

A NOTE ON STRONGLY *-CLEAN RINGS

JIAN CUI AND ZHOU WANG

ABSTRACT. A $*$ -ring R is called (strongly) $*$ -clean if every element of R is the sum of a projection and a unit (which commute with each other). In this note, some properties of $*$ -clean rings are considered. In particular, a new class of $*$ -clean rings which called strongly π - $*$ -regular are introduced. It is shown that R is strongly π - $*$ -regular if and only if R is π -regular and every idempotent of R is a projection if and only if $R/J(R)$ is strongly regular with $J(R)$ nil, and every idempotent of $R/J(R)$ is lifted to a central projection of R . In addition, the stable range conditions of $*$ -clean rings are discussed, and equivalent conditions among $*$ -rings related to $*$ -cleanness are obtained.

1. Introduction

Rings in which every element is the product of a unit and an idempotent are said to be *unit regular*. Recall that an element of a ring R is *clean* if it is the sum of an idempotent and a unit, and R is *clean* if every element of R is clean (see [12]). Clean rings were introduced by Nicholson in relation to exchange rings and have been extensively studied since then. Recently, Wang et al. [16] showed that unit regular rings have idempotent stable range one (i.e., whenever $aR + bR = R$ with $a, b \in R$, there exists $e^2 = e \in R$ such that $a + be \in U(R)$, written $\text{isr}(R) = 1$ for short), and rings with $\text{isr}(R) = 1$ are clean. In 1999, Nicholson [13] called an element of a ring R *strongly clean* if it is the sum of a unit and an idempotent that commute with each other, and R is *strongly clean* if each of its elements is strongly clean. Clearly, a strongly clean ring is clean, and the converse holds for an abelian ring (that is, all idempotents in the ring are central). Local rings and strongly π -regular rings are well-known examples of strongly clean rings.

A ring R is a *$*$ -ring* (or *ring with involution*) if there exists an operation $*$: $R \rightarrow R$ such that for all $x, y \in R$

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad \text{and} \quad (x^*)^* = x.$$

An element p of a $*$ -ring is a *projection* if $p^2 = p = p^*$. Obviously, 0 and 1 are

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projections of any \ast -ring. A \ast -ring R is \ast -regular [2] if for every x in R there exists a projection p such that $xR = pR$. Following Vaš [15], an element of a \ast -ring R is (strongly) \ast -clean if it can be expressed as the sum of a unit and a projection (that commute), and R is (strongly) \ast -clean if all of its elements are (strongly) \ast -clean. Clearly, \ast -clean rings are clean and strongly \ast -clean rings are strongly clean. It was shown in [7, 11] that there exists a clean \ast -ring but not \ast -clean, and unit regular \ast -regular rings (which called \ast -unit regular rings in [7]) need not be strongly \ast -clean, which answered two questions raised by Vaš in [15].

In this note, we continue the study of (strongly) \ast -clean rings. In Section 2, several basic properties of (strongly) \ast -clean rings are investigated. Motivated by the close relationship between strong π -regularity and strong cleanness, we introduce the concept of strongly π - \ast -regular rings in Section 3. The structure of strongly π - \ast -regular rings is considered and some properties of extensions are discussed. As we know, it is still an open question that whether a strongly clean ring has idempotent stable range one, or even has stable range one (see [13]). In Section 4, we extend $\text{isr}(R) = 1$ to the \ast -version. We call a \ast -ring R have *projection stable range one* (written $\text{psr}(R) = 1$) if, for any $a, b \in R$, $aR + bR = R$ implies that $a + bp$ is a unit of R for some projection $p \in R$. It is shown that if R is strongly \ast -clean, then $\text{psr}(R) = 1$, and if $\text{psr}(R) = 1$, then R is \ast -clean. Furthermore, several equivalent conditions among (strongly) clean rings, (strongly) \ast -clean rings and \ast -rings with projection (idempotent) stable range one are obtained.

Throughout this paper, rings are associative with unity. Let R be a ring. The set of all idempotents, all nilpotents and all units of R are denoted by $\text{Id}(R)$, R^{nil} and $U(R)$, respectively. For $a \in R$, the commutant of a is denoted by $\text{comm}(a) = \{x \in R : ax = xa\}$. We write $M_n(R)$ for the ring of all $n \times n$ matrices over R whose identity element we write as I_n . Let \mathbb{Z}_n be the ring of integers modulo n . For a \ast -ring R , the symbol $P(R)$ stands for the set of all projections of R .

2. \ast -clean rings

In this section, some basic properties of \ast -clean rings are discussed, and several examples related to \ast -cleanness are given.

Example 2.1. (1) Units, elements in $J(R)$ and nilpotents of a \ast -ring R are \ast -clean.

