The Journal of The Institute of Internet, Broadcasting and Communication (IIBC)
Vol. 15, No. 3, pp.25-33, Jun. 30, 2015. pISSN 2289-0238, eISSN 2289-0246

http://dx.doi.org/10.7236/JIIBC.2015.15.3.25
JIIBC 2015-3-4

Jacket 행렬의 성질과 특성

Properties and Characteristics of Jacket Matrices

양재승*, 박주용**, 이문호***

Jae-Seung Yang*, Ju-Yong Park, Moon-Ho Lee*****

요 약 양면을 뒤집어 입을 수 있는 Jacket처럼, 내부 및 외부 양 쪽 모두 호환이 가능한 행렬을 Jacket 행렬이라 한다. element-wise inverse와 block-wise inverse 과정을 통해 Jacket 행렬은 안 쪽 요소와 바깥쪽 요소 모두를 가진다. 이 개념은 1989년에 저자 중 한 명인 이문호 교수에 의해 이루어진 것으로서, 2000년에는 최종적으로 Jacket 행렬이라 부르게 되었다. 이것은 잘 알려진 Hadamard 행렬의 가장 일반적인 확장으로서, 직교와 비직교 행렬에 대한 성질을 포함하고 있다. Jacket 행렬은 정보 및 통신 분야 이론의 많은 문제들을 해석하는데 이용된다. 본 논문에서는 Jacket 행렬의 성질과 특성, 예를 들어 determinants와 eigenvalues, Kronecker product에 대해서 다룬다. 이 연산들은 신호 처리와 직교 코드 디자인에 매우 유용하다. 또한, 본 논문은 복잡성이 낮은 매우 간단한 수학적 모델을 통해 이들의 유용성을 계산한 결과를 제시한다.

Abstract As a reversible Jacket is having the compatibility of two sided wearing, the matrix that both the inside and the outside are compatible is called Jacket matrix, and the matrix is having both inside and outside by the processes of element-wise inverse and block-wise inverse. This concept had been completed by one of the authors Moon Ho Lee in 1989, and finally that resultant matrix has been christened as Jacket matrix, in 2000. This is the most generalized extension of the well known Hadamard matrices, which includes both orthogonal and non-orthogonal matrices. This matrix addresses many problems in information and communication theories. we investigate the properties of the Jacket matrix, i.e. determinants, eigenvalues, and kronecker product. These computations are very useful for signal processing and orthogonal codes design. In our proposal, we provide some results to calculate these values by using a very simple mathematical model with less complexity.

Key Words : Generalized Hadamard matrix, Center Weighted Hadamard, Jacket Matrix.

Ⅰ. Introduction

Hadamard matrices and Hadamard transforms have been a great deal of interest and have been applied to communications signaling, image processing, signal representation and error correction coding theory[1-4]. Investigations of Hadamard matrices were connected initially with linear algebra problems, such as finding maximum of determinant. It is well known that for an $n \times n$ Hadamard matrix H_n, $|\det H_n| = n^{n/2}$. Lee *et al.* proposed the idea of center weighted Hadamard matrices and center weighted Hadamard transform.

*정회원, 대진대학교 컴퓨터공학과
**정회원, 신경대학교 인터넷정보통신학과
***정회원, 전북대학교 전자정보공학부(교신저자)
접수일자 2015년 2월 11일, 수정완료 2015년 4월 25일
게재확정일자 2015년 6월 12일

Received: 11 February, 2015 / Revised: 25 April, 2015 / Accepted: 12 June, 2015
***Corresponding Author: moonho@jbnu.ac.kr
Dept: Division of Electronic Engineering, Chonbuk National University, korea

Futhermore, Lee *et al.* proposed the following new class of matrices: Jacket transforms that generalize real Hadamard transforms, Turyn-type Hadamard transforms, Butson-type Hadamard transform, complex Hadamard transforms and center weighted Hadamard transforms[4].

