned with the direction of the target velocity. It is assumed that the closing velocity, the X-component of the missile velocity, is constant and the line-of-sight (LOS) angle σ is small. Let z and v denote the relative position and velocity of the target with respect to the missile along the Z -axis. Then, the state variable associated with the homing kinematics can be defined as

$$x = \begin{bmatrix} z & \nu \end{bmatrix}^T \tag{2.19}$$

and the equation of motion is obtained as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a_t - a_m \end{bmatrix} \tag{2.20}$$

where a_t and a_m denote the Z-axis acceleration of the target and missile, respectively. Note that $a_t = 0$ for a non-maneuvering target.

Let $u \triangleq a_M$ and $a_t = 0$, then the optimal guidance problem is formulated as follows:

$$\text{Minimize} \quad J = \frac{1}{2}\int_{t_0}^{t_f} u^2 dt \tag{2.21}$$

$$\text{subject to (2.20) and} \quad \psi \equiv z_f = [1 \quad 0]\, x_f = 0 \tag{2.22}$$

The rationale for the cost function (2.21) is that excessive missile lateral maneuvers are not desirable since they reduce the missile speed drastically, degrading the intercept performance.

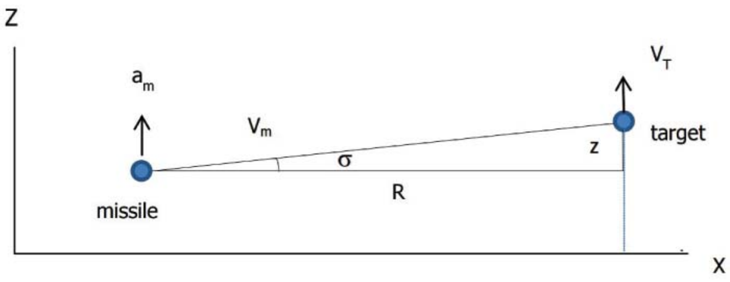

FIGURE 1. Simplified Homing Guidance Geometry

The quadratic form of (2.21) is chosen to utilize the benefit of the LQ formulation. Note that $c = 1,\ B = 1$ and $D = [1 \quad 0]$. Hence, (2.16) and (2.18) can be easily integrated as

$$R(t) = \begin{bmatrix} 1 \\ t_{go} \end{bmatrix}, \quad Q(t) = -\frac{1}{3}t_{go}^3 \tag{2.23}$$

where $t_{go} \triangleq t_f - t$, which is called the time to go. Substituting (2.23) into (2.17), we obtain the LQ optimal guidance law stated as

$$u^* = \frac{3}{t_{go}^2}(z + \nu t_{go}) \tag{2.24}$$

The guidance law of (2.24) requires the time-to-go information which is not readily available from passive seekers such as infrared or vision sensors. However, (2.24) can be rewritten as a function of variables that are directly measured by passive seekers: For small LOS angles, we see that $\sigma \approx z/R$ where R is the range to go. Since the missile velocity is assumed constant and the LOS angle is small, the missile velocity along the X-axis is also assumed constant. Then, the range to go is expressed as $R = V_c t_{go}$ where V_c is the closing velocity, and the LOS angle can be written in terms of z and t_{go} as

$$\sigma = \frac{z}{V_c t_{go}} \tag{2.25}$$

Then differentiation of (2.25) gives that

$$\dot{\sigma} = \frac{1}{V_c t_{go}^2}(z + \nu t_{go}) \tag{2.26}$$

and comparison of (2.24) and (2.26) shows that

$$u^* = 3V_c\dot{\sigma} \tag{2.27}$$

It is very interesting that the optimal guidance law of (2.27) takes the form of classical proportional navigation (PN) known as

$$u_{PN} = NV_c\dot{\sigma} \tag{2.28}$$

where the navigation constant (or the guidance gain), N , was previously chosen from 3 to 5 by experience. Proportional navigation has been widely used since the birth of guided missile technology in the 1950's. Note that LQ optimal guidance proves by theory that proportional guidance of $N = 3$ is optimal in the LQ sense for the intercept of non-maneuvering targets [3].