(2) Idempotents of a \ast -regular rings are \ast -clean.

Proof. (1) It is obvious.

(2) Let R be \ast -regular and $e \in \text{Id}(R)$. Then there exists a projection p such that $(1 - e)R = pR$. So we have $1 - e = p(1 - e)$ and $p = (1 - e)p$, and hence $ep = 0$. Note that $(e - p)(e - p) = e - ep - pe + p = e + p(1 - e) = e + (1 - e) = 1$. So $e - p \in U(R)$, and $e = p + (e - p)$ is \ast -clean in R . \square

By Example 2.1, every local ring with involution $*$ is $*$ -clean. In [15], Vaš asked whether there is an example of a $*$ -ring that is clean but not $*$ -clean. It was answered affirmatively in [7] and [11]. In fact, one can construct some counterexamples based on the following.

Example 2.2. Let R be a boolean $*$ -ring. Then R is $*$ -clean if and only if $*$ = 1_R is the identity map of R . In particular, $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with $(a, b)^* = (b, a)$ is clean but not $*$ -clean.

Proof. Note that every boolean ring is clean. Suppose that R is $*$ -clean. Given any $a \in R$. Then $-a = p + u = p + 1 = p - 1$ for some $p \in P(R)$. So we have $a = 1 - p \in P(R)$. Thus, $a^* = a$, which implies $*$ = 1_R . Conversely, if $*$ = 1_R , then every idempotent of R is a projection. Thus, R is $*$ -clean. \square

Lemma 2.3. Let R be a $*$ -ring. If $2 \in U(R)$, then for any $u^2 = 1$, $u^* = u \in R$ if and only if every idempotent of R is a projection.

Proof. (\Rightarrow) Let $e \in Id(R)$. Then $(1 - 2e)^2 = 1$. So we have $2e = 2e^*$, and thus $2(e - e^*) = 0$. Since $2 \in U(R)$, $e = e^*$. As desired.

(\Leftarrow) Given $u \in R$ with $u^2 = 1$. Then $\frac{u+1}{2} \in Id(R)$ since $(\frac{u+1}{2})^2 = \frac{u^2+2u+1}{4} = \frac{u+1}{2}$. Since every idempotent of R is a projection, it follows from $(\frac{u+1}{2})^* = \frac{u+1}{2}$ that $u^* = u$. \square

The $*$ -ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in Example 2.2 reveals that “ $2 \in U(R)$ ” in Lemma 2.3 cannot be removed.

Corollary 2.4. Let R be a $*$ -ring with $2 \in U(R)$. The following are equivalent:

- (1) R is clean and every unit of R is self-adjoint (i.e., $u^* = u$ for every $u \in U(R)$).
- (2) R is $*$ -clean and $*$ = 1_R .

Proof. (2) \Rightarrow (1) is trivial.

(1) \Rightarrow (2). Let $a \in R$. Then $a = e + u$ for some $e \in Id(R)$ and $u \in U(R)$. Note that $(1 - 2e)^2 = 1$. By Lemma 2.3, $e^* = e$. Thus $a \in R$ is $*$ -clean and $a^* = a$, and so $*$ = 1_R . \square

Recall that an element t of a $*$ -ring R is self-adjoint square root of 1 if $t^2 = 1$ and $t^* = t$.

Theorem 2.5. Let R be a $*$ -ring, the following are equivalent:

- (1) R is $*$ -clean and $2 \in U(R)$.
- (2) Every element of R is a sum of a unit and a self-adjoint square root of 1.

Proof. (1) \Rightarrow (2). Let $a \in R$. Then $\frac{1+a}{2} = p + u$ for some $p \in P(R)$ and $u \in U(R)$. It follows that $a = (2p-1)+2u$ where $(2p-1)^* = 2p-1$, $(2p-1)^2 = 1$ and $2u \in U(R)$.

(2) \Rightarrow (1). We first show that $2 \in U(R)$. By hypothesis, $1 = x + v$ with $x^2 = 1$ and $v \in U(R)$. So we have $(1 - v)^2 = x^2 = 1$, which implies that

$v^2 = 2v$. Since v is a unit, $v = 2 \in U(R)$. Given any $a \in R$, then there exist $y, w \in R$ satisfying $2a - 1 = y + w$ with $y^* = y$, $y^2 = 1$ and $w \in U(R)$. Thus, $a = \frac{y+1}{2} + \frac{w}{2}$ is a $*$ -clean expression since $(\frac{y+1}{2})^* = \frac{y+1}{2}$, $(\frac{y+1}{2})^2 = \frac{y+1}{2}$ and $\frac{w}{2} \in U(R)$. \square

Camillo and Yu [5] showed that if R is a ring in which 2 is a unit, then R is clean if and only if every element of R is the sum of a unit and a square root of 1. Indeed, by the proof of Theorem 2.5, the condition $2 \in U(R)$ is also necessary.