Definition 1: Let $A=\left(a_{jk}\right)$ be an $n\times n$ matrix whose elements are in a field F(including real fields, complex fields, finite fields, etc.). Denote the transpose matrix of the element-wise inverse of A by $A^{\dagger}$, that is, $A=\left(a_{jk}^{-1}\right)$. Then A is called a Jacket matrix if $AA^{\dagger}=A^{\dagger}A=nI_n$, where I_n is the identity matrix over the field F.

Motivation is the center weighted Hadamard transform. It was first discovered as a typical case of Jacket matrices. Therefore, in this paper, we present some of the basic properties of the center weighted Hadamard transform, including the notation of matrix factorization, and multiplication. Based on the interesting invertible properties of the center weighted Hadamard transform, we define Jacket matrices and some of the properties related to Hadamard matrices and Kronecker product.

The remainder of this paper is organized as follows. In Section Ⅱ, we discuss determinants and eigenvalues of Jacket matrix. Section Ⅲ presents Kronecker product of Jacket matrices. Conclusions are drawn in Ⅳ Section.

Ⅱ. The Determinants and Eigenvalues of Jacket Matrix

We presented some methods to obtain the different Jacket matrices. Now, in this chapter, we will investigate the properties of the Jacket matrix, i.e. determinants and eigenvalues. These computations are very useful for signal processing and orthogonal codes design. In our proposal, we provide some results to calculate these values by using a very simple mathematical model with less complexity.

Given a symmetric pattern of rank 2,

$$[S]_2=\begin{bmatrix} a & b \\ b & -c \end{bmatrix}, \quad (1)$$

the determinant is

$$\det([S]_2)=(-ac-b^2)=-(ac+b^2)\,. \quad (2)$$

It is clear that the largest determinant can be obtained only if $ac>0,\ b^2>0$[1]-[2].

Given a symmetric Jacket pattern of rank 2,

$$[J]_2=\begin{bmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{bmatrix}, \quad (3)$$

the determinant is

$$\det([J]_2)=(-ac-ac)=-(ac+ac)=-2ac\,. \quad (4)$$

It is evident that maximizing the ac, means maximizing the determinant. Otherwise we can prove that

$$\det([J]_2)=-2ac\le a^2+c^2\,. \quad (5)$$

Proof:

$$a^2+c^2\ge -2ac\,, \quad (6)$$

since

$$a^2+2ac+c^2=(a+c)^2\ge 0\,. \quad (7)$$

Thus the maximum determinant of $[J]_2$ is a^2+c^2. The Hadamard matrices are a special case of the Jacket matrices introduced in this chapter. Therefore, the 2-by-2 Hadamard matrix is given by $[H]_2=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. The eigenvalue is obtained by

$$Eig([H]_2) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \tag{8}$$

and the eigenvector is given by

$$EV([H]_2) = \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix}. \tag{9}$$

All these values can be calculated with $a = c = 1$.

The order of $[H]_N$ is 2^k, and the eigenvalues of Hadamard matrix is distributed as in (10) below

$$Eig([H]_{2^k}) = \begin{bmatrix} +2^{k/2} & 0 \\ 0 & 2^{k/2} \end{bmatrix}, \tag{10}$$

The eigenvalues of $[H]_4$ is also given as below

$$Eig([H]_4) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}. \tag{11}$$

The eigenvectors of $[H]_N$ can be easily calculated by using a function

$$EV([H]_N) = [H]_N + Eig([H]_N) \ . \tag{12}$$

Proof: The eigenvectors have

$$\begin{aligned} &[H]_N EV([H]_N) = (EV([H]_N) Eig([H]_N)), \\ &[H]_N([H]_N + Eig([H]_N)) \\ &= (([H]_N + Eig([H]_N)) Eig([H]_N)) \\ &\Rightarrow N[I]_N + [H]_N Eig([H]_N) \\ &\quad = ([H]_N Eig([H]_N) + Eig([H]_N) Eig([H]_N)). \\ &\Rightarrow N[I]_N + [H]_N Eig([H]_N) \\ &\quad = [H]_N Eig([H]_N) + N[I]_N \ . \end{aligned}$$

The (12) is proved. ■

Example:

$$[H]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ Eig([H]_2) = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix},$$

then we have

$$EV([H]_2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix}. \tag{13}$$

Let $\lambda = Eig([H]_2)$, which denotes the eigenvalue of Hadamard matrix, and $A = [H]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we have

$$\begin{aligned} &-(1-\lambda)(1+\lambda) - 1 = 0 \\ &\quad \rightarrow \lambda^2 = 2, \\ &\quad \rightarrow \lambda = \pm\sqrt{2} \end{aligned} \quad . \tag{14}$$

So the eigenvalues are obtained.

