

NOTE ON LOCAL BOUNDEDNESS FOR WEAK SOLUTIONS OF NEUMANN PROBLEM FOR SECOND-ORDER ELLIPTIC EQUATIONS

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ABSTRACT. The goal of this note is to provide a detailed proof for local boundedness estimate near the boundary for weak solutions for second order elliptic equations with bounded measurable coefficients subject to Neumann boundary condition.

1. INTRODUCTION AND MAIN RESULTS

The goal of this note is to provide a detailed proof for local boundedness estimates near the boundary for weak solutions for second order elliptic equations in divergence form with bounded measurable coefficients subject to Neumann boundary condition. Local Hölder continuity as well as local boundedness of weak solutions are very well known and usually referred to as De Giorgi-Moser-Nash theory. There is rich literature regarding this theory; for elliptic equations with Dirichlet boundary condition, one of most popular reference is a book by Gilbarg and Trudinger [1]. In contrast to Dirichlet problem, literatures dealing with Neumann (or conormal) problem are limited. This is partly because pure conormal problem, as compared to oblique derivative problem, is easy to handle, and our guess is that experts might have considered it unworthy of elaborating.

In a recent paper [2], we stated without proof that weak solutions of second order elliptic equations in divergence form with measurable coefficients in a Lipschitz domain with smooth conormal data on its boundary satisfy a certain local boundedness estimate. This statement is of course well known to experts. However, it turned out that it is very hard to locate a specific reference in the existing literature. There are a few classical and modern books discussing conormal boundary conditions (for example, [3, 4, 5, 6]), but none of them contains the exact local boundedness estimate as asserted in [2]; on the contrary, the corresponding estimate for Dirichlet problem is easily found in [1, Theorem 8.25]. As a matter of fact, since the publication of [2], we have received inquiries about its exact reference, and being unable to identify a

Received by the editors April 20 2015; Revised May 16 2015; Accepted in revised form May 16 2015; Published online June 12 2015.

2000 *Mathematics Subject Classification.* 35B45, 35J25.

Key words and phrases. De Giorgi estimate, Neumann boundary condition, measurable coefficients.

S. Kim is partially supported by NRF-2012R1A1A2040411.

satisfactory source, we reluctantly made a short note containing a detailed proof and circulated; eventually, an idea of making it accessible to public occurred to us. By no means we claim this note is original but we hope it serve as a good reference to non-experts.

Let Ω be a domain in \mathbb{R}^d ($d \geq 2$) such that Sobolev embedding and the trace embedding are both available; i.e., if $1 \leq p < d$, then there exist positive constants $\gamma_0 = \gamma_0(d, p, \Omega)$ and $\gamma_1 = \gamma_1(d, p, \partial\Omega)$ such that for any $u \in W^{1,p}(\Omega)$ we have

$$\|u\|_{L^{dp/(d-p)}(\Omega)} \leq \gamma_0 \|u\|_{W^{1,p}(\Omega)} \quad (1.1)$$

and

$$\|u\|_{L^{p(d-1)/(d-p)}(\partial\Omega)} \leq \gamma_1 \|u\|_{W^{1,p}(\Omega)}, \quad (1.2)$$

where we use notation

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^d \|D_i u\|_{L^p(\Omega)}.$$

It is a well known fact that the above inequalities (1.1), (1.2) are satisfied when Ω is a bounded Lipschitz domain.

In this note, we consider the Neumann problem

$$\begin{cases} -D_i(a^{ij} D_j u) = \operatorname{div} \mathbf{F} - f & \text{in } \Omega, \\ (a^{ij} D_j u + F^i) n_i = g & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where a^{ij} are coefficients defined on Ω satisfying the uniform ellipticity and boundedness condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a^{ij} \xi_i \xi_j, \quad \sum_{i,j=1}^d |a^{ij}|^2 \leq \Lambda^2, \quad (1.4)$$

for some positive constants λ and Λ . We of course assume the compatibility condition

$$\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, d\sigma \quad \text{if } |\Omega| < \infty. \quad (1.5)$$

We say that $u \in W^{1,2}(\Omega)$ is a weak solution of (1.3) if the following identity holds:

$$\int_{\Omega} (a^{ij} D_j u D_i v + F^i D_i v + f v) \, dx = \int_{\partial\Omega} g v \, d\sigma, \quad \forall v \in C_c^1(\bar{\Omega}),$$

where $C_c^1(\bar{\Omega})$ is the set of all C^1 functions with compact support in $\bar{\Omega}$; if Ω is bounded, then $C_c^1(\bar{\Omega})$ agrees with $C^1(\bar{\Omega})$.