Proposition 2.6. *The following are equivalent for a $*$ -ring R :*

- (1) R is $*$ -clean and $0, 1$ are the only projections.
- (2) R is clean ring and $0, 1$ are the only idempotents.
- (3) R is a local ring.

Proof. (2) \Rightarrow (3) follows from [14, Lemma 14] and (3) \Rightarrow (1) follows by Example 2.1.

(1) \Rightarrow (2). It suffices to show that the only idempotents in R are 0 and 1. For $e^2 = e \in R$, the hypothesis implies that $e = p + u$ where $p \in P(R) = \{0, 1\}$ and $u \in U(R)$. If $p = 0$, then $e = u$ is a unit, so $e = 1$. If $p = 1$, then $1 - e = -u \in U(R)$, and hence $e = 0$. As required. \square

Let I be an ideal of a $*$ -ring R . We call I is $*$ -invariant if $I^* \subseteq I$. In this case, the involution $*$ of R can be extended to the factor ring R/I , which is still denoted by $*$.

Lemma 2.7. *Let R be $*$ -clean. If I is a $*$ -invariant ideal of R , then R/I is $*$ -clean. In particular, $R/J(R)$ is $*$ -clean.*

Proof. Since the homomorphism image of a projection (resp., unit) is also a projection (resp., unit), the result follows.

Next we only need to prove that $J(R)$ is $*$ -invariant. For any $a^* \in (J(R))^*$, we show that $a^* \in J(R)$. Note that $a \in J(R)$. Take any $x \in R$. Then $1 - x^*a \in U(R)$. Thus $1 - a^*x = (1 - x^*a)^*$ is a unit of R , as desired. \square

Let R be a $*$ -ring. Then $*$ induces an involution of the power series ring $R[[x]]$, denoted by $*$, where $(\sum_{i=0}^{\infty} a_i x^i)^* = \sum_{i=0}^{\infty} a_i^* x^i$.

Proposition 2.8. *Let R be a $*$ -ring. Then $R[[x]]$ is $*$ -clean if and only if R is $*$ -clean.*

Proof. Suppose that $R[[x]]$ is $*$ -clean. Note that $R \cong R[[x]]/(x)$ and (x) is a $*$ -invariant ideal of $R[[x]]$. By Lemma 2.7, R is $*$ -clean. Conversely, assume that R is $*$ -clean. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Write $a_0 = p + u$ with $p \in P(R)$ and $u \in U(R)$. Then $f(x) = p + (u + \sum_{i=1}^{\infty} a_i x^i)$, where $p \in P(R) \subseteq P(R[[x]])$ and $u + \sum_{i=1}^{\infty} a_i x^i \in U(R[[x]])$. Hence $f(x)$ is $*$ -clean in $R[[x]]$. \square

According to [14, Proposition 13], the polynomial ring $R[x]$ is never clean. Hence, $R[x]$ is not $*$ -clean for any involution $*$.

3. Strongly π -*-regular rings

Strong π -regularity is closely related to strong cleanness. In this section, we introduce the notion of strongly π -*-regular rings which can be viewed as *-versions of strongly π -regular rings. The structure and properties of strongly π -*-regular rings are given.

Lemma 3.1 ([11, Lemma 2.1]). *Let R be a *-ring. If every idempotent of R is a projection, then R is abelian.*

Due to [7], an element a of a *-ring R is *strongly *-regular* if $a = pu = up$ with $p \in P(R)$ and $u \in U(R)$; R is *strongly *-regular* if each of its elements is strongly *-regular. By [7, Proposition 2.8], any strongly *-regular element is strongly *-clean.

Theorem 3.2. *Let R be a *-ring. Then the following are equivalent for $a \in R$:*

- (1) *There exist $e \in P(R)$, $u \in U(R)$ and an integer $m \geq 1$ such that $a^m = eu$ and a, e, u commute with each other.*
- (2) *There exist $f \in P(R)$, $v \in U(R)$ such that $a = f + v$, $fv = vf$ and $af \in R^{\text{nil}}$.*
- (3) *There exists $p \in P(R)$ such that $p \in \text{comm}(a)$, $ap \in U(pRp)$ and $a(1 - p) \in R^{\text{nil}}$.*
- (4) *There exists $b \in \text{comm}(a)$ such that $(ab)^* = ab$, $b = bab$ and $a - a^2b \in R^{\text{nil}}$.*

Proof. (1) \Rightarrow (2). Write $f = 1 - e$. Clearly, $f \in P(R)$ and $a^m - f \in U(R)$ with the inverse $u^{-1}e - f$. From $af = fa$, we have $a - f := v$ is a unit of R (since $(a - f)(a^{m-1} + a^{m-2}f + \dots + af + f) = a^m - f \in U(R)$) and $fv = vf$. It is clear that $(af)^m = a^m f = 0$.