For eigenvectors we have

$$(A - \lambda_1 I)x_1 = 0, \tag{15}$$

$$(A - \lambda_2 I)x_2 = 0. \tag{16}$$

Let $\lambda_1 = \sqrt{2}$, $\lambda_2 = -\sqrt{2}$, then we have

$$\begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0 \ , \tag{17}$$

$$\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = 0 \ , \tag{18}$$

from (17) and (18), we can have

$$(1-\sqrt{2})x_{11} + x_{21} = 0, \tag{19}$$

$$(1+\sqrt{2})x_{12} + x_{22} = 0, \tag{20}$$

Let's $x_{21} = 1$, $x_{12} = 1$, we get

$$x_{11} = (1+\sqrt{2}), \text{ and } x_{22} = (-1-\sqrt{2}) \ . \tag{21}$$

They shows that

$$(1-\sqrt{2})(1+\sqrt{2}) + 1 = 1 - 2 + 1 = 0, \tag{22}$$

$$(1+\sqrt{2}) \times 1 + (-1-\sqrt{2}) = 0. \tag{23}$$

Then the eigenvectors can be written as

$$X=\begin{bmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{bmatrix}=\begin{bmatrix}1+\sqrt{2}&1\\1&-1-\sqrt{2}\end{bmatrix}. \quad (24)$$

Next, we calculate the eigenvectors by

$$(A-\lambda I)X=0, \quad (25)$$

where X gives eigenvectors of A. Thus we have

$$\begin{aligned}&AX-(\lambda IX)=0\\&=> AX=X\lambda.\end{aligned} \quad (26)$$

For example,
$AX=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1+\sqrt{2}&1\\1&-1-\sqrt{2}\end{bmatrix}$, where X is from (24),

$$AX=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1+\sqrt{2}&1\\1&-1-\sqrt{2}\end{bmatrix}=\begin{bmatrix}2+\sqrt{2}&-\sqrt{2}\\\sqrt{2}&2+\sqrt{2}\end{bmatrix},$$

$$X\lambda=\begin{bmatrix}1+\sqrt{2}&1\\1&-1-\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2}&0\\0&-\sqrt{2}\end{bmatrix}=\begin{bmatrix}2+\sqrt{2}&-\sqrt{2}\\\sqrt{2}&2+\sqrt{2}\end{bmatrix}.$$

Thus we have $AX-X\lambda=(A-I\lambda)X$.

Given a symmetrical Jacket pattern of rank 4 Case,

$$[J]_4=\begin{bmatrix}a&b&b&a\\b&-c&c&-b\\b&c&-c&-b\\a&-b&-b&a\end{bmatrix}, \quad (27)$$

the determinant can be calculated as

$$\det([J]_4)=\det(C)\det(C-B^TA^{-1}B), \quad (28)$$

where

$$A=\begin{bmatrix}a&b\\b&-c\end{bmatrix}, B=\begin{bmatrix}b&a\\c&-b\end{bmatrix}, C=\begin{bmatrix}-c&-b\\-b&a\end{bmatrix},$$
and $[J]_4=\begin{bmatrix}A&B\\B^T&C\end{bmatrix}$.