We are now ready to state our main theorem.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^d$ be such that Sobolev embedding and trace inequalities (1.1), (1.2) are available. Assume that*

$$\mathbf{F} \in L^{p_1}(\Omega)^d, \quad f \in L^{p_2}(\Omega), \quad g \in L^{p_3}(\partial\Omega),$$

where $p_1 > d$, $p_2 > d/2$, and $p_3 > d - 1$. Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1.3) with conditions (1.4) and (1.5). Then u is locally bounded in $\bar{\Omega}$ and there exists a positive

constant C that can be determined quantitatively a priori only in terms of the set of parameters $\{d, \lambda, \Lambda, p_1, p_2, p_3\}$ and Ω via the constants γ_0 and γ_1 such that for any $x_0 \in \overline{\Omega}$ and $0 < r \leq 1$, we have

$$\begin{aligned} \|u\|_{L^\infty(\Omega_{r/2})} \leq & C \left(r^{-\frac{d}{2}} \|u\|_{L^2(\Omega_r)} + r^{1-\frac{d}{p_1}} \|\mathbf{F}\|_{L^{p_1}(\Omega_r)} \right. \\ & \left. + r^{2-\frac{d}{p_2}} \|f\|_{L^{p_2}(\Omega_r)} + r^{1-\frac{d-1}{p_3}} \|g\|_{L^{p_3}(\Sigma_r)} \right). \end{aligned} \tag{1.7}$$

where we denote $\Omega_r = \Omega \cap B_r(x_0)$ and $\Sigma_r = \partial\Omega \cap B_r(x_0)$.

Finally, several remarks are in order.

Remark 1.8. We say that $u \in W^{1,2}(\Omega)$ is a weak sub-solution (super-solution) of (1.3) if the following identity holds:

$$\int_{\Omega} (a^{ij} D_j u D_i v + F^i D_i v + f v) \, dx \leq (\geq) \int_{\partial\Omega} g v \, d\sigma, \quad \forall v \in C_c^1(\overline{\Omega}).$$

The same proof of the theorem will show that if u is a weak sub-solution (super-solution) of the Neumann problem (1.3), then the estimates (1.7) remains valid with u replaced by $u_+ = \max(u, 0)$ ($u_- = \max(-u, 0)$).

Remark 1.9. It should be clear from the proof that in the case when $g \equiv 0$, the constant C in the theorem does not depend on the parameter γ_1 . Therefore, in the case when $g \equiv 0$, it is not required to assume that the domain Ω enjoy the Sobolev trace inequality (1.2).

Remark 1.10. It should be also clear from the proof that (1.7) holds for any $0 < r \leq r_0$, if we allow the constant C to depend on r_0 as well. In particular, if Ω is a bounded Lipschitz domain, then we obtain condition (LB) of [2] with $C_1 = C_1(d, \lambda, \Lambda, \Omega, \text{diam } \Omega)$.

Remark 1.11. If Ω is a Lipschitz epigraph domain (i.e., a domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$), then for $1 \leq p < d$ we have

$$\|u\|_{L^{dp/(d-p)}(\Omega)} \leq \gamma \|Du\|_{L^p(\Omega)}$$

where γ is a constant depending only on d, p , and the Lipschitz constant K of the domain (see [2, Lemma 6.8]). Therefore, if $g = 0$, then the restriction $r \leq 1$ is not needed in (2.3) and (2.6). So, in the case when Ω is a Lipschitz epigraph domain, we also get the condition (LB') of [2] with $C_1 = C_1(d, \lambda, \Lambda, K)$.

Remark 1.12. We say that u is a weak solution of

$$\begin{cases} -D_i(a^{ij} D_j u) = \text{div } \mathbf{F} - f & \text{in } \Omega_r, \\ (a^{ij} D_j u + F^i) n_i = g & \text{on } \Sigma_r, \end{cases} \tag{1.13}$$

if the following identity holds for any $v \in C_c^1(\Omega_r \cup \Sigma_r)$:

$$\int_{\Omega_r} (a^{ij} D_j u D_i v + F^i D_i v + f v) \, dx = \int_{\Sigma_r} g v \, d\sigma.$$

We note that the estimate (1.7) is local in nature. In fact, if $u \in W^{1,2}(\Omega_r)$ is a weak solution of (1.13), then the same proof will show that the estimate (1.7) still holds.