Generally speaking, a missile operated in the atmosphere relies on the aerodynamic forces for lateral (perpendicular to the velocity vector) maneuvers. In the 3-dimensional space, we can define two wind angles, angle of attack and sideslip angle, to denote the direction of the velocity vector with respect to the X-axis of the missile body. The accelerations produced by the angle of attack and the sideslip angle are called normal acceleration and lateral acceleration, respectively, or just lateral accelerations, collectively. The motion associated with normal acceleration is called longitudinal motion and that associated with lateral acceleration called lateral motion or directional motion. For axis-symmetric missiles, the dynamic characteristics of the longitudinal motion and the directional motion are identical so that two independent guidance laws of the same structure are used for 3-dimensional guidance. For example, one channel (pitch channel) of the guidance algorithm handles the guidance in the vertical plane while the other channel (yaw channel) is responsible for the guidance in the horizontal plane. (The directions of normal and lateral accelerations may not be aligned with the vertical and horizontal directions, depending on applications.)

2.4. **LQ Optimal Homing Guidance for Intercept with Specified Impact Angle.** The LQ optimal guidance law discussed above can be extended to handle the terminal impact-angle constraint by a simple modification in the problem formulation: Suppose that we want the missile to approach to the target with $\sigma(t) = 0$ as $t_{go} \to 0$. This objective can be achieved by imposing $\sigma(t_f) = 0$ and $\dot{\sigma}(t_f) = 0$, and (2.25) and (2.26) translate these conditions to $z(t_f) = 0$ and $\nu(t_f) = 0$. The new terminal condition on the state gives

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.29}$$

then $R(t)$ and $Q(t)$ are integrated as

$$R(t) = \begin{bmatrix} 1 & 0 \\ t_{go} & 1 \end{bmatrix}, \quad Q(t) = -\begin{bmatrix} t_{go}^3/3 & t_{go}^2/2 \\ t_{go}^2/2 & t_{go} \end{bmatrix} \tag{2.30}$$

Finally, the optimal guidance law for intercept and zero relative impact angle is obtained in feedback form as

$$u^* = \frac{1}{t_{go}^2}(6z + 4\nu t_{go}) \tag{2.31}$$

For stationary targets, this guidance law can be expressed in terms of angle variables as

$$u^* = \frac{1}{t_{go}}(6\sigma - 4\gamma) \tag{2.32}$$

where γ is the flight path angle of the missile relative to the target; $\gamma = -\nu/V_c$.

3. Extension of Optimal Guidance Laws

In this section various extensions of the basic LQ optimal guidance laws described above are introduced. First, we discuss guidance laws for which the cost function is weighted by the time to go or some arbitrary function of the time to go. The next topic is about how to handle the dynamic lag produced by the missile's dynamics and the autopilot, which is the controller to produce the lateral acceleration as commanded by the guidance law. Then various extensions of LQ optimal guidance for the control of impact angle, impact time, or both of them are introduced. The last topic of this section is integrated guidance which merges the autopilot design with the guidance law design.

3.1. **Weighted Cost Function.** The quadratic cost function of (2.21) is a simple integration of the control energy over the engagement time interval. However, excessive missile maneuver near the impact point may not be desirable since the target information from the seeker is not much reliable at a very close range. Furthermore, the time to go information may not be very accurate due to various reasons so that a guidance law written in terms of the time to go can produce large steering errors if this is the case. To circumvent this difficulty we may modify the cost function as

$$J = \frac{1}{2}\int_{t_0}^{t_f} \frac{u^2}{t_{go}^k} dt, \quad k \geq 0 \tag{3.1}$$

Since large weights given by $1/t_{go}^k$ discourage the control activities near the impact, we can expect the optimal guidance command $u^*(t)$ to converge to 0 as $t_{go} \to 0$.

