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OPTIMALITY CONDITIONS AND DUALITY IN FRACTIONAL ROBUST OPTIMIZATION PROBLEMS

Moon Hee Kim and Gwi Soo Kim

Abstract. In this paper, we consider a fractional robust optimization problem (FP) and give necessary optimality theorems for (FP). Establishing a nonfractional optimization problem (NFP) equivalent to (FP), we formulate a Mond-Weir type dual problem for (FP) and prove duality theorems for (FP).

1. Introduction

Consider a fractional robust optimization problem:

$$\text{(FP)} \quad \inf_{x \in \mathbb{R}^n} \left\{ \frac{f(x)}{g(x)} : h_j(x, v_j) \leq 0, \ \forall v_j \in V_j, \ j = 1, \cdots, m \right\},$$

where v_j are uncertain parameters and $v_j \in V_j, \ j = 1, \cdots, m$ for some convex compact sets $V_j \subset \mathbb{R}^q, \ j = 1, \cdots, m$ and $f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, \ j = 1, \cdots, m$ are continuously differentiable functions. Assume that $f(x) \geq 0$ and $g(x) > 0$.

Let $F := \{x \in \mathbb{R}^n : h_j(x, v_j) \leq 0, \ \forall v_j \in V_j, \ j = 1, \cdots, m\}$ be the robust feasible set of (FP). Then we say that x^* is a robust solution of (FP) if $x^* \in F$ and $\frac{f(x)}{g(x)} \geqq \frac{f(x^*)}{g(x^*)}$ for any $x \in F$.

Consider the following nonfractional optimization problem:

$$\begin{array}{lll} \text{(NFP)} & \text{Minimize} & p \\ & \text{subject to} & f(x) - pg(x) \leq 0, \\ & & h_j(x, v_j) \leq 0, \quad \forall v_j \in V_j, \ j = 1, \cdots, m. \end{array}$$

Following the approaches in [8], we can establish an equivalent relationship between (FP) and (NFP) as follows:

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Proposition 1.1. *Let $\bar{x} \in F$.*
(1) If $\bar{x}$ is a robust solution of (FP), then $(\bar{x}, \bar{p})$ is a solution of (NFP), where $\bar{p} = \frac{f(\bar{x})}{g(\bar{x})}$.
(2) If $(\bar{x}, \bar{p})$ is an solution of (NFP), then $\bar{x}$ is a robust solution of (FP).

Robust optimization provides a tool for handling the uncertainty related with the optimization problems ([1, 2, 3]).

Recently, Jeyakumar, Li and Lee [4] established necessary optimality theorems and robust duality theorems for a generalized convex programming problem in the face of data uncertainty. Recently, Kuroiwa and Lee [6] extended the necessary optimality theorems to a multiobjective robust optimization problem. Furthermore, Kim and Lee [5, 7] extended the robust duality theorems to a multiobjective robust optimization problem.

In this paper, we consider a fractional robust optimization problem (FP) and prove necessary optimality theorems for (FP). Establishing a nonfractional optimization problem (NFP), which is equivalent to (FP), we formulate a Mond-Weir type dual problem for (FP) and obtain duality theorems for (FP).

2. Optimality theorems and duality theorems

In this section, we give necessary optimality conditions for the fractional robust optimization problem (FP).

Let $\bar{x} \in F$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in V_j \text{ s.t. } h_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Let $V_j^0 = \{v_j \in V_j \mid h_j(\bar{x}, v_j) = 0\}$ for $j \in J_1(\bar{x})$. For a continuously differentiable function $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, we use $\nabla_1 h$ to denote the derivative of h with respect to the first variable.

Now we say that an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$ for (FP) if there exists $d \in \mathbb{R}^n$ such that for any $j \in J_1(\bar{x})$ and any $v_j \in V_j^0$,

$$\nabla_1 h_j(\bar{x}, v_j)^T d < 0.$$

Now we present a necessary optimality theorem for a solution of (FP). For the proof of the following theorem, we follow the approaches for Theorem 3.1 in [4].