Remark 1.14. In addition to the local boundedness estimate (1.7) for a weak solution of (1.3), local Hölder continuity estimate (near the boundary) is also available; we leave it to interested reader to fill the details.

2. PROOF OF THE THEOREM

We prove the theorem by adapting the idea of De Giorgi. For $n = 1, 2, \dots$, we denote

$$r_n = \frac{r}{2} + \frac{r}{2^n}, \quad k_n = k \left(2 - \frac{1}{2^{n-1}} \right), \quad A_n = \{x \in \Omega_{r_n} : u(x) > k_n\},$$

where k is a nonnegative constant to be chosen later. We denote

$$v_n = (u - k_n)_+,$$

and let $\eta = \eta_n$ be a smooth function in \mathbb{R}^d satisfying

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{r_{n+1}}, \quad \text{supp } \eta \subset B_{r_n}, \quad |D\eta| \leq 2^{n+2}r^{-1}.$$

By testing with $\eta^2 v_n$ in (1.3), we get

$$\begin{aligned} \int_{\Omega} \eta^2 a^{ij} D_j v_n D_i v_n dx &= - \int_{\Omega} a^{ij} D_j v_n 2\eta D_i \eta v_n dx \\ &\quad - \int_{\Omega} (F^i \eta^2 D_i v_n + F^i 2\eta D_i \eta v_n) dx - \int_{\Omega} f \eta^2 v_n dx + \int_{\partial\Omega} g \eta^2 v_n d\sigma. \end{aligned}$$

By using the ellipticity and the properties of η , we get

$$\begin{aligned} \lambda \int_{A_n} \eta^2 |Dv_n|^2 dx &\leq 2\Lambda \int_{A_n} \eta |Dv_n| |D\eta| v_n dx \\ &\quad + \int_{A_n} |\mathbf{F}| \eta |Dv_n| dx + 2 \int_{A_n} |\mathbf{F}| |D\eta| v_n dx + \int_{\Omega} |f| \eta v_n dx + \int_{\partial\Omega} |g| \eta v_n d\sigma, \end{aligned}$$

and thus, by Cauchy's inequality and Hölder's inequality, we have

$$\begin{aligned} \int_{A_n} \eta^2 |Dv_n|^2 dx &\leq C(\lambda, \Lambda) \frac{4^n}{r^2} \int_{A_n} v_n^2 dx \\ &\quad + C(\lambda) \left(\int_{A_n} |\mathbf{F}|^2 dx + \int_{\Omega} |f| \eta v_n dx + \int_{\partial\Omega} |g| \eta v_n d\sigma \right). \end{aligned} \quad (2.1)$$

By Hölder's inequality we get

$$\int_{A_n} |\mathbf{F}|^2 dx \leq \|\mathbf{F}\|_{L^{p_1}(\Omega_r)}^2 |A_n|^{1 - \frac{2}{p_1}}. \quad (2.2)$$

When $d > 2$ we set $p = 2$ and when $d = 2$ we take any $p \in [1, 2)$ satisfying $\frac{3}{2} - \frac{1}{p_2} \geq \frac{1}{p}$. Then by Hölder's inequality, (1.1), and Cauchy's inequality, we have

$$\begin{aligned} \int_{\Omega} |f|\eta v_n \, dx &\leq \|f\|_{L^{dp/(dp-d+p)}(A_n)} \|\eta v_n\|_{L^{dp/(d-p)}(\Omega)} \\ &\leq \gamma_0 \|f\|_{L^{p_2}(A_n)} |A_n|^{1-\frac{1}{p}+\frac{1}{d}-\frac{1}{p_2}} \|\eta v_n\|_{W^{1,p}(\Omega)} \\ &\leq \gamma_0 \|f\|_{L^{p_2}(A_n)} |A_n|^{\frac{1}{2}+\frac{1}{d}-\frac{1}{p_2}} \|\eta v_n\|_{W^{1,2}(\Omega)} \\ &\leq (\gamma_0^2/4\epsilon) \|f\|_{L^{p_2}(\Omega_r)}^2 |A_n|^{1+\frac{2}{d}-\frac{2}{p_2}} + \epsilon \|\eta v_n\|_{W^{1,2}(\Omega)}^2 \quad \text{for any } \epsilon > 0, \end{aligned}$$