It is easily proved that

$$\det\left(\begin{bmatrix}I&-A^{-1}B\\0&I\end{bmatrix}\right)=1, \quad (29)$$

by

$$\det\left(\begin{bmatrix}A&0\\B^T&C\end{bmatrix}\right)=\det\left(\begin{bmatrix}A&B\\0&C\end{bmatrix}\right)=\det(A)\det(C), \quad (30)$$

therefore, by $\det(AB)=\det(A)\det(B)$ [4].

$$\begin{aligned}&\det\left(\begin{bmatrix}A&B\\B^T&C\end{bmatrix}\right)=\det\left(\begin{bmatrix}A&B\\B^T&C\end{bmatrix}\begin{bmatrix}I-A^{-1}B\\0\quad I\end{bmatrix}\right)\\&=\det\left(\begin{bmatrix}A&0\\B^T&C-B^TA^{-1}B\end{bmatrix}\right)\\&=\det(A)\det(C-B^TA^{-1}B).\end{aligned} \quad (31)$$

The numerical result is

$$\det([J]_4)=16ab^2c. \quad (32)$$

Given a symmetric pattern $[S]_2=\begin{bmatrix}a&b\\b&-c\end{bmatrix}$, its eigenvalues is

$$Eig([S]_2)=\frac{1}{2}\begin{bmatrix}-c+a+\sqrt{c^2+2ac+a^2+4b^2}&0\\0&-c+a-\sqrt{c^2+2ac+a^2+4b^2}\end{bmatrix}, \quad (33)$$

and the eigenvector is

$$EV([S]_2)=\begin{bmatrix}\dfrac{-\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}\sqrt{c^2+2ac+a^2+4b^2}}{b}&\dfrac{-c+a-\sqrt{c^2+2ac+a^2+4b^2}}{b}\\1&1\end{bmatrix}. \quad (34)$$

Simply, the Jacket pattern $[J]_2=\begin{bmatrix}a&\sqrt{ac}\\\sqrt{ac}&-c\end{bmatrix}$ has

$$Eig([J]_2)=\begin{bmatrix}-\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}\sqrt{c^2+a^2+6ac}&0\\0&-\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}\sqrt{c^2+a^2+6ac}\end{bmatrix}, \quad (35)$$

and the eigenvector is

$$EV([J]_2) = \begin{bmatrix} \dfrac{\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}\sqrt{c^2+6ac+a^2}}{\sqrt{ac}} & \dfrac{\frac{1}{2}c+\frac{1}{2}a+\frac{1}{2}\sqrt{c^2+6ac+a^2}}{\sqrt{ac}} \\ 1 & 1 \end{bmatrix}. \tag{36}$$

Furthermore, the symmetrical Jacket pattern has

$$[J]_4 = \begin{bmatrix} a & b & b & a \\ b & -c & c & -b \\ b & c & -c & -b \\ a & -b & -b & a \end{bmatrix}, \tag{37}$$

and

$$Eig([J]_4) = \begin{bmatrix} -2b & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & -2c & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}, \tag{38}$$

its eigenvector is

$$EV([J]_4) = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \tag{39}$$

and it can be easily proved by using the eigen decomposition as

$$\begin{aligned} & EV([J]_4)Eig([J]_4)(EV([J]_4))^{-1} \\ &= \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2b & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & -2c & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2b & 0 & 0 & 0 \\ 0 & 2b & 0 & 0 \\ 0 & 0 & -2c & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix} \begin{bmatrix} -1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -1/4 \\ 0 & -1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} a & b & b & a \\ b & -c & c & -b \\ b & c & -c & -b \\ a & -b & -b & a \end{bmatrix} = [J]_4. \end{aligned} \tag{40}$$

Ⅲ. The Kronecker Product of Jacket Matrices

A classical center weighted Jacket pattern is derived from the center weighted Hadamard matrix [4],

$$[J]_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w & w & -1 \\ 1 & w & -w & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{41}$$

and its inverse is

$$[J]_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w^{-1} & w^{-1} & -1 \\ 1 & w^{-1} & -w^{-1} & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{42}$$

where $[J]_4$ is size of 4×4 square matrix, and w denotes the weighted factors, which can be arbitrary numbers.

Theorem 1 : the higher order Jacket matrices can be obtained by using the Kronecker product [Appendix] as

$$[J]_N = [J]_4 \otimes [H]_{N/4} \tag{43}$$

where $[H]_N$ is a $N\times N$ Hadamard matrix, and $N=2^n$, with $n=2,3,4,\ldots$.

