

# Zeroth-Order Shear Deformation Micro-Mechanical Model for Periodic Heterogeneous Beam-like Structures

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**Abstract:** This paper discusses a new model for investigating the micro-mechanical behavior of beam-like structures composed of various elastic moduli and complex geometries varying through the cross-sectional directions and also periodically-repeated along the axial directions. The original three-dimensional problem is first formulated in an unified and compact intrinsic form using the concept of decomposition of the rotation tensor. Taking advantage of two smallness of the cross-sectional dimension-to-length parameter and the micro-to-macro heterogeneity and performing homogenization along dimensional reduction simultaneously, the variational asymptotic method is used to rigorously construct an effective zeroth-order beam model, which is similar a generalized Timoshenko one (the first-order shear deformation model) capable of capturing the transverse shear deformations, but still carries out the zeroth-order approximation which can maximize simplicity and promote efficiency. Two examples available in literature are used to demonstrate the consistence and efficiency of this new model, especially for the structures, in which the effects of transverse shear deformations are significant.

**Key Words :** Homogenization, Dimensional Reduction, Spanwise Heterogeneity, Variational Asymptotic Method, Transverse Shear Deformation

## 1. Introduction

Due to high-strength and low-weight, superior noise and energy absorption, and high-temperature resistance characteristics, composite materials have demonstrated excellently practical potential and rapidly increasing popularity in various engineering applications. Furthermore, extensive analytic understandings and elaborate experimental techniques predicting and controlling their properties are even possible to manufacture new microstructure-based materials and structures to achieve the

ever-increasing performance requirements. With the help of the phenomenal power of present day computer facilities the full three-dimensional (3D) finite element analysis (FEA) is widely accepted and used for analysis of such materials and structures by meshing all the details of constituent microstructures. However, it is not an efficient and convenient way because of the inordinate requirements of computational cost to capture the micro-scale mechanical characteristics.

For this reason, research attentions devoted to an alternative approach of the full 3D FEA, especially, by using a unit cell (UC) have received considerable attention in the past several decades. It allows for a practical definition of the building block of the heterogeneous material, and leads to

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replacing the original heterogeneous structures with a homogeneous one with a set of effective material properties if the size of UC ( $d$ ) is much smaller than the size of the structure ( $L$ ) (i.e.  $\eta = d/L \ll 1$ ). See Kanouté *et al.*<sup>5)</sup> for a review. However, most of these approaches are not suitable for engineering analyses of dimensionally reducible structures<sup>7)</sup>, i.e. those with one or two dimensions much smaller than others. Composite beam-like structures are one of examples with the cross-sectional dimension  $h$  much smaller than the axial dimension (i.e.  $e = h/L \ll 1$ ).

In this work, we propose to use the variational asymptotic method (VAM)<sup>1)</sup> to carry out simultaneous homogenization and dimensional reduction, to construct an engineering model suitable for beam made of spanwisely periodic and heterogeneous microstructures, and to extend the previous work<sup>6)</sup> by producing a non-classical model that includes transverse shear effects, but still carries out the zeroth-order approximation which can maximize simplicity and promote efficiency. Considering both smallness of the cross-sectional dimension-to-length parameter( $e$ ) and heterogeneity( $\eta$ ), we use VAM to rigorously decouple the original 3D anisotropic, heterogeneous problem into a nonlinear one-dimensional (1D) beam analysis on the macroscopic level and a linear 3D micro-mechanical analysis. The micro-mechanical analysis can be implemented in COMSOL MULTIPHYSICS™ (COMSOL), a finite element based simulation and modeling tool. As a preliminary validation of the present approach, two examples, in which the transverse shear deformation is especially significant, are used to demonstrate the application and accuracy of this new model.

## 2. Beam kinematics and 3D formulation with homogenization

Geometrically, when a 3D elastic body has one

dimension much larger than the other two, it can be mathematically modeled as a beam with a 1D reference line  $r$  measured by the axial coordinate  $x_1$  and the two-dimensional (2D) reference plane  $A$  by cross-sectional Cartesian coordinates  $x_\alpha$  (Here and throughout the paper, Greek indices assume values 2 and 3 while Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated). Let us now consider a heterogeneous beam-like structure composed of periodically-repeated unit cells (UCs), denoted by  $\Omega_s$ , over the axial coordinate  $x_1$  along  $r$  (see Fig. 1).