where $\gamma_0 = \gamma_0(d, \Omega, p_2)$. Let $q = dp_3/(dp_3 - d + 1)$. Then similarly we get

$$\begin{aligned} \int_{\partial\Omega} |g|\eta v_n \, d\sigma &\leq \|g\|_{L^{p_3}(\Sigma_r)} \|\eta v_n\|_{L^{p_3/(p_3-1)}(\partial\Omega)} \leq \gamma_1 \|g\|_{L^{p_3}(\Sigma_r)} \|\eta v_n\|_{W^{1,q}(\Omega)} \\ &\leq \gamma_1 \|g\|_{L^{p_3}(\Sigma_r)} \|\eta v_n\|_{W^{1,2}(\Omega)} |A_n|^{\frac{1}{2}-\frac{d-1}{dp_3}} \\ &\leq (\gamma_1^2/4\epsilon) \|g\|_{L^{p_3}(\Sigma_r)}^2 |A_n|^{1-\frac{2(d-1)}{dp_3}} + \epsilon \|\eta v_n\|_{W^{1,2}(\Omega)}^2, \end{aligned}$$

where $\gamma_1 = \gamma_1(d, \Omega, p_3)$. On the other hand, we have

$$\|\eta v_n\|_{W^{1,2}(\Omega)}^2 = \int_{\Omega} |\eta v_n|^2 + |D(\eta v_n)|^2 \, dx \leq \int_{\Omega} |\eta v_n|^2 + 2|D\eta|^2 v_n^2 + 2\eta^2 |Dv_n|^2 \, dx.$$

Therefore, we get (use $r \leq 1$)

$$\begin{aligned} \int_{\Omega} |f|\eta v_n \, dx + \int_{\partial\Omega} |g|\eta v_n \, d\sigma &\leq \frac{\gamma_0^2}{4\epsilon} \|f\|_{L^{p_2}(\Omega_r)}^2 |A_n|^{1+\frac{2}{d}-\frac{2}{p_2}} \\ &\quad + \frac{\gamma_1^2}{4\epsilon} \|g\|_{L^{p_3}(\Sigma_r)}^2 |A_n|^{1-\frac{2(d-1)}{dp_3}} + 2\epsilon \left(\frac{4^{n+3}}{r^2} \int_{A_n} v_n^2 \, dx + 2 \int_{A_n} \eta^2 |Dv_n|^2 \, dx \right). \quad (2.3) \end{aligned}$$

Let $\delta = \min\left(\frac{2}{d} - \frac{2}{p_1}, \frac{4}{d} - \frac{2}{p_2}, \frac{2}{d} - \frac{2(d-1)}{dp_3}\right) > 0$. Then we get from (2.1), (2.2), and (2.3) that

$$\begin{aligned} \int_{A_n} \eta^2 |Dv_n|^2 \, dx &\leq C(\lambda, \Lambda) \frac{4^n}{r^2} \int_{A_n} v_n^2 \, dx + C(\lambda) |A_n|^{1+\delta-\frac{2}{d}} |B_r|^{\frac{2}{d}-\frac{2}{p_1}-\delta} \|\mathbf{F}\|_{L^{p_1}(\Omega_r)}^2 \\ &\quad + C(\lambda, \gamma_0, \gamma_1) |A_n|^{1+\delta-\frac{2}{d}} \left(|B_r|^{\frac{4}{d}-\frac{2}{p_2}-\delta} \|f\|_{L^{p_2}(\Omega_r)}^2 + |B_r|^{\frac{2}{d}-\frac{2(d-1)}{dp_3}-\delta} \|g\|_{L^{p_3}(\Sigma_r)}^2 \right). \end{aligned}$$

Therefore, if we set

$$M := |B_r|^{-\frac{1}{p_1}} \|\mathbf{F}\|_{L^{p_1}(\Omega_r)} + |B_r|^{\frac{1}{d}-\frac{1}{p_2}} \|f\|_{L^{p_2}(\Omega_r)} + |B_r|^{-\frac{d-1}{dp_3}} \|g\|_{L^{p_3}(\Sigma_r)},$$

then we have

$$\int_{A_n} \eta^2 |Dv_n|^2 \, dx \leq C(\lambda, \Lambda) \frac{4^n}{r^2} \int_{A_n} v_n^2 \, dx + C(\lambda, \gamma_0, \gamma_1) |A_n|^{1+\delta-\frac{2}{d}} |B_r|^{\frac{2}{d}-\delta} M^2. \quad (2.4)$$

