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ON ERROR ESTIMATES OF AN IMPLICIT ITERATION SCHEME

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Abstract. The purpose of this note is to study the estimation of errors of the implicit Mann iterative process with random errors.

1. Introduction

In 1995, Liu [3] introduced the Mann iteration process with errors. In [7], Xu pointed out that Liu's definition [3] depend on the convergence of error terms is not compatible with randomness because the occurence of error terms is always random. During the last few years or so many authors under certain conditions have employed the Mann and Ishikawa iteration methods with errors for the itrative approximation of the solution of nonlinear equations with eccretive operator and the fixed points of psedocontractive mappings.

Let K be a nonempty convex subset of a real Banach space X. The sequence $\{x_n\}$ called implicit Mann iteration process [1, 5-6] is defined as

$$\begin{cases} x_0 \in K, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_n,\ n \geq 1, \end{cases} \tag{1}$$

where $T : K \longrightarrow K$ is a mapping and $\{\alpha_n\}$ is a real sequence satisfying some conditins.

In this paper we study that the implicit Mann iterative process (1) is influenced by the random errors. We show that the accumulative errors in the iterative process are bounded and the errors are controllable in a permissible range if we select $\{\alpha_n\}$ appropriately.

2. Main Results

Following the approach of [8], suppose that $T : K \longrightarrow K$ is a mapping. For any $x_n \in K$ $(n \geq 1)$ define the error of Tx_n by $u_n = Tx_n - \overline{Tx_n}$, where $\overline{Tx_n}$

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is an exact value of Tx_n, in other words Tx_n is an approximate value of $\overline{Tx_n}$. It is easy to know that $\{\|u_n\|\}$ is bounded from the theory of errors. Hence we set $M = \sup\{\|u_n\| : n \geq 1\}$ which is called the bounds on absolute errors of $\{Tx_n\}$. In the implicit Mann iterative process the errors come from Tx_n and the errors of last iteration will affect next. So we have

$$\begin{aligned}
x_0 &= \overline{x_0}, \\
x_1 &= (1-\alpha_1)x_0 + \alpha_1 Tx_1 \\
&= (1-\alpha_1)\overline{x_0} + \alpha_1(\overline{Tx_1} + u_1) \\
&= \overline{x_1} + \alpha_1 u_1, \\
x_2 &= (1-\alpha_2)x_1 + \alpha_2 Tx_2 \\
&= (1-\alpha_2)\overline{x_1} + \alpha_2(\overline{Tx_2} + u_2) + (1-\alpha_2)\alpha_1 u_1 \\
&= \overline{x_2} + (1-\alpha_2)\alpha_1 u_1 + \alpha_2 u_2, \\
x_3 &= (1-\alpha_3)x_2 + \alpha_3 Tx_3 \\
&= (1-\alpha_3)\overline{x_2} + \alpha_3(\overline{Tx_3} + u_3) + (1-\alpha_3)(1-\alpha_2)\alpha_1 u_1 + (1-\alpha_3)\alpha_2 u_2 \\
&= \overline{x_3} + (1-\alpha_3)(1-\alpha_2)\alpha_1 u_1 + (1-\alpha_3)\alpha_2 u_2 + \alpha_3 u_3.
\end{aligned}$$

Hence by induction, we have

$$\begin{aligned}
x_n &= (1-\alpha_n)x_{n-1} + \alpha_n Tx_n \\
&= \overline{x}_n + (1-\alpha_n)(1-\alpha_{n-1})...(1-\alpha_2)\alpha_1 u_1 \\
&\quad +(1-\alpha_n)(1-\alpha_{n-1})...(1-\alpha_2)\alpha_2 u_2 + ... \\
&\quad +(1-\alpha_{n-2})\alpha_{n-1}u_{n-1} + \alpha_n u_n \\
&= \overline{x}_n + \sum_{j=1}^{n} \alpha_j u_j \prod_{i=j+1}^{n} (1-\alpha_i),
\end{aligned}$$

for all $n \geq 1$.
Putting

$$S_n = x_n - \overline{x}_n = \sum_{j=1}^{n} \alpha_j u_j \prod_{i=j+1}^{n} (1-\alpha_i), \tag{2}$$

for all $n \geq 1$. Obvious, the errors of iterative process, after $n+1$ times iterations, are added up to S_n.

Now we prove a main result as follows.

Theorem 2.1. *Let T and S_n be as above , then there exists a constasnt $k \in (0,1)$ such that $\|S_n\| \leq kM$, $n \geq 1$.*

Proof. In fact, from (2) we have

$$\begin{aligned} \|S_n\| &\leq M\sum_{j=1}^{n}\alpha_j\prod_{i=j+1}^{n}(1-\alpha_i) \\ &= M\left(1-\prod_{i=1}^{n}(1-\alpha_i)\right), \end{aligned} \tag{2}$$

for all $n \geq 1$.

Putting $k = 1 - \prod_{i=0}^{n}(1-\alpha_i)$, since $\prod_{i=0}^{n}(1-\alpha_i) \in (0,1)$, therefore, $k \in (0,1)$ and $\|S_n\| \leq kM$, $n \geq 1$. □

From this theorem, we obtain some results as follows:

Remark 1. 1. The accumulative errors in the Mann iterative process is bounded and it is not more than the bounds on absolute error of $\{Tx_n\}$.

2. If $\sum_{j=0}^{n}\alpha_j = +\infty$ then $\prod_{i=0}^{n}(1-\alpha_i) = 0$. It is implies that $\|S_n\| \leq M$, $n \geq 1$.

3. If $\sum_{j=0}^{n}\alpha_j < +\infty$ then $\prod_{i=0}^{n}(1-\alpha_i) \in (0,1]$, i.e., $k \in (0,1)$. It implies that $\|S_n\| \leq kM$, $n \geq 1$.

3. Applications

(1) Taking $\alpha_i = 1/(i+2)^2$, $n \geq 1$ then $\prod_{i=0}^{n}(1-\alpha_i) = 1/2$, i.e., $k = 1/2$. It imlies that $\|S_n\| \leq M/2$, $n \geq 1$.

(2) In particular, for any $\varepsilon \in (0,1)$, taking $\alpha_i = \varepsilon/2^{i+2}$,

$$\prod_{i=0}^{n}(1-\alpha_i) \geq 1 - \sum_{j=0}^{n}\alpha_j = 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

Consequently, $k < \varepsilon$. It implies that $\|S_n\| \leq \varepsilon M$, $n \geq 1$. Hence, the random errors is controllable in a permissible range if we can select an $\{\alpha_i\}$ appropriately.

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