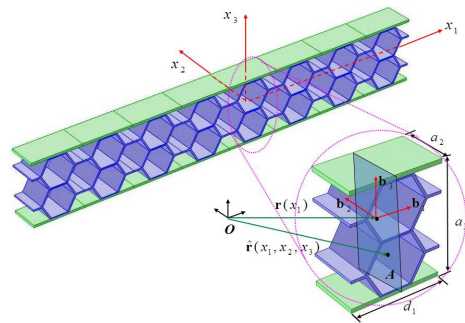


Fig. 1 A heterogeneous beam-like structure with spanwise-repeated unit cell

To implement the homogenization procedure into the present approach, we need to assume the existence of a distinct scale separation between two types of spatial variations, and describe the rapid change in the material characteristics along the axial direction by one so-called ‘fast’ variation  $y_1$  parallel to a ‘slow’ variation  $x_1$ . These two sets of variations are related as  $y_1 = x_1/\eta$ .

In order to homogenize the heterogeneous beam-like structures with representative UCs, there are only two indispensable assumptions associated with the micro-mechanical analysis through the homogenization procedure<sup>6)</sup>. First, we assume that

the exact solution of the field variables have volume averages over  $\Omega$ . Second, due to the existence of a distinct scale separation between two types of spatial variations described by  $y_1$  and  $x_1$ , and the assessment and checkup of the orders of all the quantities in the formulation, the derivative of a function,  $f_i$ , defined in  $\Omega$ , with respect to  $x_1$  can be evaluated as

$$\begin{aligned} \frac{\partial f_i(x_1, x_2, x_3; y_1)}{\partial x_1} &= \frac{\partial f}{\partial x_1} \Big|_{x_1 = const} + \frac{1}{\eta} \frac{\partial f}{\partial x_1} \Big|_{x_1 = const} \\ &\equiv f_i' + \frac{1}{\eta} f_{i|1} \end{aligned} \quad (1)$$

Note that in real derivation,  $\eta$  is not a number but denoting the order of the term it is associated with.

For the 3D beam kinematic description, letting  $\mathbf{b}_i$  denote an orthonormal reference triad along the coordinate lines of the undeformed beam, one can then describe the position of any material point by its position vector  $\hat{\mathbf{r}}$  relative to a point  $O$  fixed in an inertial frame, such that

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + x_\alpha \mathbf{b}_\alpha(x_1) \quad (2)$$

where  $\mathbf{r}$  is the position vector of a point located by  $x_1$  on the reference line and  $\mathbf{r}' = \mathbf{b}_1$ .

When the beam deforms, the particle has the corresponding position vector  $\hat{\mathbf{R}}$  in the deformed configuration. To determine the latter uniquely by the deformation of 3D body, a new orthonormal triad  $\mathbf{B}_i$  is first introduced for the deformed beam as unit base vectors, which is just a tool to represent vectors and tensors in their component form during the derivation. The relation between  $\mathbf{B}_i$  and  $\mathbf{b}_i$  can be specified by an arbitrary large rotation in terms of the matrix of direction cosines  $C(x_2, x_3)$ . On the previous work<sup>6)</sup>, instead of  $\mathbf{B}_i$ , we introduced another triad  $\mathbf{T}_i$  with  $\mathbf{T}_1$  tangent to

the deformed beam reference line and  $\mathbf{T}_\alpha$  determined by a rotation about  $\mathbf{T}_1$ . However, these restrictions are released in the present approach for capturing transverse shear effects on beam-like structures. That is, the difference between two types of orthonormal triads is due to small rotations associated with transverse shear deformation.