(2) \Rightarrow (3). Set $p = 1 - f$. Then $p \in P(R)$, $ap = pa = vp \in U(pRp)$ and $a(1 - p) = af \in R^{\text{nil}}$.

(3) \Rightarrow (4). By (3), $aw = wa = p$ for some $w \in U(pRp)$. So we obtain $[a - (1 - p)][w - (1 - p)] = 1 - a(1 - p) \in U(R)$ since $a(1 - p)$ is nilpotent, which implies that $a - (1 - p) \in U(R)$. Let $b = [a - (1 - p)]^{-1}p$. Then $b \in \text{comm}(a)$, $bp = b$ and $ab = [a - (1 - p)]b = p \in P(R)$. Thus $(ab)^* = ab$, $b = bp = bab$ and $a - a^2b = a(1 - ab) = a(1 - p) \in R^{\text{nil}}$.

(4) \Rightarrow (1). Let $e = ab$. Then $(ab)^* = ab$ implies $e^* = e$, and $bab = b$ yields $e^2 = e$. So $e \in P(R)$. As $a - a^2b \in R^{\text{nil}}$, $a^m = a^m e$ for some integer $m \geq 1$. Take $u = a^m + (1 - e)$ and $u' = b^m e + (1 - e)$. Then $uu' = u'u = 1$. Hence, $u \in U(R)$ and $a^m = a^m e = ue$ with a, e, u commuting with each other. \square

Recall that an element a of a ring R is *strongly π -regular* if $a^n \in a^{n+1}R \cap Ra^{n+1}$ for some $n \geq 1$ (equivalently, $a^n = eu$ with $e \in Id(R)$, $u \in U(R)$ and a, e, u all commute [13]); R is *strongly π -regular* if every element of R is strongly π -regular. Based on the above, we introduce the following concept.

Definition 3.3. Let R be a $*$ -ring. An element $a \in R$ is called strongly π - $*$ -regular if it satisfies the conditions in Theorem 3.2; R is called strongly π - $*$ -regular if every element of R is strongly π - $*$ -regular.

Corollary 3.4. Any strongly $*$ -regular element is strongly π - $*$ -regular, and any strongly π - $*$ -regular element is strongly $*$ -clean.

Example 3.5. (1) Let $R = \mathbb{Z}_4$ and $*$ = 1_R . Then R is strongly π - $*$ -regular. However, $2 \in R$ is not strongly $*$ -regular.

(2) Let R be a local domain with involution $*$ and $J(R) \neq 0$. Note that $P(R) = Id(R) = \{0, 1\}$. So R is strongly $*$ -clean by Proposition 2.6, but any power of a nonzero element in $J(R)$ can not expressed as the product of a projection and a unit.

Recall that a ring R is π -regular if for any $a \in R$, there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. Strongly π -regular rings and regular rings are π -regular (see [13]). A ring R is *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. Abelian rings are directly finite.

Theorem 3.6. The following are equivalent for a $*$ -ring R :

- (1) R is strongly π - $*$ -regular.
- (2) R is π -regular and every idempotent of R is a projection.
- (3) For any $a \in R$, there exist $n \geq 1$ and $p \in P(R)$ such that $a^n R = pR$, and R is abelian.
- (4) For any $a \in R$, there exists $n \geq 1$ such that a^n is strongly $*$ -regular.
- (5) For any $a \in R$, there exist $p \in P(R)$ and $u \in U(R)$ such that $a = p + u$, $ap \in R^{\text{nil}}$; and $v^{-1}qv$ is a projection for all $v \in U(R)$ and all $q \in P(R)$.

Proof. (1) \Rightarrow (2). Note that every strongly π - $*$ -regular ring is strongly π -regular and strongly $*$ -clean. Thus R is a π -regular ring. By [11, Theorem 2.2], every idempotent of R is a projection.

(2) \Rightarrow (3). For any $a \in R$, there exists $n \geq 1$ such that $a^n = a^n x a^n$ for some $x \in R$. Write $a^n x = p$. Then $p \in P(R)$ and $a^n = p a^n$. It is clear that $a^n R = pR$. In view of Lemma 3.1, R is abelian.

(3) \Rightarrow (4). Let $e \in Id(R)$. Then $eR = pR$ for some $p \in P(R)$. Since R is abelian, we have $e = pe = ep = p$. Thus, every idempotent of R is a projection. Given $a \in R$, there exist $n \geq 1$ and $q \in P(R)$ such that $a^n R = qR$. So one gets $a^n = q a^n$ and $q = a^n x$ for some $x \in R$, which implies $a^n = a^n x a^n$. Next we show that $a^n - (1 - q)$ is invertible. Note that $[a^n - (1 - q)][xq - (1 - q)] = 1$. Then $a^n - (1 - q) := u \in U(R)$ since R is directly finite. Multiplying the equation $a^n - (1 - q) = u$ by p yields $a^n = a^n q = uq = qu$, which implies that a^n is strongly $*$ -regular.