Cho [4] considered this problem for the first time to obtain the closed-form solution expressed as

$$u^* = \frac{1}{t_{go}^2}[(k+2)(k+3)z + 2(k+2)\nu t_{go}] \tag{3.2}$$

Note that the exponent k is a parameter that the designer can freely choose to shape the missile trajectory. Ryoo et al. [5] extended the guidance problem subject to time-to-go weighted cost functions to include impact angle constraints. Furthermore, Lee et al. [6] was successful to derive the solution of the LQ guidance problem with arbitrary weighting functions given as

$$J = \frac{1}{2}\int_{t_0}^{t_f} W(t)u(t)^2 dt \tag{3.3}$$

It is surprising that the optimal guidance command for arbitrary weighting functions is still written in the simple form of the basic LQ guidance laws:

$$u^* = \frac{k_1(t_{go})}{t_{go}^2} z + \frac{k_2(t_{go})}{t_{go}} \nu \tag{3.4}$$

although $k_1(t_{go})$ and $k_2(t_{go})$ are to be calculated by backward integration of three differential equations associated with the weighting function. Recent application of this development for trajectory shaping of anti-tank missiles can be found in Ryu et al. [6]. In this work, the weight function is chosen to put more control energy during the initial climbing phase and the terminal homing phase while restricting maneuvers in the middle.

3.2. **Dynamic Lag.** In Section 2 we assume that the acceleration command generated by the guidance law is realized instantly without any dynamic lag or time delay. In reality, it is not possible at all so that there will be guidance errors if the dynamic lag produced by the autopilot, which is the commonly-used terminology for the controller of missile systems, and the missile dynamics are not properly addressed in the guidance law design. To the author's knowledge, Cottrell [8] was the first to treat this problem to derive the optimal guidance law in feedback form. The lag model used in this study is a first-order model expressed as

$$\dot{a}_M = \frac{1}{\tau}(u - a_M) \tag{3.5}$$

where a_M is the realized lateral acceleration, u the commanded lateral acceleration, and τ the time constant of the dynamic lag. Later Ryoo et al. [9] have proposed a generalized formulation of optimal guidance with impact angle constraints for a constant-speed missile with an arbitrary system order, showing that the optimal guidance command is expressed as a linear combination of the step and the ramp responses of the missile acceleration. In Section 3.5, we will include the dynamics of the missile (and the autopilot, if necessary) in the guidance problem to show the effect of the missile dynamics on the guidance gains.

While the optimal guidance laws based on various dynamic lag models are optimal under reasonable assumptions, they have two drawbacks in practical implementation. First, the feedback of the additional states associated with the dynamic lag is required. As the degree of the dynamic lag model increases, the number of the state variables to be measured or observed increases. Second, very accurate estimation of the time to go is required. It is because that the optimal guidance gain is no longer constant but a function of the time to go for this class of the optimal guidance laws. Specifically, the guidance gains experience rapid changes when t_{go} is of the order of the time constant of the dynamic lag. Since the guidance commands produced during this period is critical, inaccurate estimation of t_{go} may result in degradation of the guidance performance rather than improvements. This observation leads to the study of developing accurate and reliable time-to-go estimation techniques, as observed in a number of works. For example, Tahk et al. [10] have suggested a recursive time-to-go computation method for proportional navigation which can compensate time-to-go errors due to the path curvature, which are dependent on the initial heading error. Ryoo et al. [11] have proposed two methods of time-to-go calculation for impact-angle-control guidance laws by approximating the curved path as a 3rd-order polynomial function. As a matter of fact, the time to go depends on the guidance law; for example, proportional navigation and an impact-angle-control guidance law will produce different time-to-go history even if the initial conditions are exactly same. Therefore, any new guidance law should be proposed with a proper time-to-go estimation method if the time to go is used for guidance command generation.

Alternative approach to treat the dynamic lag is to exclude the dynamic lag from the formulation but design the guidance law in such a way that the guidance command becomes 0 or very small near the intercept point. This approach relies on trajectory shaping to avoid excessive maneuvers at the final phase of homing guidance, resulting in guidance performance less sensitive to the time-to-go errors. Polynomial guidance to be discussed in Section 4 is used for such purpose.

The extreme opposite of the trajectory shaping such as polynomial guidance is integrated guidance that will be discussed in Section 3.5. Conventionally, the guidance law assumes that the missile has an autopilot system for realization of the acceleration commands generated by the guidance algorithm. In other words, the guidance law and the autopilot are designed separately. However, it can be beneficiary to treat them simultaneously if the dynamic lag is large or a very small miss distance is required to achieve the mission objectives. Since the full dynamic model of the missile (and the autopilot if used) is included in the formulation of LQ optimal control, the full state feedback is required for integrated guidance.