Now any material point in the deformed beam can be represented by its position vector  $\hat{\mathbf{R}}$

$$\hat{\mathbf{R}}(x_i; y_1) = \mathbf{R}(x_1) + x_\alpha \mathbf{B}_\alpha(x_1) + w_j(x_i; y_1) \mathbf{B}_j(x_1) \quad (3)$$

where  $\mathbf{R} = \mathbf{r} + \mathbf{u}$  denotes the position vector of the reference line for the deformed structure,  $\mathbf{u} = u_i \mathbf{b}_i$  is the displacement vector of the reference line from the undeformed configuration, and  $w_i$  denotes the undetermined fluctuating functions describing the deformation not captured by  $\mathbf{R}$  and  $\mathbf{B}_i$ . Due to the existence of a distinct scale separation between two types of spatial variations described by  $y_1$  and  $x_1$ ,  $w_i$  are periodic functions in  $y_1$ , that is

$$w_i(x_1, x_2, x_3; d_1/2) = w_i(x_1, x_2, x_3; -d_1/2) \quad (4)$$

In order to ensure a one-to-one mapping between  $\hat{\mathbf{R}}$  and  $(\mathbf{R}, \mathbf{B}_i, w_i)$  in Eq. (3), six constraints are needed. If we define  $\mathbf{R} = \langle \hat{\mathbf{R}} \rangle$ , then we have the following three constraints

$$\langle w_i \rangle \equiv \frac{1}{\Omega} \int_{\Omega} w_i d\Omega = 0 \quad (5)$$

It means that fluctuating function does not contribute to the rigid-body displacement of the UC. Also, the following constraint

$$\langle w_{2,3} - w_{3,2} \rangle = 0 \quad (6)$$

is chosen related to twisting associated with the rotation of the UC about  $\mathbf{B}_1$ . Following Yu *et al.*<sup>7)</sup>, there is now a need to impose two additional

constraints on the unknown fluctuating function because  $\mathbf{B}_1$  is not necessarily parallel to the tangent line due to the two extra degrees of freedom associated with transverse shear deformation. Thus, in addition to the four classical constraints in Eqs. (5) and (6), the fluctuating function satisfies two additional ones

$$\langle x_\alpha w_1 \rangle = 0 \quad (7)$$

to make the formulation in Eq. (3) unique.

For the purpose of formulating our problem in the intrinsic form at the global level, the 1D generalized strain measures including transverse shear strains can be defined using the partial derivatives  $\mathbf{R}$  and  $\mathbf{B}_i$  with respect to  $x_1$ , such as

$$\mathbf{R}' = (1 + \gamma_{11})\mathbf{B}_1 + 2\gamma_{1\alpha}\mathbf{B}_\alpha \quad (8)$$

and

$$\mathbf{B}_i' = \mathbf{K} \times \mathbf{B}_i \text{ with } \mathbf{K} = \mathbf{k} + \kappa \quad (9)$$

where  $\gamma_{11}$  is the extensional strain,  $2\gamma_{1\alpha}$  are the transverse shear strains,  $\mathbf{K}$  is the curvature vector of the deformed reference line,  $\mathbf{k}$  is the curvature vector of the undeformed one,  $\kappa_1$  is the twist, and  $\kappa_\alpha$  are the bending curvatures. Here for simplicity, we restrict the beam structure to be prismatic such that  $k_i = 0$  and  $\mathbf{K} = \kappa$ .

Based on the concept of decomposition of rotation tensor<sup>3)</sup>, the 3D Jauman-Biot-Cauchy strain components for small local rotation are given by

$$\Gamma_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) - \delta_{ij} \quad (10)$$

where  $\delta_{ij}$  is the Kronecker symbol, and  $F_{ij}$  the mixed-basis component of the deformation gradient tensor such that

$$F_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j \quad (11)$$

Here  $\mathbf{g}^j$  are the 3D contravariant base vector of the undeformed configuration and in a prismatic case,  $\mathbf{g}^j = \mathbf{g}_i = \mathbf{b}_j$ , while  $\mathbf{G}_i$  are the 3D covariant basis vectors of the deformed configuration, which can be obtained in the following way:

$$\mathbf{G}_1 = \frac{\partial \widehat{\mathbf{R}}}{\partial x_1} = \widehat{\mathbf{R}}' + \frac{1}{\eta} \widehat{\mathbf{R}}_{|1} \text{ and } \mathbf{G}_\alpha = \frac{1}{\eta} \frac{\partial \widehat{\mathbf{R}}}{\partial x_\alpha} = \frac{1}{\eta} \widehat{\mathbf{R}}_{|\alpha} \quad (12)$$

Here the second expressions of Eq. (12) are attributed to the consideration of the distinct scale separation between fast spatial variations ( $y_1$  and  $x_\alpha$ ) and a slow spatial one ( $x_1$ ) during the derivation.

