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BOUNDARY BEHAVIOR OF HOLOMORPHIC DISCS IN CONVEX FINITE TYPE DOMAINS

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Abstract. In this paper, we study holomorphic discs in a domain with a plurisubharmonic peak function at a boundary point. The aim is to describe boundary behavior of holomorphic discs in convex finite type domains in the complex Euclidean space in term of a special local neighborhood system at a boundary point.

1. Introduction

We will describe the boundary behavior of holomorphic discs in a domain (connected open set) in $\mathbb{C}^n$ in terms of a certain local neighborhood system at a boundary point. This research has its origin in author's thesis ([5]) for strongly pseudoconvex domains in almost complex manifolds. An aim of [5] was to study a convergence of the scaling sequence in almost complex manifolds. In order to get the convergence, it needs to consider a special neighborhood system at a strongly pseudoconvex boundary point which is invariant under the non-isotropic dilation. Let Ω be a domain in $\mathbb{C}^n$ which has the hyperplane $\{z \in \mathbb{C}^n : \operatorname{Re} z_1 = 0\}$ as a tangent plane at the strongly pseudonvex boundary point $0 \in \partial\Omega$. Then we consider the local neighborhood system $\{Q(0,\delta) : \delta > 0\}$ of 0 where

$$Q(0,\delta) = \{z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : |z_1| < \delta, |z'| < \delta^{1/2}\},$$

which is invariant under the dilation $\mathcal{D}_t(z_1, z') = (tz_1, t^{1/2}z')$ $(t > 0)$, that is, $\mathcal{D}_t(Q(0,\delta)) = Q(0, t\delta)$ for any t and δ. Proposition 3.2 in [5] states that if a holomorphic disc $u : \Delta \to \Omega$ satisfies $u(0) \in Q(0,\delta)$ for a sufficiently small δ, then $u(\Delta_r) \subset Q(0, C\delta)$ for some constant C which is independent of u. Here we denote by $\Delta_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ and $\Delta = \Delta_1$. This result is based on a localization lemma of holomorphic discs due to Ivashkovich-Rosay (Lemma 2.2

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in [4]) which says that a certain estimate of a plurisubharmonic peak function of a boundary point may well control holomorphic discs whose origin is close to the point.

In this paper, we will give a similar result for domains in $\mathbb{C}^n$ of finite type in the sense of D'Angelo [1]. In Section 2, we shall give a localization result (Lemma 2.1) from a plurisubharmonic peak function with a certain boundary estimate. Then the boundary behavior of holomorphic discs in a convex finite type domain will be described in terms of a suitable local neighborhood system in Section 3.

2. A localization

Let Ω be a domain in $\mathbb{C}^n$ and p be a boundary point of Ω. If there is a real-valued function φ_p on $\overline{\Omega} \cap U$ for a neighborhood U of p such that φ_p is plurisubharmonic on $\Omega \cap U$, $\varphi_p < 0$ on $\overline{\Omega} \cap U \setminus \{p\}$ and $\varphi_p(p) = 0$, then we call φ_p a *local plurisubharmonic peak function at* p. It is well-known that if $p \in \partial\Omega$ admits a plurisubharmonic peak function, then for any neighborhood V of p and any real number r with $0 < r < 1$ there is a neighborhood W of p such that $u(\Delta_r) \subset V$ for any holomorphic disc $u : \Delta \to \Omega$ with $u(0) \in W$ (see Lemma 2.1 in [4]).

Lemma 2.1. *Let p be a boundary point of a domain Ω in $\mathbb{C}^n$ admitting a local plurisubharmonic peak function φ_p defined on $\overline{\Omega} \cap U$ for a neighborhood U of p such that*

$$-A\,|z-p|^{\lambda} \le \varphi_p(z) \le -B\,|z-p|^{2k\lambda} \tag{1}$$

for some positive integer k and positive real numbers A, B, λ with $2k\lambda \ge 2$. Then there is a positive real number c_r for each $0 < r < 1$ such that for every holomorphic disc $u : \Delta \to \Omega$ with its oringin $u(0)$ sufficiently close to p,

$$|u(0) - u(\zeta)| \le c_r\,|u(0) - p|^{1/2k}$$

if $\zeta \in \Delta_r$.

Proof. Let us assume that the neighborhood U has a diameter less than 1. Given r, fix $r < r_1 < 1$. Since $2k\lambda > 2$, the function $|u-p|^{2k\lambda}$ is plurisubharmonic on Δ for any holomorphic disc $u : \Delta \to \mathbb{C}^n$. Applying the Poisson integral formula to $|u-p|^{2k\lambda}$, we have a constant $C = C(r, r_1)$ such that

$$|u(\zeta) - p|^{2k\lambda} \le C \int_0^{2\pi} \left|u(r_1 e^{i\theta}) - p\right|^{2k\lambda} \frac{d\theta}{2\pi}$$

for any $\zeta \in \Delta_r$.