3.3. **Impact Angle Control.** Impact angle control is important for many missile applications. Anti-tank missiles try to hit the top part of the hostile tank where the armor is a lot thinner than the front and lateral sides. By doing so, the kill probability of a single shot can be maximized. For anti-ship missiles, attacking the target ship along the least defended direction is crucial since the modern ship defense systems have a good capability of neutralizing incoming missiles. For certain hard targets, hitting the target with a right angle to the surface is a requirement. Or target visibility of the seeker can be heavily dependent on the impact angle.

Since the impact angle is an important parameter for the effectiveness of missile systems, there have been numerous studies on guidance laws that can satisfy the impact angle requirements. As demonstrated in Section 2.4, an LQ optimal guidance law can be derived for impact angle control by simply adding an extra terminal condition on the missile velocity. Examples of recent studies on impact-angle-control guidance laws based on LQ optimal control can be found in [5, 6, 7, 9, 11, 12, 13]. Polynomial guidance to be discussed in Section 4 basically includes impact angle control as found in [14, 15, 16, 17]. Previous studies also include a class of generalized impact-angle-control guidance laws studied in [18, 19, 20]. Impact angle control is also an important topic in the study of other guidance methodology such as sliding mode control. One recent example can be found in [21].

Impact-angle-control guidance laws are also useful in the mid-course phase of missile guidance. For long-range missile applications, the missile needs to approach the hand-over region, where the transition from the mid-course phase to the terminal homing phase takes place, with a set of proper conditions on the flight path angle and the look angle (the angle between the X-axis of the missile and the LOS) for the seeker to find the target without difficulty. Since impact angle control can produce a highly curved trajectory, the target may not be within the field of view (FOV) of the seeker when the seeker is turned on. Jeon et al. [22] have proposed the optimal impact-angle-control guidance law that takes care of the seeker FOV, and then Park et al. [23] studied the same problem including the flight path angle constraint.

3.4. **Impact Time Control.** Until the salvo attack of multiple missiles is seriously considered, the impact time of the intercept engagement was not an issue. If many missiles are going to attack a single target heavily defended by various defense systems, however, the arrival time of each missile can be an important parameter in view of weapon effectiveness. Suppose that a close-in weapon system (CIWS) of the target vessel has a very high kill probability against hostile anti-ship missiles. Sending several missiles one after another in a serial fashion will not be quite successful in this situation. But if several missiles are attacking the same target simultaneously, the CIWS gun may have to allow some of the missiles to penetrate the defense system to reach the ship since the gun is only able to engage with a single target at a time.

The first published study on impact time control has been conducted by Jeon et al. [24]. In this study, a bias u_F is introduced to the guidance command as

$$u = u_B + u_F \tag{3.6}$$

and the control energy of u_B is optimized for an arbitrary constant u_F by applying the LQ optimal guidance technique. For target intercept with zero miss, the optimal solution turns out to be

$$u^* = u_{PN} - \frac{1}{2}u_F \tag{3.7}$$

Note that the intercept condition is automatically satisfied for any u_F. Thus, we can utilize the freedom in choosing u_F to meet the impact time requirement. The magnitude of u_F can be determined from the relationship between the estimated time-to-go error of PN and its sensitivity to u_F. The impact-time-control guidance law consists of two feedback paths; one for conventional PN and another for time-to-go correction.

The use of a bias term has been again adopted by Lee et al. [25] for derivation of the guidance law for impact time control together with impact angle control. To obtain more degree of freedom to satisfy the impact angle constraint, the state vector is augmented to include the lateral acceleration and the lateral jerk (time derivative of acceleration) is defined as the guidance command. By assigning a suitable impact angle to each missile, this guidance law can make the ship defense more vulnerable to a salvo attack.

If the missile is equipped with a certain propulsion system, the arrival time can be controlled by adjusting the thrust level. However, it can be done more easily by trajectory shaping proposed in these studies while the speed is kept constant. Strictly speaking, however, the impact-time-control guidance laws are not control energy optimal since the control energy of u_B is optimized instead of $u_B + u_F$.