(4) \Rightarrow (5). For $e \in Id(R)$, $e = qv = vq$ for some $q \in P(R)$ and $v \in U(R)$ by the assumption. Then $e = qv = e^2 = qv^2$, and so we obtain $q = qv = e$, which implies that every idempotent of R is a projection. Clearly, $v^{-1}qv$ is a projection for all $v \in U(R)$ and all $q \in P(R)$. Given $a \in R$ as in (4), $a^n =$

$(1 - p)w = w(1 - p)$ for some $p \in P(R)$ and $w \in U(R)$. Note that R is abelian. So we have $a^n p = (ap)^n = 0$ and $(a - p)[a^{n-1}w^{-1}(1 - p) - \sum_{i=0}^{n-1} a^i p] = 1$, and hence $a - p \in U(R)$ as R is directly finite.

(5) \Rightarrow (1). By (5), every element of R is *-clean. In view of [11, Theorem 2.2], R is abelian. Thus R is strongly π -*-regular by Theorem 3.2(2). \square

Corollary 3.7. *Let R be a *-ring. The following are equivalent:*

- (1) R is strongly π -*-regular.
- (2) $R/J(R)$ is strongly π -*-regular with $J(R)$ nil, every projection of R is central and every projection of $R/J(R)$ is lifted to a projection of R .
- (3) $R/J(R)$ is strongly *-regular with $J(R)$ nil, and every idempotent of $R/J(R)$ is lifted to a central projection of R .

Proof. Write $\bar{R} = R/J(R)$. By Lemma 2.7, \bar{R} is a *-ring.

(1) \Rightarrow (2). Clearly, \bar{R} is strongly π -*-regular. As R is strongly π -regular, for any $a \in J(R)$, there exist $m \geq 1$, $e \in Id(R)$ and $u \in U(R)$ such that $e = a^m u \in J(R)$. So $a^m = eu^{-1} = 0$, which implies that $J(R)$ is nil. Note that R is strongly *-clean. So the rest follows from [11, Corollary 2.11].

(2) \Rightarrow (3). By virtue of [11, Corollary 2.11], \bar{R} is reduced (i.e., $\bar{R}^{nil} = 0$), and every idempotent of \bar{R} is lifted to a central projection of R . So we only need to prove that \bar{R} is strongly *-regular. Given any $x \in \bar{R}$. By Theorem 3.2, there exist $p \in P(\bar{R})$ and $v \in U(\bar{R})$ such that $a = p + v$, $vp = pv$ and $ap \in \bar{R}^{nil} = 0$. It follows that $a = a(1 - p) = v(1 - p) = (1 - p)v$ is strongly *-regular in \bar{R} .

(3) \Rightarrow (1). Since \bar{R} is strongly regular, it is reduced clean. By [11, Corollary 2.11], every idempotent of R is a projection. Note that $J(R)$ is nil and \bar{R} is π -regular. So R is π -regular by [1, Theorem 4]. In view of Theorem 3.6, R is strongly π -*-regular. \square

Corollary 3.8. *Let R be a *-ring. Then R is strongly *-clean and π -regular if and only if R is strongly π -*-regular.*

Proof. If R is strongly *-clean and π -regular, by [11, Theorem 2.2], idempotents of R are projections. So R is strongly π -*-regular by Theorem 3.6. The other direction is clear. \square

For a *-ring R , the matrix ring $M_n(R)$ has a natural involution inherited from R : if $A = (a_{ij}) \in M_n(R)$, A^* is the transpose of (a_{ij}^*) (i.e., $A^* = (a_{ij}^*)^T = (a_{ji}^*)$). Henceforth we consider $M_n(R)$ as a *-ring with respect to this natural involution.

Corollary 3.9. *Let R be a *-ring. Then $M_n(R)$ is not strongly π -*-regular for any $n \geq 2$.*

Let R be a *-ring and $S = pRp$ with $p \in P(R)$. Then the restriction of $*$ on S will be an involution of S , which is also denoted by $*$.

Corollary 3.10. *If R is strongly π -*-regular, then so is eRe for any $e \in Id(R)$.*

Proof. Let $S = eRe$ with $e \in Id(R)$. By hypothesis, e is a projection of R . So S is a $*$ -ring. It is well known that S is strongly π -regular (see also [4, Lemma 39]). Clearly, every idempotent of $S (\subseteq R)$ is a projection. So the result follows by Theorem 3.6. \square

Let RG be the group ring of a group G over a ring R . According to [11, Lemma 2.12], the map $*$: $RG \rightarrow RG$ given by $(\sum_g a_g g)^* = \sum_g a_g^* g^{-1}$ is an involution of RG , and is denoted by $*$ again.