Let $u : \Delta \to \Omega$ be a holomorphic disc whose origin $u(0)$ is sufficiently close to p. Since p admits the plurisubharmonic peak function φ_p, we may assume that $u(\overline{\Delta}_{r_1}) \subset U$. Thus we can consider the subharmonic function $\varphi_p \circ u$ defined on $\overline{\Delta}_{r_1}$. Equation (1) implies the inequality

$$-A\,|u(\zeta) - p|^{\lambda} \le \varphi_p \circ u(\zeta) \le -B\,|u(\zeta) - p|^{2k\lambda} \tag{2}$$

for $|\zeta| \le r_1$. By the second inequality of (2) and the mean value inequality of the subharmonic function $\varphi_p \circ u$, it follows that

$$C\int_0^{2\pi} \left|u(r_1e^{i\theta}) - p\right|^{2k\lambda} \frac{d\theta}{2\pi} \le -\frac{C}{B}\int_0^{2\pi} \varphi_p \circ u(r_1e^{i\theta})\frac{d\theta}{2\pi} \le -\frac{C}{B}\varphi_p \circ u(0)\ .$$

From the first inequality of (2), we have

$$|u(\zeta) - p|^{2k\lambda} \le \frac{AC}{B}\,|u(0) - p|^{\lambda}$$

for any $\zeta \in \Delta_r$. Thus we obtain that

$$|u(\zeta) - u(0)|^{2k} \le (|u(0) - p| + |u(\zeta) - p|)^{2k} \le c_r\,|u(0) - p|$$

for some c_r depending only on r, r_1 and (1). This proves the lemma. □

Suppose that Ω is strongly pseudoconvex at $p \in \partial\Omega$. Then we can choose a local defining function $\rho : U \to \mathbb{R}$ on a neighborhood U of p, a smooth function with $\Omega \cap U = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, which is strictly plurisubharmonic on U. Taking small $\varepsilon > 0$ so that $\varphi_p(z) = \rho(z) - \varepsilon\,|z - p|^2$ is also strictly plurisubharmonic near p, we have the local plurisubharmonic peak function φ_p at p with $-A\,|z - p| \le \varphi_p(z) \le -B\,|z - p|^2$. Thus we can apply Lemma 2.1 for $k = 1$.

Let $\Omega \subset\subset \mathbb{C}^n$ be a convex domain with smooth boundary and of finite type $2k$. Let $0 < \lambda < 1$. Then by Fornaess-Sibony [2], Ω admits a global plurisubharmonic peak function φ_p at each p such that

$$|\varphi_p(z) - \varphi_p(z')| \le A\,|z - z'|^{\lambda} \quad \text{for any } z, z' \in U,$$

and

$$\varphi_p(z) \le -\frac{1}{A}\,|z - p|^{2k\lambda} \quad \text{for any } z \in \Omega \cap U,$$

for some real positive number A which is independent of a choice of p. Combining these conditions, we get

$$-A\,|z - p|^{\lambda} \le \varphi_p(z) \le -\frac{1}{A}\,|z - p|^{2k\lambda}\ .$$

Since this estimate is uniform for $p \in \partial\Omega$, we have

Corollary 2.2. *Let Ω be a convex domain with smooth boundary and of finite type $2k$ and let $0 < r < 1$. Then there is a constant c_r such that for any holomorphic disc $u : \Delta \to \Omega$ whose origin $u(0)$ is sufficiently close to the boundary $\partial\Omega$,*

$$|u(0) - u(\zeta)| \le c_r\,(\mathrm{dist}(u(0), \partial\Omega))^{1/2k}$$

if $\zeta \in \Delta_r$.

If Ω be a (not necessarily convex) domain of finite type $2k$ in $\mathbb{C}^n$, every boundary point p of Ω admits a plurisubharmonic peak function φ_p with

$$-A\,|z-p| \le \varphi_p(z) \le -\,|z-p|^{2k} \ .$$

for some uniform constant A (Theorem A in [2]). Thus Corollary 2.2 also holds for finite type bounded domains in $\mathbb{C}^n$.

3. Boundary behavior of holomorphic discs on convex finite type domains

Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth boundary. For each point $p \in \partial\Omega$, let $\nu_p \in \mathbb{C}^n$ be the outward unit normal vector of $\partial\Omega$ at p. Then we decompose the complex vector space $\mathbb{C}^n$ by

$$\mathbb{C}^n = N_p \oplus T_p$$

where N_p is a complex 1-dimensional vector space generated by ν_p and T_p is its orthogonal complement. We denote by $\pi_1 : \mathbb{C}^n \to N_p$ and $\pi_2 : \mathbb{C}^n \to T_p$ corresponding orthogonal projections. For $\delta > 0$, let us define

$$Q^k_\Omega(p,\delta) = \{z \in \mathbb{C}^n : |\pi_1(z-p)| < \delta, |\pi_2(z-p)| < \delta^{1/2k}\} \ .$$

If Ω is of finite type $2k$ at p, then $\{Q^k_\Omega(p,\delta)\}$ is a suitable local neighborhood system at p in the sense of the following.