Theorem 2 : If a symmetric Jacket matrix of rank 2 exists, the higher order symmetric Jacket matrices can be represented by

$$[J]_N = [H]_{N/2} \otimes [J]_2. \tag{44}$$

For example, by considering $N=4$, a symmetric Jacket matrix can be written by

$$[J]_4 = [J]_2 \otimes [H]_2, \tag{45}$$

where $[J]_2$ is

$$[J]_2 = \begin{bmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{bmatrix}, \tag{46}$$

and

$$[H]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{47}$$

thus we have

$$\begin{aligned} [J]_4 &= \begin{bmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a & a & \sqrt{ac} & \sqrt{ac} \\ a & -a & \sqrt{ac} & -\sqrt{ac} \\ \sqrt{ac} & \sqrt{ac} & -c & -c \\ \sqrt{ac} & -\sqrt{ac} & -c & c \end{bmatrix}. \end{aligned} \tag{48}$$

From (48) we have

$$\begin{aligned} [J]_4^{-1} &= \begin{bmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{bmatrix}^{-1} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 1/a & 1/\sqrt{ac} \\ 1/\sqrt{ac} & -1/c \end{bmatrix} \otimes \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 1/a & 1/a & 1/\sqrt{ac} & 1/\sqrt{ac} \\ 1/a & -1/a & 1/\sqrt{ac} & -1/\sqrt{ac} \\ 1/\sqrt{ac} & 1/\sqrt{ac} & -1/c & -1/c \\ 1/\sqrt{ac} & -1/\sqrt{ac} & -1/c & 1/c \end{bmatrix}. \end{aligned} \tag{49}$$

Consequently, the pattern of the **Theorem 2** is different from the center weighted Jacket pattern.

In general, using the Kronecker product, we can easily identify the different patterns of Jacket matrices, such that

$$[J]_{N_1+N_2+\ldots} = [J]_{N_1} \otimes [J]_{N_2} \otimes \ldots. \tag{50}$$

Proof: Similar to (43), the inverse of the multiplications of their elements are the same as the multiplications of their inverse elements. Thus the (50) is proved.

For example, Let us write

$$\begin{aligned} [J]_{N_1+N_2} &= [J]_{N_1} \otimes [J]_{N_2} \\ &= \begin{bmatrix} j_{1,1}\cdot[J]_{N_2} & j_{1,2}\cdot[J]_{N_2} & \cdots & j_{1,N_1}\cdot[J]_{N_2} \\ j_{2,1}\cdot[J]_{N_2} & j_{2,2}\cdot[J]_{N_2} & \cdots & j_{2,N_1}\cdot[J]_{N_2} \\ \vdots & \vdots & \cdots & \vdots \\ j_{N_1,1}\cdot[J]_{N_2} & j_{N_2,1}\cdot[J]_{N_2} & \cdots & j_{N_1,N_1}\cdot[J]_{N_2} \end{bmatrix}, \end{aligned} \tag{51}$$

its inverse has

$$\begin{aligned} [J]_{N_1+N_2}^{-1} &= [J]_{N_1}^{-1} \otimes [J]_{N_2}^{-1} \\ &= \begin{bmatrix} \frac{1}{j_{1,1}}\cdot[J]_{N_2}^{-1} & \frac{1}{j_{1,2}}\cdot[J]_{N_2}^{-1} & \cdots & \frac{1}{j_{1,N_1}}\cdot[J]_{N_2}^{-1} \\ \frac{1}{j_{2,1}}\cdot[J]_{N_2}^{-1} & \frac{1}{j_{2,2}}\cdot[J]_{N_2}^{-1} & \cdots & \frac{1}{j_{2,N_1}}\cdot[J]_{N_2}^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{j_{N_1,1}}\cdot[J]_{N_2}^{-1} & \frac{1}{j_{N_2,1}}\cdot[J]_{N_2}^{-1} & \cdots & \frac{1}{j_{N_1,N_1}}\cdot[J]_{N_2}^{-1} \end{bmatrix}. \end{aligned} \tag{52}$$

Since $[J]_{N_2}^{-1}$ is an element inverse Jacket matrix, $\frac{1}{j_{m,n}}\cdot[J]_{N_2}^{-1}$ is the element inverse of $j_{m,n}\cdot[J]_{N_2}$, with $m,n \in 1,2,\ldots,N_1$. As a result, (52) is the element inverse matrix of (51).