Corollary 3.11. *Let R be a $*$ -ring with artinian prime factors, $2 \in J(R)$ and G be a locally finite 2-group. Then R is strongly π - $*$ -regular if and only if RG is strongly π - $*$ -regular.*

Proof. Assume that R is strongly π - $*$ -regular. Then $Id(R) = P(R)$. In particular, R is abelian. So idempotents of R coincide with idempotents in RG by [8, Lemma 11], and hence every idempotent of RG is a projection. Since R is a ring with artinian prime factors and G is a locally finite 2-group, RG is a strongly π -regular ring by [10, Theorem 3.3]. In view of Theorem 3.6, RG is strongly π - $*$ -regular.

Conversely, R is strongly π -regular by [10, Proposition 3.4]. Note that $Id(R) \subseteq Id(RG)$ and all idempotents of RG are projections. By Theorem 3.6, R is strongly π - $*$ -regular. \square

Let \mathbb{C} be the complex field. It is well known that for any $n \geq 1$, the matrix ring $M_n(\mathbb{C})$ is strongly π -regular. However, $M_n(\mathbb{C})$ is not strongly π - $*$ -regular whenever $n \geq 2$ by Corollary 3.9. So it is interesting to determine when a matrix of $M_n(\mathbb{C})$ is strongly π - $*$ -regular. The set of all $n \times 1$ matrices over \mathbb{C} is denoted by \mathbb{C}^n .

Example 3.12. Let $S = M_n(\mathbb{C})$ with $*$ the transpose operation. Then A is strongly π - $*$ -regular if and only if there exist $e_1, e_2, \dots, e_n \in \mathbb{C}^n$ such that $e_i^* e_j = 0$ for $i = 1, \dots, r; j = r + 1, \dots, n$, and $A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1}$ with $P = (e_1, e_2, \dots, e_n) \in U(S)$, $C \in U(M_r(\mathbb{C}))$ and $N \in [M_{n-r}(\mathbb{C})]^{nil}$. In particular, any real symmetric matrix is strongly π - $*$ -regular.

Proof. Given $A \in S$. Assume that $rank(A) = r$. By the Jordan canonical decomposition, there exists $P = (e_1, e_2, \dots, e_n) \in U(S)$ such that

$$A = P \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

where $e_i \in \mathbb{C}^n$ for all i , $C \in U(M_r(\mathbb{C}))$ and $N \in [M_{n-r}(\mathbb{C})]^{nil}$. Write

$$B = P \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

Then one easily gets that $BA = AB$, $B = BAB$ and $A - A^2B = P \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} P^{-1}$ is nilpotent. Note that B satisfies the above conditions is unique (see [3]). In

view of Theorem 3.2, A is strongly π -*-regular if and only if $(AB)^* = AB$. Notice that $AB = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$ and

$$\begin{aligned} (AB)^* &= AB \\ \Leftrightarrow (P^{-1})^* \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^* &= P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \\ \Leftrightarrow (P^*P)^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (P^*P) &= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \\ \Leftrightarrow \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} (P^*P) &= (P^*P) \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \\ \Leftrightarrow P^*P &= \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \text{ with } V_1 \in U(M_r(\mathbb{C})) \text{ and } V_2 \in U(M_{n-r}(\mathbb{C})) \\ \Leftrightarrow e_i^*e_j &= 0 \text{ for all } i \in \{1, 2, \dots, r\}, j \in \{r + 1, r + 2, \dots, n\}, \end{aligned}$$

where

$$\begin{aligned} V_1 &= (e_1^*, e_2^*, \dots, e_r^*)^T (e_1, e_2, \dots, e_r); \\ V_2 &= (e_{r+1}^*, e_{r+2}^*, \dots, e_n^*)^T (e_{r+1}, e_{r+2}, \dots, e_n). \end{aligned}$$

If $A \in S$ is a real symmetric matrix, then there exists an orthogonal matrix P (i.e., $P^{-1} = P^T = P^*$) such that $A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$. So the result follows. \square

In view of [2, Proposition 3], the involution of a *-regular ring R is proper (i.e., $x^*x = 0$ implies that $x = 0$ for all $x \in R$).

Remark 3.13. If R is strongly π -*-regular, then for any $x \in R$, $x^*x = 0$ implies $x \in R^{\text{nil}}$. Indeed, by Theorem 3.2, there exist $p \in P(R)$ and $u \in U(R)$ such that $x^m = pu = up$ for some $m \geq 1$. Then $0 = (x^*)^m x^m = (x^m)^* x^m = u^* pu$, and thus $p = 0$, whence $x^m = 0$.