Now, denote

$$Y_n := \int_{A_n} \eta_n^2 v_n^2 dx,$$

and observe that

$$Y_n \geq \int_{A_{n+1}} v_n^2 dx \geq \frac{k^2}{4^n} |A_{n+1}|. \quad (2.5)$$

Then it follows from (1.1) and Hölder's inequality that (use $r \leq 1$)

$$\begin{aligned} Y_{n+1} &\leq \gamma'_0 \|\eta_{n+1} v_{n+1}\|_{W^{1,2d/(d+2)}(\Omega)}^2 \leq \gamma'_0 |A_{n+1}|^{\frac{2}{d}} \|\eta_{n+1} v_{n+1}\|_{W^{1,2}(\Omega)}^2 \\ &\leq \gamma'_0 |A_{n+1}|^{\frac{2}{d}} \left(\frac{4^{n+3}}{r^2} \int_{A_{n+1}} v_{n+1}^2 dx + 2 \int_{A_{n+1}} \eta_{n+1}^2 |Dv_{n+1}|^2 dx \right), \end{aligned} \quad (2.6)$$

where $\gamma'_0 = \gamma'_0(d, \Omega)$. This together with (2.4) implies that (use $v_n \geq v_{n+1}$)

$$Y_{n+1} \leq C_0 |A_{n+1}|^{\frac{2}{d}} \left(\frac{4^n}{r^2} \int_{A_{n+1}} v_n^2 dx + |A_{n+1}|^{1+\delta-\frac{d}{2}} |B_r|^{\frac{2}{d}-\delta} M^2 \right),$$

where $C_0 = C_0(\lambda, \Lambda, \gamma_0, \gamma'_0, \gamma_1) = C_0(d, \lambda, \Lambda, \Omega, p_2, p_3)$. Therefore, by (2.5) we obtain (use $0 < \frac{2}{d} - \delta$)

$$\begin{aligned} Y_{n+1} &\leq C_0 |B_r|^{\frac{2}{d}-\delta} \left(\frac{4^n}{r^2} |A_{n+1}|^\delta Y_n + |A_{n+1}|^{1+\delta} M^2 \right) \\ &\leq C_0 |B_r|^{\frac{2}{d}-\delta} \left(\frac{4^n}{r^2} \left(\frac{4^n}{k^2} \right)^\delta + \left(\frac{4^n}{k^2} \right)^{1+\delta} M^2 \right) Y_n^{1+\delta} \\ &\leq C_0 |B_r|^{\frac{2}{d}-\delta} \frac{4^{2n}}{r^2} k^{-2\delta} \left(1 + \frac{r^2 M^2}{k^2} \right) Y_n^{1+\delta}. \end{aligned}$$

We take

$$k := 4^{1/\delta^2} (2C_0 |B_1|^2)^{1/2\delta} |B_r|^{-\frac{1}{2}} \|u\|_{L^2(\Omega_r)} + rM \quad \text{and} \quad K := 2C_0 r^{-2} |B_r|^{\frac{2}{d}-\delta} k^{-2\delta}.$$

Then we have ($rM \leq k$)

$$Y_{n+1} \leq 16^n K Y_n^{1+\delta}$$

and

$$Y_1 \leq \int_{\Omega_r} |u|^2 dx \leq 16^{-1/\delta^2} K^{-1/\delta}.$$

The following lemma is taken from [6, Lemma 15.1, p. 319].

Lemma 2.7. *Let $\{Y_n\}$ be a sequence of positive numbers linked by the recursive inequalities*

$$Y_{n+1} \leq b^n K Y_n^{1+\sigma}$$

for some $b > 1$, $K > 0$, and $\delta > 0$. If

$$Y_1 \leq b^{-1/\sigma^2} K^{-1/\delta},$$

then $\{Y_n\} \rightarrow 0$ as $n \rightarrow \infty$.

By the lemma, we have $Y_n \rightarrow 0$ as $n \rightarrow \infty$, and thus, we get

$$u \leq 2k \quad \text{on } \Omega_{r/2}.$$

By applying the same argument to $-u$, we obtain the estimate (1.7) from the definition of M and k . ■

ACKNOWLEDGMENTS

I thank Jongkeun Choi who helped me with first draft of this note.

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