It is noted that an alternative approach for impact time control of salvo attacks has been proposed by Jeon et al. [26], which relies on the communication between the missiles during the engagement. In this method, the missiles use PN for guidance but the guidance gains are continuously tuned to synchronize the arrival times. The larger the guidance gain, the shorter the flight time since the missile turns to the target earlier.

3.5. **Integrated Guidance.** For integrated guidance, the state vector includes three groups of state variables; x_k for the kinematics of the engagement, x_t for the target dynamics, and x_m for the target dynamics. The homing kinematics is already given by (2.19) and (2.20). Now suppose that the target dynamics is modeled as

$$\begin{aligned} \dot{x} &= F_t x_t + G_t u_t \\ a_t &= H_t x_t \end{aligned} \tag{3.8}$$

where u_t , the maneuver input of the target, is usually modeled as a white noise process. This assumption is justified since u_t is neither known nor predictable in real homing engagements. Similarly, the missile dynamics is represented by

$$\begin{aligned} \dot{x}_m &= F_m x_m + G_m u_m \\ a_m &= H_m x_m \end{aligned} \tag{3.9}$$

where u_m is the guidance command input. Note that u_m is either the lateral acceleration command or the control surface deflection command. The former case requires an autopilot system.

The overall system dynamics for homing guidance is then represented as

$$\dot{x} = Fx + Gu + Lw \tag{3.10}$$

where $u = u_m,\ w = u_t$ and the state variable x is defined as

$$x = \left[\begin{array}{ccc} x_k & x_t & x_m \end{array}\right]^T \tag{3.11}$$

The matrices $F,\ G,\ L$ are given as

$$\begin{gathered} x = \left[\begin{array}{ccc} x_k & x_t & x_m \end{array}\right]^T \\ F = \begin{bmatrix} F_k & F_{ct} & F_{cm} \\ 0 & F_t & 0 \\ 0 & 0 & F_m \end{bmatrix} \\ F_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_{ct} = \begin{bmatrix} 0 \\ H_t \end{bmatrix}, \quad F_{cm} = \begin{bmatrix} 0 \\ -H_m \end{bmatrix}, \\ G = \begin{bmatrix} 0 \\ 0 \\ G_m \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ G_t \\ 0 \end{bmatrix}, \end{gathered} \tag{3.12}$$

Then, LQ optimal guidance discussed in Section 2 is applied with the cost function of (2.21) and the terminal condition

$$R_f^T x_f = 0, \quad R_f = \left[\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \end{array}\right]^T \tag{3.13}$$

for zero miss distance. The optimal guidance command is obtained in a straight manner as described in Section 2:

$$u = G^T R Q^{-1} R^T x \tag{3.14}$$

$$\dot{R} = -F^T R, \quad R(t_f) = R_f \tag{3.15}$$

$$\dot{Q} = R^T G G^T R, \quad Q(t_f) = 0 \tag{3.16}$$

Let ZEM denote the zero-effort miss, which is defined as the miss obtained by setting the guidance command to zero from the current time to the terminal time. Using (3.15), we can show that

$$ZEM = R^T x = R_k^T x_k + R_t^T x_t + R_m^T x_m \tag{3.17}$$

Define a scalar variable $p(t)$ as

$$p = G^T R = G_m^T R_m \tag{3.18}$$

and denote $Q(t)$ as $q(t)$. Then, (3.16) is rewritten as

$$\dot{q} = R^T G G^T R = R_m^T G_m G_m^T R_m, \quad q(t_f) = 0 \tag{3.19}$$

or

$$\dot{q} = p^2, \quad q(t_f) = 0 \tag{3.20}$$

The guidance gain $\Lambda(t)$ is defined as

$$\frac{\Lambda}{t_{go}^2} = \frac{p}{q} \tag{3.21}$$

Then, the optimal guidance command is expressed as

$$u^* = \frac{p}{q}(ZEM) = \frac{\Lambda}{t_{go}^2}(ZEM). \tag{3.22}$$

Closed-Form Solution of the Optimal Homing Guidance:

The solution to the optimal homing guidance problem is obtained by integrating (3.15) and (3.16). A tedious calculation gives $p(t)$ and $q(t)$ as [27]

$$p(t_{go}) = G_m^T[I - e^{F_m^T t_{go}}](F_m^{-2})^T H_m^T + t_{go} G_m^T (F_m^{-1})^T H_m^T \tag{3.23}$$

$$\begin{aligned} q(t_{go}) = & -k_2^2 t_{go} - k_1 k_2 t_{go}^2 - \frac{1}{3} k_1^2 t_{go}^3 - 2k_2 H_m F_m^{-3}(I - e^{F_m t_{go}}) G_m \\ & + 2k_1 \{ H_m F_m^{-3} e^{F_m t_{go}} + H_m F_m^{-4}(I - e^{F_m t_{go}}) \} G_m \\ & + (H_m F_m^{-2}) X (H_m F_m^{-2})^T \end{aligned} \tag{3.24}$$

where $k_1 = H_m F_m^{-1} G_m$ and $k_2 = H_m F_m^{-2} G_m$, and the matrix X satisfies

$$F_m X + X F_m^T = G_m G_m^T - e^{F_m t_{go}} G_m G_m^T e^{F_m^T t_{go}} \tag{3.25}$$

The optimal guidance gain $\Lambda(t_{go})$ is directly computed form the solutions of $p(t_{go})$ and $q(t_{go})$. It is noted that $p(t_{go})$ and $q(t_{go})$ are irrelevant to the target dynamics, and so is $\Lambda(t_{go})$. However, ZEM is dependent on the target dynamics as well as the missile dynamics. For small LOS angles, the optimal guidance command can be expressed as

$$u^* = \Lambda(t_{go}) \left[\dot{\sigma} V_c + \frac{R_t^T x_t + R_m^T x_m}{t_{go}^2} \right] \tag{3.26}$$

It is observed that the first term of the optimal guidance law of (3.31) takes the form of proportional navigation but the guidance gain is, however, time varying. The closed-form solution of the optimal guidance law does not require iterative numerical methods but requires the evaluation of matrix exponential functions and a Lyapunov equation solver.

Optimal Guidance Gain for Infinitely Large Time-To-Go:

Form (3.23) it is seen that

$$p_\infty = G_m^T R_{m\infty} \approx (H_m F_m^{-1} G_m)^T t_{go} \tag{3.27}$$

where the subscript ∞ denotes the limit value of the variable for the case $t_{go} \to \infty$. The steady-state gain from u to a_M of the missile dynamics is defined as

$$k_s = -H_m F_m^{-1} G_m \tag{3.28}$$

Then, it is easy to see that

$$p_\infty \approx -k_s t_{go}, \quad q_\infty \approx -\frac{1}{3} k_{ss}^2 t_{go}^3 \tag{3.29}$$

As $t_{go} \to \infty$, the optimal guidance command can be approximated as

$$u_\infty^* = \frac{p_\infty}{q_\infty}(Z + \nu t_{go} + R_{m\infty}^T x_m) \approx \frac{3}{k_s t_{go}^2}(z + \nu t_{go} + (H_m F_m^{-1})^T x_m t_{go}) \tag{3.30}$$

The guidance gain is given as

$$\Lambda_\infty = \frac{3}{k_s} \tag{3.31}$$

If an autopilot system is employed, then $k_s = 1$ and we have PN of a $\Lambda_\infty = 3$ as expected.

Optimal Guidance Gain for Very Small Time-To-go:

Note that $e^{F_m^t t_{go}}$ can be expressed as a series of t_{go} for an arbitrarily small time-to-go.

$$e^{F_m^t t_{go}} = I + F_m^t t_{go} + \frac{1}{2}(F_m^t t_{go})^2 + \frac{1}{6}(F_m^t t_{go})^3 + \cdots \tag{3.32}$$

Substitute (3.32) into (3.23) to obtain that

$$p_o = G_m^T R_{m_o} = -\sum_{i=1}^{\infty} \frac{1}{(i+1)!} \mu_i t_{go}^{(i+1)} \tag{3.33}$$

where μ_i is defined as $\mu_i = H_m F_m^{i-1} G_m$ which is identified as the i-th Markov parameter of the missile dynamic model represented by $(F_m,\ G_m,\ H_m)$ [28]. Suppose that μ_k is the first nonzero Markov parameter. Then, we can show that, as the time-to-go goes to zero,

$$\Lambda_o = \frac{p_o}{q_o} t_{go}^2 \approx \frac{[2(k+1)+1](k+1)!}{\mu_k t_{go}^k} \tag{3.34}$$