Theorem 2.1. *Let $\bar{x} \in F$ be a robust solution of (FP). Suppose that $h_j(\bar{x}, \cdot)$ is concave on V_j, $j = 1, \cdots, m$. Then there exist $\lambda \geq 0$, $\mu_j \geq 0$, $j = 1, \cdots, m$, not all zero, $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ such that*

$$\lambda \left[\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})} \nabla g(\bar{x}) \right] + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0,$$

$$\mu_j h_j(\bar{x}, \bar{v}_j) = 0, \; j = 1, \cdots, m.$$

Moreover, if we assume that the Extended Mangasarian-Fromovitz constraint qualification then we have (EMFCQ) holds, then

$$\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}\nabla g(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0,$$
$$\mu_j h_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.$$

Proof. Suppose that $\bar{x}$ is a robust solution of (FP). Then for any $x \in F$, $f(x) - \frac{f(\bar{x})}{g(\bar{x})}g(x) \geqq f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}g(\bar{x}) = 0$. So, it follows from Theorem 3.1 in [4] that the conclusions hold. □

Using the equivalent relationship in Proposition 1.1, we formulate a Mond-Weir type robust dual problem (FD) for (FP).

$$\begin{aligned} \text{(FD)} \quad & \text{maximize} && p \\ & \text{subject to} && \nabla f(x) - p\nabla g(x) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(x, v_j) = 0, \qquad (1) \\ &&& f(x) - pg(x) \geq 0, \\ &&& \sum_{j=1}^{m} \mu_j h_j(x, v_j) \geq 0, \\ &&& v_j \in V_j,\ \mu_j \geq 0,\ j = 1, \cdots, m. \end{aligned}$$

Let $V = V_1 \times \cdots \times V_m$.

Theorem 2.2. *(Weak Duality) Let $x \in F$ and $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p}) \in \mathbb{R}^n \times V \times \mathbb{R}^m \times \mathbb{R}$ be feasible for (FD). Suppose that $f(\cdot)$ is convex, $g(\cdot)$ is concave and $h_j(\cdot, \bar{v}_j),\ j = 1, \cdots, m$ are convex, then*

$$\frac{f(x)}{g(x)} \geq \bar{p}.$$

Proof. Suppose to the contrary that there exist a feasible solution x of (FP) and a feasible solution $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ of (FD), such that

$$\frac{f(x)}{g(x)} - \bar{p} < 0, \text{ that is, } \quad f(x) - \bar{p}g(x) < 0.$$

Since $f(\bar{x}) - \bar{p}g(\bar{x}) \geq 0$, $f(x) - \bar{p}g(x) < f(\bar{x}) - \bar{p}g(\bar{x})$. By the convexity of $f(\cdot) - \bar{p}g(\cdot)$ at $\bar{x}$,

$$\left[\nabla f(\bar{x}) - \bar{p}\nabla g(\bar{x})\right]^T (x - \bar{x}) < 0. \tag{2}$$

Since $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}, \bar{v}_j) \geqq \sum_{j=1}^{m} \bar{\mu}_j h_j(x, \bar{v}_j)$, by the convexity $h_j(\cdot, \bar{v}_j)$ at $\bar{x}$,

$$\left[\sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_j(\bar{x}, \bar{v}_j)\right]^T (x - \bar{x}) \leq 0. \tag{3}$$

From (2) and (3),

$$\left[\nabla f(\bar{x}) - \bar{p}\nabla g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_j(\bar{x}, \bar{v}_j)\right]^T (x - \bar{x}) < 0,$$

which contradicts (1). □

Theorem 2.3. *(Strong Duality) Let $\bar{x}$ be a robust solution of (FP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$. Then, there exist $(\bar{u}, \bar{v}, \bar{\mu})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{p})$ is feasible for (FD) and the objective values of (FP) and (FD) are equal. If $f(\cdot)$ is convex, $g_j(\cdot)$, $j = 1, \cdots, m$ are concave, $h_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are convex on V_j, then $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a solution of (FD).*

Proof. By Theorem 2.1, there exist $\bar{\mu}_j \geq 0,\ j = 1, \cdots, m,\ \bar{v}_j \in V_j,\ j = 1, \cdots, m$ such that

$$\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}\nabla g(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0,$$
$$\bar{\mu}_j h_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.$$

Let $\bar{p} = \frac{f(\bar{x})}{g(\bar{x})}$. Then $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a feasible for (FD). By Theorem 2.2, $\frac{f(\bar{x})}{g(\bar{x})} \geq \widetilde{p}$, for any feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu}, \widetilde{p})$ for (FD). Since $\frac{f(\bar{x})}{g(\bar{x})} = \bar{p}$, $\bar{p} \geq \widetilde{p}$. Hence $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a solution of (FD). □

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Moon Hee Kim
Department of Refrigeration Engineering, Tongmyong University, Busan 608-711, Korea
E-mail address: mooni@tu.ac.kr

Gwi Soo Kim
Department of Applied Mathematics, Pukyung National University, Pusan 608-737, Korea.
E-mail address: gwisoo1103@hanmail.net