Based on some basic Jacket matrices, a variety of different patterns can be approached easily using Kronecker product.

We compared conventional DFT, DCT, Hadamard and Jacket matrix based on Kronecker as shown table 1.

Then

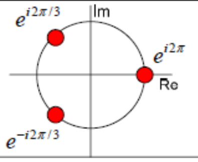
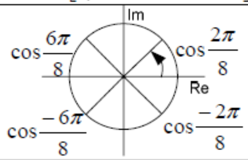
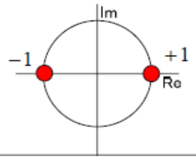
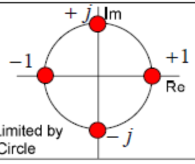
① DFT case : DFT ⊗ DFT is not DFT, and matrix size is 2^p (p=prime).

② DCT case : DCT ⊗ DCT is not DCT, and matrix size is 2^n.

③ Hadamard case : Hadamard ⊗ Hadamard is Hadamard, and matrix size is 2^n, $4n$

The Journal of The Institute of Internet, Broadcasting and Communication (IIBC)
Vol. 15, No. 3, pp.25-33, Jun. 30, 2015. pISSN 2289-0238, eISSN 2289-0246

표 1. DFT, DCT 및 Hadamard Jacket 행렬의 Kronecker product 비교
Table 1. The comparison of DFT, DCT and Hadamard Jacket matrix with Kronecker product.

	DFT (1822) J. Fourier	DCT(1974) N. Ahmed, K.R. Rao,et.	Hadamard (1893) J. Hadamard	Jacket(1989)* Moon Ho Lee
Formua	$X(n)=\sum_{k=0}^{N-1}x(k)w^{nk}$ $n=0,1...N-1, w=e^{-j2\pi/N}$	$[C_N]_{m,n}=\sqrt{\frac{2}{N}}k_m\cos\frac{m(n+\frac{1}{2})\pi}{N}$ $m,n=0,1,\ldots,N-1$	$[H]_2=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ $[H]_n=[H]_{n/2}\otimes[H]_2$	$[J]_4=\begin{bmatrix}1&1&1&1\\1&-j&j&-1\\1&j&-j&1\\1&-1&-1&1\end{bmatrix}$ $[J]_n=[J]_{n/2}\otimes[H]_2 \quad n>4$
Forward	$N=3$ $F_3=\begin{bmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{bmatrix}$ $w=e^{-i2\pi/3}$	$N=4$ $[C]_4=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\C_8^1&C_8^3&C_8^5&C_8^7\\C_8^2&C_8^6&C_8^6&C_8^2\\C_8^3&C_8^7&C_8^1&C_8^5\end{bmatrix}$ $C_8^i=\cos\frac{i\pi}{8}$	$(-1)^{\oplus_{k=0}^{n-1}i_kj_k}$ $[H]_4=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$	$(-1)^{\oplus_{k=0}^{n-1}i_kj_k}w^{(i_{n-2}\oplus i_{n-1})(j_{n-2}\oplus j_{n-1})}$ $[J]_4=\begin{bmatrix}1&1&1&1\\1&-w&w&-1\\1&w&-w&-1\\1&-1&-1&1\end{bmatrix}$ w=1: Hadamard w=2: Center Weighted Hadamard
Inverse	Element-Wise Inverse $F_3^{-1}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&w^{-1}&w^{-2}\\1&w^{-2}&w^{-1}\end{bmatrix}$	Non-Element-Wise Inverse $[C]_4^{-1}=\frac{1}{2}\begin{bmatrix}\frac{1}{\sqrt2}&C_8^1&C_8^2&C_8^3\\\frac{1}{\sqrt2}&C_8^3&C_8^6&C_8^7\\\frac{1}{\sqrt2}&C_8^5&C_8^6&C_8^1\\\frac{1}{\sqrt2}&C_8^7&C_8^2&C_8^5\end{bmatrix}$	Element-Wise Inverse $[H]_4^{-1}=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$	Element-Wise Inverse or Block-Wise Inverse $[J]_4^{-1}=\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-1/w&1/w&-1\\1&1/w&-1/w&-1\\1&-1&-1&1\end{bmatrix}$
Circle				
Kronecker	$DFT\otimes DFT\neq DFT$	$DCT\otimes DCT\neq DCT$	$H\otimes H=Hadamard$	$J\otimes J=Jacket$
Size	$2^n, p=prime$	2^n	$2^n, 4n$	$Arbitrary$

④ Jacket matrix case : Jacket ⊗ Jacket is Jacket matrix, and matrix size is arbitrary.