4. Stable range conditions

In [13], Nicholson asked whether every strongly clean ring has stable range one, and it is still open. Recall that a ring R is said to *have idempotent stable range one* (written $\text{isr}(R)=1$) provided that for any $a, b \in R$, $aR + bR = R$ implies that $a + be \in U(R)$ for some $e \in \text{Id}(R)$ (see [6, 16]). If e is an arbitrary element of R (not necessary an idempotent), then R is said to *have stable range one*. Clearly, if $\text{isr}(R) = 1$, then R is clean and has stable range one. We extend the notion of $\text{isr}(R) = 1$ to *-versions.

Definition 4.1. A *-ring R is said to have projection stable range one (written $\text{psr}(R) = 1$) if for any $a, b \in R$, $aR + bR = R$ implies there exists $p \in P(R)$ such that $a + bp \in U(R)$.

The following result is motivated by [6, Proposition 2].

Proposition 4.2. *Let R be a *-ring. The following are equivalent:*

- (1) $\text{psr}(R) = 1$.
- (2) For any $a, b \in R$, $aR + bR = R$ implies there exists $p \in P(R)$ such that $a + bp$ is right invertible.

- (3) For any $a, b \in R$, $aR + bR = R$ implies there exists $p \in P(R)$ such that $a + bp$ is left invertible.

Proof. The proof is similar to that of [6, Proposition 2].

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (3). Let $a, b \in R$ with $aR + bR = R$. Then there is a projection $p \in R$ such that $a + bp = u$ is right invertible. Assume that $uw = 1$ for some $w \in R$. Then $wR + (1 - wu)R = R$. So the hypothesis implies there exists $q \in P(R)$ such that $w + (1 - wu)q$ is right invertible. Note that $u[w + (1 - wu)q] = 1$. Thus $w + (1 - wu)q$ is also left invertible, and hence invertible. This implies that $u \in U(R)$.

(3) \Rightarrow (1). Given any $a, b \in R$ with $aR + bR = R$. Then there exists $p \in P(R)$ such that $a + bp$ is left invertible. We may let $v \in R$ with $v(a + bp) = 1$. Then $vR + 0R = R$. By hypothesis, we can find a projection q such that $v + 0q = v$ is left invertible. So v is a unit, which implies that $a + bp \in U(R)$. Therefore, $\text{psr}(R) = 1$. \square

For a $*$ -ring R , it is clear that if $\text{psr}(R) = 1$, then $\text{isr}(R) = 1$. However, there exists a $*$ -ring with $\text{isr}(R) = 1$ but not satisfies $\text{psr}(R) = 1$.

Example 4.3. Define the involution of \mathbb{Z}_2 by $*$: $x \mapsto x$. Let $S = M_2(\mathbb{Z}_2)$. Then S is a $*$ -ring. In view of [16, Corollary 3.4], $\text{isr}(S) = 1$ since S is unit regular. Notice that $P(S) = \{O, I_2, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S = S$. However, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P$ is not invertible for any $P \in P(S)$. Hence, $\text{psr}(S) \neq 1$.

From Example 4.3, one can also find that the projection stable range one property cannot be inherited to the matrix ring.

Proposition 4.4. Let R be a $*$ -ring. If $\text{psr}(R) = 1$, then R is $*$ -clean.

Proof. For any $a \in R$, the equation $aR + (-1)R = R$ implies that $a + (-1)p = u \in U(R)$ for some $p \in P(R)$. So $a = p + u$, and hence R is $*$ -clean. \square

According to [15, Proposition 4], the ring in Example 4.3 is $*$ -clean. So we conclude that the converse of Proposition 4.4 is not true.

Following Nicholson [12], a ring R is *exchange* if for every $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$. Clean rings are exchange, the converse holds whenever the rings are abelian. A $*$ -ring R is called *$*$ -abelian* if every projection of R is central [15].

Theorem 4.5. Let R be a $*$ -ring. The following are equivalent:

- (1) $\text{psr}(R) = 1$ and R is $*$ -abelian.
- (2) For any $a, b \in R$, $aR + bR = R$ implies there exists a projection $p \in \text{comm}(a)$ such that $a + bp \in U(R)$.
- (3) $\text{isr}(R) = 1$ and every idempotent of R is a projection.
- (4) R is clean (or exchange) and every idempotent of R is a projection.
- (5) R is $*$ -clean and $*$ -abelian.
- (6) R is strongly $*$ -clean.

(7) For every $a \in R$, there exists a projection $p \in aR$ such that $1 - p \in (1 - a)R$.

Proof. (1) \Rightarrow (2) and (3) \Rightarrow (4) are clear; (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) follows from [11, Theorem 2.2].