Theorem 3.1. *Let Ω be a convex domain with smooth boundary and of finite type $2k$ and let $p \in \partial\Omega$. For each $0 < r < 1$, there are a positive real number C_r such that if $u : \Delta \to \Omega$ is a holomorphic disc with $u(0) \in Q^k_\Omega(p,\delta)$ for a sufficiently small δ, then*

$$u(\Delta_r) \subset Q^k_\Omega(p, C_r\delta) \ .$$

Proof. Taking a complex rigid motion of $\mathbb{C}^n$, we may assume that $p = 0$ and $\nu_p = (1,0,\ldots,0)$. Then there are an open neighborhood U of 0 and a local defining function $\rho : U \to \mathbb{R}$ of Ω such that

$$\rho(z) = \operatorname{Re} z_1 + \varepsilon(z)$$

where $\varepsilon(z) = O(|z|^2)$ and $\varepsilon \ge 0$. Simultaneously we have

$$Q^k_\Omega(0,\delta) = \{z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : |z_1| < \delta, |z'| < \delta^{1/2k}\} \ .$$

Let $r < r_1 < r_2 < 1$ and let $u = (u_1, u') : \Delta \to \Omega$ be a holomorphic disc with $u(0) \in Q^k_\Omega(0,\delta)$ for a sufficiently small δ. Since $\operatorname{dist}(u(0), \partial\Omega) < \delta$ and $|u(0)|^2 = |u_1(0)|^2 + |u'(0)|^2 \le \delta^2 + \delta^{1/k} \le (C'\delta^{1/2k})^2$ for some uniform constant K, Corollary 2.2 implies that

$$\begin{aligned}|u(\zeta)| \le |u(\zeta) - u(0)| + |u(0)| &\le c_{r_2}\left(\operatorname{dist}(u(0), \partial\Omega)\right)^{1/2k} + |u(0)| \\ &\le c_{r_2}\delta^{1/2k} + K\delta^{1/2k} = (c_{r_2} + K)\delta^{1/2k}\end{aligned}$$

for $|\zeta| < r_2$. Therefore it suffices to prove that there is a constant C depending on r such that $|u_1(\zeta)| < C\delta$ for $|\zeta| < r$.

Let us see $\operatorname{Re} u_1$. Since the origin 0 admits a plurisubharmonic peak function, we may assume that a holomorphic disc $u : \Delta \to \Omega$ with $u(0) \in Q^k(0, \delta)$ (δ is sufficiently small) satisfies $u(\overline{\Delta}_{r_2}) \subset U$. Then $\operatorname{Re} u_1$ is a negative harmonic function on $\overline{\Delta}_{r_2}$. Applying Harnack's inequality to $\operatorname{Re} u_1$, we have

$$\frac{r_2 - r_1}{r_2 + r_1} \operatorname{Re} u_1(0) \le \operatorname{Re} u_1(\zeta)$$

for $|\zeta| < r_1$. Since $\operatorname{Re} u_1(0) > -\delta$, we have

$$-\frac{r_2 - r_1}{r_2 + r_1}\delta < \operatorname{Re} u_1(\zeta) < 0 \tag{3}$$

for $|\zeta| < r_1$.

It remains to study $\operatorname{Im} u_1$. From the interior estimates of derivatives for the harmonic function $\operatorname{Re} u_1$ (Theorem 2.10 in [3]), there is a constant $L > 0$ such that

$$\sup_{\Delta_r} |d(\operatorname{Re} u_1)| \le L \sup_{\Delta_{r_1}} |\operatorname{Re} u_1| \le L\frac{r_2 - r_1}{r_2 + r_1}\delta \ .$$

Since $|d(\operatorname{Re} u_1)| = |d(\operatorname{Im} u_1)|$ from the Cauchy-Riemann equation for the holomorphic function u_1, we have

$$\sup_{\Delta_r} |d(\operatorname{Im} u_1)| \le L\frac{r_2 - r_1}{r_2 + r_1}\delta \ .$$

Since $|\operatorname{Im} u_1(0)| < \delta$, the mean value theorem implies that there is a constant $C = C(r, r_1, r_2)$ such that $|\operatorname{Im} u_1(\zeta)| < C\delta$ for $|\zeta| < r$. With (3) this completes the proof. □

Using the interior derivative estimates for $\operatorname{Im} u_1$ and the Cauchy estimates for $u_2, \dots, u_n$, we have the following under the same setting of the proof.

Corollary 3.2. *There is a constant C for r such that*

$$\|\pi_1 \circ u\|_{C^1(\Delta_r)} < C\delta \ ,$$
$$\|\pi_2 \circ u\|_{C^1(\Delta_r)} < C\delta^{1/2k} \quad (j = 2, \dots, n) \ ,$$

for any holomorphic disc u in Ω with $u(0) \in Q^k_\Omega(0, \delta)$ for sufficiently small δ.

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