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SOLVABILITY FOR A SYSTEM OF MULTI-POINT BOUNDARY VALUE PROBLEMS ON AN INFINITE INTERVAL

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Abstract. The existence of at least one solution to a system of multi-point boundary value problems on an infinite interval is investigated by using the Alternative of Leray-Schauder.

1. Introduction

The boundary value problems on an infinite interval arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous media and theory of draining flows (see, e.g., [1, 2, 6]). The study on nonlocal elliptic boundary value problems was investigated by Bicadze and Samarskiĭ ([3]), and later continued by Il'in and Moiseev ([10]) and Gupta ([9]). Since then, the existence of solutions for nonlocal boundary value problems has received a great deal of attention in the literature. For more recent results, we refer the reader to [5, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24] and the references therein.

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In this paper, we consider the following system of second-order nonlinear differential equations with coupled boundary conditions

$$\begin{cases} (w_1\varphi_p(u'))'(t) + f(t, u(t), v(t), u'(t), v'(t)) = 0, \ a.e. \ t \in (0, \infty), \\ (w_2\varphi_p(v'))'(t) + g(t, u(t), v(t), u'(t), v'(t)) = 0, \ a.e. \ t \in (0, \infty), \\ u(0) = \sum_{j=1}^{n}(a_j u(\xi_j) + b_j v(\xi_j)), \ \lim_{t\to\infty}(\varphi_p^{-1}(w_1)u')(t) = l_1, \\ v(0) = \sum_{j=1}^{n}(c_j u(\xi_j) + d_j v(\xi_j)), \ \lim_{t\to\infty}(\varphi_p^{-1}(w_2)v')(t) = l_2, \end{cases} \tag{P}$$

where $p > 1$, $\varphi_p(s) = |s|^{p-2}s$ for $s \in \mathbb{R}$, $a_j, b_j, c_j, d_j \in \mathbb{R}$, $\xi_j \in (0, \infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \infty$, $n \in \mathbb{N}$, $l_1, l_2 \in \mathbb{R}$, $w_1, w_2 : (0, \infty) \to (0, \infty)$ are continuous functions, and $f, g : [0, \infty) \times \mathbb{R}^4 \to \mathbb{R}$ are Carathéodory functions such that $f = f(t, u, v, y, z)$ and $g = g(t, u, v, y, z)$ are Lebesgue measurable in t for all $(u, v, y, z) \in \mathbb{R}^4$ and continuous in (u, v, y, z) for almost all $t \in [0, \infty)$. We further assume the following conditions hold:

(H) $\left(1 - \sum_{j=1}^{n} a_j\right)\left(1 - \sum_{j=1}^{n} d_j\right) - \sum_{j=1}^{n} b_j \sum_{j=1}^{n} c_j \neq 0$;

(W) for $i = 1, 2$, $\varphi_p^{-1}\left(\frac{1}{w_i}\right) \in L^1_{loc}[0, \infty)$, and let

$$\theta_i(t) := \int_0^t \varphi_p^{-1}\left(\frac{1}{w_i(s)}\right) ds, \ t \in (0, \infty);$$

(F) for $i = 1, 2$, there exist nonnegative measurable functions α_i, β_i, γ such that

$$(1 + \theta_i)^{p-1}\alpha_i, \ \frac{\beta_i}{w_i}, \ \gamma \in L^1(0, \infty)$$

and, for almost all $t \in [0, \infty)$ and all $(u, v, y, z) \in \mathbb{R}^4$,

$$|f(t, u, v, y, z)| \le \alpha_1(t)|u|^{p-1} + \alpha_2(t)|v|^{p-1} + \beta_1(t)|y|^{p-1} + \beta_2(t)|z|^{p-1} + \gamma(t) \tag{1}$$

and

$$|g(t, u, v, y, z)| \le \alpha_1(t)|u|^{p-1} + \alpha_2(t)|v|^{p-1} + \beta_1(t)|y|^{p-1} + \beta_2(t)|z|^{p-1} + \gamma(t). \tag{2}$$