With the assumption that the 1D generalized strains are small compared to unity which is sufficient for geometrical nonlinear analysis, we can neglect all the terms that are products of the 3D fluctuating functions and the 1D generalized strains, and obtain the 3D strain field. Therefore, The strain energy stored in the heterogeneous beams can be generally calculated as:

$$U = \frac{1}{2} \int_{x_1} \langle \Gamma^T D \Gamma \rangle dx_1 \quad (13)$$

where  $\Gamma = [\Gamma_{11} \ 2\Gamma_{12} \ 2\Gamma_{13} \ \Gamma_{12} \ 2\Gamma_{22} \ \Gamma_{33}]^T$  and  $D$  is the 3D  $6 \times 6$  material matrix, which consists of elements of the fourth-order elasticity tensor expressed in the cross-sectional coordinate system  $x_\alpha$  and the local axial coordinate system  $y_1$ .

To deal with the applied loads, we alternatively develop the virtual work of the applied loads. However, according to Yu. *et al.*<sup>7)</sup> the virtual work done by the external forces can be negligible in the zeroth-order approximation because the applied loads are of higher order.

Now, the complete statement of the problem up to the zeroth-order approximation can be expressed in terms of the principle of virtual work, such that

$$\delta U = 0 \quad (14)$$

Thus, one can pose the problem that governs the only fluctuating functions as the minimization of a total potential functional

$$\delta \Pi = 0 \quad \text{with } \Pi = U \quad (15)$$

### 3. Dimensional Reduction

To rigorously and efficiently reduce the original 3D problem to an effective 1D beam model with spanwise heterogeneity, VAM will be used to mathematically reproduce the 3D energy stored in the heterogeneous structure into a 1D intrinsic formulation, which is asymptotically correct up to the desired order taking advantage of the small parameters inherent in the structure. Here three small parameters are introduced into the problem:  $\hat{\epsilon}$  denoting the smallness of generalized strains,  $e$  denoting the smallness of the cross-sectional dimensions-to-length parameter and  $\eta$  denoting the smallness of heterogeneity<sup>6)</sup>.

Following Yu. *et al.*<sup>7)</sup> the quantities of interest assess and keep track of the following determined orders in the formulation:

$$\gamma_{11} \sim h\kappa_i \sim \gamma_{1\alpha} \sim \hat{\epsilon} \quad (16)$$

Here unlike the previous work<sup>6)</sup> we consider that the transverse shear effects are not the correction (or higher-order terms) to a classical beam theory; they are the effects of the leading order, which are introduced into the zeroth-order approximation of the present approach.

In order to obtain the strain energy for the zeroth-order approximation, the corresponding 3D strain field can be used in the following matrices such as

$$\Gamma = \Gamma_h w + \Gamma_\epsilon \epsilon \quad (17)$$

where  $w = [w_1, w_2, w_3]^T$ ,

$\epsilon = [\gamma_{11}, 2\gamma_{12}, 2\gamma_{13}, \kappa_1, \kappa_2, \kappa_3]^T$ , and

$$\Gamma_h = \frac{1}{\eta} \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 & 0 \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial y_1} \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \quad \Gamma_\epsilon = \begin{bmatrix} 1 & 0 & 0 & x_3 - x_2 \\ 0 & 1 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (18)$$

Substituting Eq. (17) into Eq. (13), the total potential can be retained as the formally leading terms in the form

$$2\Pi_\Omega = \langle [\Gamma_h w]^T D[\Gamma_h w] + 2[\Gamma_h w]^T D[\Gamma_\epsilon \epsilon] + \epsilon^T D_{\epsilon\epsilon} \epsilon \rangle \quad (19)$$

According to the usual procedure of calculus of variations, one can obtain the result that the undetermined fluctuating functions are linearly related to  $\epsilon$ . In addition, for the general case, we need to use some numerical techniques such as FEM for calculating approximate solutions. Therefore, one can express the fluctuating functions as

$$w(x_1, x_2, x_3; y_1) = V(y_1, x_2, x_3) \epsilon(x_1) \quad (20)$$

where  $V$  is the  $3 \times 6$  matrix of the fluctuating displacement function values defined over  $\Omega$ .