For the transfer function of the missile dynamics expressed as

$$T_m(S) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n} \tag{3.35}$$

the coefficients of the numerator of $T_m(S)$ are related to the Markov parameters as follows [28];

$$b_1 = \mu_1, \quad b_2 = \mu_2 + a_1\mu_1, \quad b_3 = \mu_3 + a_1\mu_2 + a_2\mu_1 \cdots \tag{3.36}$$

Eq. (3.36) implies that if the pole excess number of $T_m(S)$ is k , then μ_k is the first nonzero Markov parameters. Thus, the larger the pole excess number is, the faster the guidance gain increases, as shown in (3.34). Also, the sign of the optimal guidance gain is determined by the sign of μ_k.

Tail-controlled and canard-controlled missiles are most common for the air-to-air missile application. For tail-controlled systems, the transfer function is non-minimum phase, and the first Markov parameter is negative. Hence, the sign of the optimal guidance gain changes to a negative value as the time to go decreases to 0. For canard-controlled systems, the optimal control gain is always positive since the transfer function is minimum phase.

4. Polynomial Guidance

In this section, the basic idea of polynomial guidance techniques is described and recent studies are introduced. Generalized impact-angle-control guidance laws that are closely related with polynomial guidance are also discussed.

4.1. **Time-to-go Polynomial Guidance.** The idea of time-to-go polynomial guidance was developed early 2007 for missile guidance with zero terminal angle of attack [14]. To maximize the effectiveness of the warhead, a certain class of missile systems requires that the angle of attack at the impact time is zero. This requirement can be rigorously handled only if the rotational dynamics of the vehicle is included in the problem formulation and the angle of attack is one of the state variables. Fortunately, the constraint on the terminal angle of attack can be replaced by a constraint on the terminal maneuver acceleration if the rotational motion of the vehicle near the target is minimal. The terminal acceleration can easily be made zero by applying the time-to-go polynomial guidance method for which the missile lateral acceleration is assumed to take the form of

$$a_M = c_m t_{go}^m + c_n t_{go}^n \tag{4.1}$$

Note that the zero terminal acceleration condition is already satisfied as long as m and n are positive and the coefficients c^m and c^n are finite. These coefficients are determined to satisfy the conditions on terminal miss and velocity; for example, $z(t_f) = 0$ and $\nu(t_f) = 0$. Once these coefficients are calculated, we can express the missile acceleration a_M in a feedback form

$$a_M = \frac{k_1}{t_{go}^2} z + \frac{k_2}{t_{go}} \nu \tag{4.2}$$

where the gravity is neglected for simplicity. It is interesting to observe that the missile acceleration takes the same form as the LQ guidance laws. The feedback gains k_1 and k_2 are dependent to m and n as shown below.

$$k_1 = (m+2)(n+2), \quad k_2 = (m+n+3) \tag{4.3}$$

The values of k_1 and k_2 for several combinations of m and n are given in Table 1. Note that the time-to-go polynomial guidance law with $m = 0$ and $n = 1$ is identical to the LQ optimal guidance law for impact angle control shown in (2.31).

TABLE 1. Examples of Polynomial Guidance Laws

Polynomial Type	k_1	k_2	Remarks
$m = 0,\ n = 1$	6	4	$a_M(t_f) \neq 0$
$m = 1,\ n = 2$	12	6	$a_M(t_f) = 0$
$m = 2,\ n = 3$	15	7	$a_M(t_f) = 0$
$m = 2,\ n = 3$	20	8	$a_M(t_f) = \dot{a}_M(t_f) = 0$