Ⅳ. Conclusion

Computations of the eigenvalues, determinants, and Kronecker product of the Jacket matrices are being investigated in this paper. Their calculations will be directly related to the kronecker product and symmetric properties.

As shown the table, they have good performances of result, especially using kronecker product. In Hadamard case, Its size is proved arbitrary. The matrix decomposition takes the form of the Kronecker products of fundamental Hadamard matrices and successively lower order weighted Hadamard matrices.

In general, using Kronecker product, we can easily identify different patterns of Jacket matrices, which are not only based on Hadamard. It is easy to prove that The Kronecker product of the two three-dimensional Jacket matrices is also a three dimensional Jacket matrix. Thus the three Jacket matrix of order 4^n can be constructed by n successive direct multiplication among three-Jacket matrices of order 4. Meanwhile, It is obvious that the Kronecker product of two four-dimensional Jacket matrices is also a four dimensional Jacket matrix.

References

[1] G. Strang, *Linear Algebra and Its Applications,* Third Edition, BROOKS/COLE, THOMSON LEARNING, 1988.

[2] R. K. Yarlagadda, and John E. Hershey, *Hadamard Matrix Analysis and Synthesis with Applications to Communications and Signal /Image Processing,* Kluwer Academic Publishers, U.S., 1997.

[3] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge, MA: Cambridge Univ. Press, UK, 1991.

[4] Moon Ho Lee, *Jacket Matrices: Construction and Its Application for Fast Cooperative Wireless Signal Processing,* Germany LAMBERT, 2012.

[5] Y. S. Im, E. Y. Kang, "MPEG-2 Video Watermarking in Quantized DCT Domain," The Journal of The Institute of Internet, Broadcasting and Communication(JIIBC), Vol. 11, No. 1, pp. 81-86, 2011.

[6] I. Jeon, S. Kang, H. Yang, "Development of Security Quality Evaluate Basis and Measurement of Intrusion Prevention System," Journal of the Korea Academia-Industrial cooperation Society(JKAIS), Vol. 11, No. 1, pp. 81-86, 2010.

[7] Y. M. Kwon, K. H. Kim, H. J. Kwon, "Digital Halftoning using DCT," The Journal of The Institute of Internet, Broadcasting and Communication(JIIBC), Vol. 13, No. 6, pp. 79-85, 2013.

Appendix

A1. Kronecker Product

Let matrix $[A]$ be of size $m\times n$ and matrix $[B]$ of size $k\times l$. The Kronecker product of matrices $[A]$ and $[B]$ is a amtrix of size $mk\times nl$ and is denoted as $[B]\otimes[A]$. It is defined as

$$[B]\otimes[A]=\begin{pmatrix} Ab_{0,0} & Ab_{0,1} & \cdots & Ab_{0,l-1} \\ Ab_{1,0} & Ab_{1,1} & \cdots & Ab_{1,l-1} \\ \vdots & \vdots & \vdots & \vdots \\ Ab_{k-1,0} & Ab_{k-1,1} & \cdots & Ab_{k-1,l-1} \end{pmatrix}, \quad \text{(A-1)}$$

where $b_{i,j}$ denotes the $i,\ j$th element of matrix B. The following are key properties of the Kronecker matrix product:

$$\text{No.1:}\ (A\otimes B)\otimes C=A\otimes(B\otimes C). \quad \text{(A-2)}$$

$$\text{No.2:}\ (A\otimes B)(C\otimes D)=(AC)\otimes(BD), \quad \text{(A-3)}$$

where C is of dimension $n\times q$ and D is of dimension $l\times s$.