In order to deal with realistic and complex UC geometries and constituent materials efficiently and conveniently, Eq. (19) is alternatively reformulated into the corresponding PDE suitable for incorporation into COMSOL such as

$$[\Gamma_h]^T \{D([\Gamma_h V] + [\Gamma_\epsilon])\} = 0 \quad \text{in } \Omega \quad (21)$$

In addition, the periodic boundary conditions in Eqs. (4) and the average constraints in Eqs. (5), (6) and (7) can be easily handled in COMSOL through the following way:

$$V(d_1/2, x_2, x_3) = V(-d_1/2, x_2, x_3) \text{ on } \partial\Omega \quad (22)$$

and

$$\langle V \rangle = 0, \quad \langle V_{2,3} - V_{3,2} \rangle = 0 \text{ and } \langle x_\alpha V_1 \rangle = 0 \quad (23)$$

Estimating the solution  $V = V_0$  of Eq.(21) from COMSOL and then substituting the solution back into Eq. (19), we can calculate the energy functional stored in the UC, asymptotically correct through the order of  $\mu\epsilon^2$  as

$$2\Pi_\Omega = \epsilon^T \langle [G_h V]^T D [G_\epsilon] + D_{\epsilon\epsilon} \rangle \epsilon = \epsilon^T \bar{S} \epsilon \quad (24)$$

where  $\bar{S}$  is the effective 1D beam stiffness calculated from the knowledge of complex geometric and material characteristics in a representative UC at the microscopic level considering the smallness of both cross-sectional dimension-to-length parameter and heterogeneity. Here the present approach is different from the 2D cross-sectional one without spanwise heterogeneity<sup>7)</sup> mainly in the following aspect. To obtain the effective 1D beam stiffness including the transverse shear stiffness, Refs. need to carry out the first-order approximation to obtain the asymptotically correct strain energy through the second-order approximation and additional transformation processes to find an equivalent Timoshenko beam model (the first-order shear deformation model). while the present approach directly derive one through the zeroth-order approximation without significant loss of accuracy and inordinary burn of computational cost, especially on 3D problem with the dominant effects of transverse shear deformations.

## 4. Validation Examples

As a preliminary validation of the present

approach, we first investigate a simple beam made of an isotropic material. Second one, two heterogeneous beams having solid inverted T-section and multi-celled box section at the microscopic level and more significant transverse shear deformation at the macroscopic level are used to demonstrate the accuracy and capability of the proposed theory and the differences between the VABS (Variational Asymptotic Beam Sectional Analysis)<sup>7)</sup> and the present approach.

### 4.1 a simple beam made of an isotropic material

The first example is a simple beam made of one isotropic material with the material axes the same as the global coordinates  $x_\alpha$  studied in Cesnik<sup>2)</sup>. It has dimensions  $a_2 = 2[\text{m}]$  by  $a_3 = 2[\text{m}]$ . The material properties of the beam are  $E = 2.6 \times 10^7 [\text{Pa}]$  and  $\nu = 0.3$ . According to the present theory, we can model this beam using two approaches: (a) 2D UC without periodicity and (b) 3D UC with spanwise periodicity along  $x_1$ . As expected, we have verified that these two modeling approaches yield the same effective 1D beam stiffnesses as the closed-form solution of the Saint-Venant stiffnesses and the transverse shear stiffness using the shear correction factors (5/6) given by Cesnik<sup>2)</sup>, with  $\bar{S}_{11} = 1.040 \times 10^8 [\text{N}]$ ,  $\bar{S}_{44} = 2.249 \times 10^7 [\text{N} \cdot \text{m}^2]$ ,  $\bar{S}_{55} = \bar{S}_{66} = 3.467 \times 10^7 [\text{N} \cdot \text{m}^2]$ , and  $\bar{S}_{22} = \bar{S}_{33} = 3.334 \times 10^7 [\text{N} \cdot \text{m}]$ .