The merit of time-to-go polynomial guidance is that the trajectory shaping can be done easily by testing several combinations of m and n. Furthermore, the guidance performance is less sensitive to the time-to-go estimation error if the lateral acceleration and its time derivatives are designed to converge to zero as t_{go} goes to 0. Extensive analysis of the characteristics of time-to-go polynomial guidance has been conducted by Lee et al. [15]. In this work, the maximum acceleration command, the range of the look angle (the angle between the missile's X-axis and the LOS) are analyzed in details, a method for selection of the polynomial type is proposed, and time-to-go estimation is addressed. Inspired by the method used in [24], Kim et al. [16] have extended polynomial guidance to control impact time as well as impact angle, using an additional constant in the form of the lateral command;

$$a_M = c_m t_{go}^m + c_n t_{go}^n + c_l \tag{4.4}$$

where c_l is determined to meet the impact-time requirement. Another study on time-to-go polynomial guidance is observability enhancement for passive seekers [17]. Since the range information is not provided by a passive seeker, the missile needs to exert additional lateral maneuvers to improve target observability. Adding a term proportional to z as

$$a_M = \frac{k_1}{t_{go}^2} z + \frac{k_2}{t_{go}} \nu + k_3 z \tag{4.5}$$

it is possible to produce continuous oscillatory motions in the lateral position. This type of perturbations prevents the LOS rate from converging to zero quickly.

Time-to-go (or range-to-go) polynomial guidance is not based on LQ optimal guidance but a polynomial guidance law is a solution of an LQ optimal guidance problem as proved by Lee et al. [18]. Further discussion on this issue is given in the next section.

4.2. **Generalized Impact-Angle-Control Guidance Laws.** Time-to-go polynomial guidance laws take the form of

$$u^* = \frac{k_1}{t_{go}^2} z + \frac{k_2}{t_{go}} \nu \tag{4.6}$$

and the LQ optimal guidance laws considered in Section 2 are reduced to the same form. Even for arbitrary weighting functions the LQ guidance laws can be expressed by this form although k_1 and k_2 are no longer constant. Lee et al. [18] have studied the inverse problem of finding the cost function associated with the guidance law of (4.6). This study proves that there exists an LQ optimal guidance problem for any arbitrary combination of k_1 and k_2. The cost function of the corresponding LQ problem has quadratic penalties on the states as well as on the control. Furthermore, this study analyzes the domain of (k_1, k_2) that produces zero miss and zero terminal impact angle, and then investigates the characteristics of the optimal trajectory. Table 2 provides the summary on the trajectory shapes determined by the choice of k_1 and k_2.

From (4.3) and Table 2 we observe that a polynomial guidance law belongs to Category 1 if the following conditions are satisfied:

$$mn \geq 0, \quad (m-n)^2 > 0, \quad m+n > 0 \tag{4.7}$$

Note that the conditions of (4.7) are satisfied if m and n are two different nonnegative numbers. In addition, we also observe that the generalized impact-angle-control guidance laws of Category 2 and 3 cannot be obtained as long as m and n are real numbers.

TABLE 2. Trajectory Shapes of Generalized Impact-Angle-Control Guidance

Category	Domain	Trajectory Shape
1	$2(k_2-1) \leq k_1 < \left(\frac{k_2+1}{2}\right)^2, \ k_2 > 3$	Polynomial
2	$k_1 = \left(\frac{k_2+1}{2}\right)^2, \ k_2 > 3$	Polynomial + Logarithmic
3	$k_1 > \left(\frac{k_2+1}{2}\right)^2, k_2 > 3$	Polynomial + Logarithmic +Trigonometric

The analytical solutions of generalized impact-angle-control guidance with dynamic lag has been obtained by Lee et al. [19, 20]. The solution of the LQ optimal guidance law for impact angle control is first obtained in [19], and then extended for the class of generalized impact-angle-control guidance law in [20]. The analytic solutions can be utilized for investigating the divergent behavior of the guidance loop near the impact time and for selecting suitable guidance laws to minimize the terminal miss produced by the dynamic lag.

5. CONCLUSION

LQ optimal guidance for missile systems is one of the most successful applications of optimal control theory. It has been able to meet various mission objectives such as zero miss (perfect intercept), specified impact angle, and specified impact time in an energy efficient way by using a simple feedback control structure. Although there still exist many unsolved problems associated with time-varying velocity, constrained maneuver capability, and constrained seeker FOV, LQ optimal guidance and its extensions are being utilized to obtain practical solutions through simplification and approximations. Due to the richness of its theory, flexibility, and application experience, LQ optimal guidance methodology is expected to remain as the most viable tool for missile guidance in the future.

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