$$\text{No.3:}\ A\otimes D=(AI_n)\otimes(I_lD)=(A\otimes I_l)(I_n\otimes D), \quad \text{(A-4)}$$

where I_j denotes a $j\times j$ identity matrix.

$$\text{No.4:}\ (A\otimes B)^T=A^T\otimes B^T, \quad \text{(A-5)}$$

where T is the transpose.

$$\text{No.5:}\ A\otimes(\alpha B)=(\alpha A)\otimes B=\alpha(A\otimes B), \quad \text{(A-6)}$$

where α is a constant.

$$\text{No.6:}\ (A\otimes B)^H=A^H\otimes B^H, \quad \text{(A-7)}$$

$$\text{No.7:}\ (A+B)\otimes C=(A\otimes C)+(B\otimes C). \quad \text{(A-8)}$$

$$\text{No.8:}\ A\otimes(B+C)=(A\otimes B)+(A\otimes C). \quad \text{(A-9)}$$

where H denotes the Hermitian of this matrix.

$$\text{No.9:}\ A\otimes(B\otimes C)=(A\otimes B)\otimes C. \quad \text{(A-10)}$$

$$\text{No.10:}\ (A\otimes B)^{-1}=A^{-1}\otimes B^{-1}. \quad \text{(A-11)}$$

No.11: If $A_1,A_2,...,A_p$ are $M\times M$ and $B_1,B_2,...B_p$ are $N\times N$, then

$$(A_1\otimes B_1)(A_2\otimes B_2)\dots(A_p\otimes B_p)=(A_1A_2\dots A_p)\otimes(B_1B_2\dots B_p). \quad \text{(A-12)}$$

No.12: If A is $M\times M$ and B is $N\times N$, then

$$\det(A\otimes B)=(\det A)^M(\det B)^N, \qquad (A\text{-}13)$$

where $\det(A)$ denotes the determinant of A.

No.13: If A is $M\times M$ and B is $N\times N$,

$$tr(A\otimes B)=(trA)(trB). \qquad (A\text{-}14)$$

No.14: If A is $M\times M$ and B is $N\times N$,

$$rank(A\otimes B)=(rankA)(rankB). \qquad (A\text{-}15)$$

$$|A\otimes B|=|A|^r\cdot|B|^s. \qquad (A\text{-}16)$$

where A and B are square matrices of size $r\times r$ and $s\times s$.

저자 소개

양 재 승(정회원)

- 1988년 : 연세대학교 금속공학과 학사
- 1995년 : 연세대학교 산업정보 석사
- 2010년 : 전북대학교 정보보호공학 박사
- 1989년 ~ 1999년 : 한국UNISYS 차장
- 2000년 ~ 2002년 : SEEC Inc. 한국 지사장
- 2001년 ~ 2010년 : 제이에스 정보 이사
- 2011년 3월 ~ 현재 : 대진대학교 컴퓨터공학과 강사

<주관심분야 : Polar Code, 정보보안>

박 주 용(정회원)

- 1982년 : 전북대학교 전자공학과 학사
- 1986년 : 전북대학교 전자공학과 석사
- 1994년 : 전북대학교 전자공학 박사
- 1991년 ~ 2007년 : 서남대학교 전기전자공학과 부교수
- 207년 3월 ~ 현재 : 신경대학교 인터넷정보통신학과 교수

<주관심분야 : 무선이동통신, 통신이론>

이 문 호(정회원)

- 1984년 : 전남대학교 전기공학과 박사, 통신기술사
- 1985년 ~ 1986년 : 미국 미네소타 대학 전기과 포스트닥터
- 1990년 : 일본동경대학 정보통신공학과박사
- 1970년 ~ 1980년 : 남양MBC 송신소장
- 1980년 10월 ~ 2010년 2월 : 전북대학교 전자공학부 교수
- 2010년 2월 ~ 2013 : WCU-2 연구책임교수
- 현재 전북대학교 전자공학부 초빙교수

<주관심분야 : 무선이동통신>

※This work was supported by MEST, 2015R1A2A1A05000977, NRF, Korea.