### 4.2 solid inverted T-sectional and multi-celled box-sectional beams

Second, let us consider two wind turbine blade models having a solid inverted T and a multi-cell box unit sections studied in Jonnalagadda and Whitcomb(JW)<sup>4)</sup>:

For the solid inverted T unit section with the geometric variables and the material properties are

given by  $h_t = 2[\text{m}]$ ,  $d_t = 1[\text{m}]$  and  $E = 300 \times 10^9 [\text{Pa}]$ ,  $\nu = 0.49$  (Fig. 2-(a))

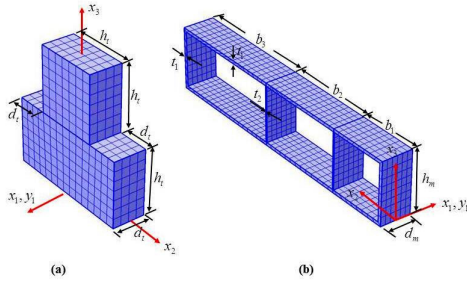


Fig. 2 Meshed (a) solid inverted T and (b) multi-cell box unit sections for wind turbine blades

For the multi-cell box unit section with the geometric variables and the material properties are given by  $d_m = 2[\text{m}]$ ,  $b_1 = h_m = 2[\text{m}]$ ,  $b_2 = 3[\text{m}]$ ,  $b_3 = 4[\text{m}]$ ,  $t_1 = 0.1[\text{m}]$ ,  $t_2 = 0.05[\text{m}]$  and  $E = 10.153 \times 10^6 [\text{Pa}]$ ,  $\nu = 0.35$  (Fig. 2-(b))

The effective 1D beam stiffnesses predicted by the approach in Jonnalagadda and Whitcomb<sup>4)</sup>, VABS based on the first-order approximation and the present approach based on the zeroth-order one are listed in Table 1.

Table 1 Effective 1D beam stiffnesses obtained by Jonnalagadda and Whitcomb (JW)<sup>4)</sup>, VABS and the present approach (PA)

	Case (a) $\times 10^{12}$			Case (b) $\times 10^7$		
	JW	VABS	PA	JW	VABS	PA
$\bar{S}_{11}$	3.60	3.60	3.60	2.38	2.38	2.38
$\bar{S}_{15}$	6.00	6.00	6.00	2.38	2.38	2.38
$\bar{S}_{22}$	0.88	0.88	0.89	0.06	0.06	0.06
$\bar{S}_{24}$	-1.28	-1.27	-1.29	-0.06	-0.06	-0.06
$\bar{S}_{33}$	0.81	0.81	0.81	0.01	0.01	0.01
$\bar{S}_{44}$	3.35	3.33	3.32	5.36	5.37	5.36
$\bar{S}_{55}$	14.4	14.4	14.4	4.18	4.18	4.18
$\bar{S}_{66}$	3.60	3.60	3.60	66.6	66.7	66.6

As expected, these results for wind turbine blade models show good agreement between three approaches. Here we digress to point out that the present approach is much more efficient because using the approach in Jonnalagadda and Whitcomb<sup>4)</sup> one needs to carry out six analyses of a 3D unit cells under six different sets of boundary conditions and load conditions and postprocess the 3D stresses to compute the beam stress resultants, while using the present approach, one only needs to carry out one analysis of a 3D UC and any postprocess is not required. Also, using VABS one need to produce a refined model, which requires the first-order approximation and additional transformation processes that include transverse shear effects, but the present approach still carries out the zeroth-order approximation which can maximize simplicity and promote efficiency, with or without spanwise periodicity.

## 5. Conclusions

The variational asymptotic method leading to simultaneous homogenization and dimensional reduction is used to construct a new model for investigating the micro-mechanical behavior of heterogeneous beam-like structures, which are composed of periodically-repeated microstructures along the axial direction. Without significant loss of simplicity and efficiency, this model serves as a rigorous link between the original 3D heterogeneous problem and the simple engineering beam models such as the zeroth-order Timoshenko beam model. As a preliminary validation of the present approach, two examples available in literature are used to demonstrate the consistence and efficiency of this new model, especially for the structures, in which the effects of transverse shear deformations are significant.

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