(2) \Rightarrow (3). We only need to show that all idempotents are projections. Let $e \in Id(R)$. Then $eR + (-1)R = R$. So there exists $p \in P(R)$ such that $ep = pe$ and $e - p \in U(R)$. Note that $(e - p)(1 - e - p) = (1 - e - p)(e - p) = 0$. Thus, $e = 1 - p \in P(R)$. Therefore, every idempotent of R is a projection.

(7) \Rightarrow (1). Let $e \in Id(R)$. Then there exists a projection $p \in eR$ such that $1 - p \in (1 - e)R$. So we obtain $p = ep$ and $1 - p = (1 - e)(1 - p)$. It follows that $e = p$, and thus $Id(R) = P(R)$. In view of Lemma 3.1, R is abelian. Note that R is exchange. Then by [6, Theorem 12], $isr(R) = 1$, and hence $psr(R) = 1$. \square

It is still unknown that whether strongly clean rings have stable range one ([13]). However, we have an affirmative answer of their *-versions.

Corollary 4.6. *If R is a strongly *-clean ring, then $psr(R) = 1$.*

The following example will reveal that the converse of Corollary 4.6 does not hold.

Example 4.7. Let $S = M_2(\mathbb{Z}_3)$. The involution of S is defined by $A \rightarrow A^*$, where A^* is the transpose of $A \in S$. Then S is not strongly *-clean by [7, Theorem 2.3]. Since S is unit regular, $isr(S) = 1$ by [16, Corollary 3.4]. In view of [9, Lemma 7], we have

$$Id(S) = \{O, I_2, \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \text{ with } yz = x - x^2\},$$

and

$$P(S) = \{O, I_2, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\}.$$

We next prove that $psr(S) = 1$. Assume on the contrary. Then there exist $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $AS + A'S = S$ but $A + A'P$ is not a unit for any $P \in P(S)$. That is,

$$\det(A + A'P) = 0.$$

This implies the following system of equations:

$$\begin{aligned} ad - bc = 0 & \quad \text{(i),} & ad' - b'c = 0 & \quad \text{(ii),} \\ a'd - bc' = 0 & \quad \text{(iii),} & a'd' - b'c' = 0 & \quad \text{(iv),} \\ ac' - a'c = bd' - b'd & \quad \text{(v).} \end{aligned}$$

On the other hand, as $isr(S) = 1$, there exists $E \in Id(S) \setminus P(S)$ such that $A + A'E \in U(S)$. Then E must be of the form $\begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}$ where $yz = x - x^2$. By Eqs. (i)-(iv), we obtain

$$\det(A + A'E) = (ac' - a'c)y - (bd' - b'd)z.$$

Next we show that $ac' - a'c = bd' - b'd = 0$.

Case 1. $c \neq 0$. Multiplying Eq. (v) by c and by substituting $b'c = ad'$, we have $(ac' - a'c)c = bd'c - b'dc = (bc - ad)d' = 0$ by Eq. (i). Thus, $ac' - a'c = bd' - b'd = 0$.

Case 2. $d \neq 0$. Multiplying Eq. (v) by d and by substituting $a'd = bc'$, we have $(bd' - b'd)d = ac'd - a'cd = (ad - bc)c' = 0$ by Eq. (i). So $ac' - a'c = bd' - b'd = 0$.

Case 3. $c = d = 0$. From Eqs. (ii) and (iii), we get $ad' = bc' = 0$. If $b \neq 0$, then $c' = 0$, it follows that $ac' - a'c = 0$. If $a \neq 0$, then $d' = 0$, and so $bd' - b'd = 0$. Thus $ac' - a'c = bd' - b'd = 0$.

Therefore, $\det(A + A'E) = (ac' - a'c)y - (bd' - b'd)z = 0$ for any case, which contradicts $A + A'E \in U(S)$. Hence, $\text{psr}(R) = 1$.

By Theorem 4.5, we have the following result immediately.

Corollary 4.8. *Let R be a $*$ -ring. If $\text{Id}(R) = P(R)$, then the following are equivalent:*

- (1) R is (strongly) clean.
- (2) R is exchange.
- (3) R is (strongly) $*$ -clean.
- (4) $\text{isr}(R) = 1$.
- (5) $\text{psr}(R) = 1$.

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JIAN CUI
DEPARTMENT OF MATHEMATICS
ANHUI NORMAL UNIVERSITY
WUHU 241000, P. R. CHINA
E-mail address: cui368@mail.ahnu.edu.cn

ZHOU WANG
DEPARTMENT OF MATHEMATICS
SOUTHEAST UNIVERSITY
NANJING 210096, P. R. CHINA
E-mail address: zhouwang@seu.edu.cn