Recently, Zhang ([24]) studied the following multipoint boundary value problems on an infinite interval in a Banach space E with uncoupled boundary conditions

$$\begin{cases} u''(t) + k_1(t, u(t), v(t), u'(t), v'(t)) = 0, \ \ a.e. \ t \in (0, \infty), \\ v''(t) + k_2(t, u(t), v(t), u'(t), v'(t)) = 0, \ \ a.e. \ t \in (0, \infty), \\ u(0) = \displaystyle\sum_{j=1}^{n} \alpha_j u(\xi_j), \ \ \lim_{t\to\infty} u'(t) = u_\infty \in \mathbb{R}, \\ v(0) = \displaystyle\sum_{j=1}^{n} \beta_j v(\xi_j), \ \ \lim_{t\to\infty} v'(t) = v_\infty \in \mathbb{R}, \end{cases}$$

where $\alpha_j, \beta_j \in [0, \infty)$, $\xi_j \in (0, \infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \infty$, $0 < \sum_{j=1}^{n} \alpha_j < 1$, $0 < \sum_{j=1}^{n} \beta_j < 1$, $\sum_{j=1}^{n} \alpha_j \xi_j / (1 - \sum_{j=1}^{n} \alpha_j) > 1$, and $\sum_{j=1}^{n} \beta_j \xi_j / (1 - \sum_{j=1}^{n} \beta_j) > 1$. Under the suitable conditions on the nonlinearities $k_1(t, u, v, y, z)$ and $k_2(t, u, v, y, z)$ which may be singular at $t = 0$, $u, v = \theta$, and/or $y, z = \theta$, the existence of positive solutions for the problem was investigated in view of cone theory with Mönch fixed point theorem and a monotone iterative technique (see, e.g., [4, 8]). Kosmatov ([14]) studied the second-order nonlinear differential equation

$$(qy')'(t) = k_3(t, y(t), y'(t)), \ \text{a.e.} \ t \in [0, \infty),$$

satisfying two sets of boundary conditions:

$$y'(0) = 0, \ \lim_{t\to\infty} y(t) = \sum_{j=1}^{n} \kappa_j y(\xi_j) \ \text{with} \ \sum_{j=1}^{n} \kappa_j = 1,$$

or

$$y(0) = 0, \ \lim_{t\to\infty} y(t) = \sum_{j=1}^{n} \kappa_j y(\xi_j) \ \text{with} \ \sum_{j=1}^{n} \kappa_j \int_0^{\xi_j} \frac{1}{q(\tau)} d\tau = \int_0^{\infty} \frac{1}{q(\tau)} d\tau,$$

where $k_3 : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies $L^1[0, \infty)$-Carathéodory conditions, $q \in C[0, \infty) \cap C^1(0, \infty)$, $1/q \in L^1[0, \infty)$, and $q(t) > 0$ for all $t \in [0, \infty)$. Using coincidence degree theory ([18]), the existence of solutions for the problems was investigated . More recently, Kim ([12]) showed the existence of at least one bounded solution for problem

$$\begin{cases} (w\varphi_p(u'))'(t) + k_4(t, u(t), u'(t)) = 0, \ \ a.e. \ t \in [0, \infty), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\xi_i), \ \ \lim_{t\to\infty} (\varphi_p^{-1}(w) u')(t) = 0; \end{cases}$$

where $\alpha_i \in \mathbb{R}$ with $\sum_{i=1}^{m-2} \alpha_i \neq 1$, $k_4 : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, and $w \in C[0, \infty)$ with $\varphi_p^{-1}(1/w) \in L^1(0, \infty)$.

Motivated by the above results, we investigate the existence of solutions to the problem (P) with coupled boundary conditions. By a solution to problem (P), we understand a function $(u, v) \in (C[0, \infty) \cap C^1(0, \infty)) \times (C[0, \infty) \cap C^1(0, \infty))$ with $(w_1\varphi_p(u'), w_2\varphi_p(v')) \in AC[0, \infty) \times AC[0, \infty)$ which satisfies (P). The main tool of this paper is the following theorem (Alternative of Leray-Schauder, see, e.g., [7, p.124]):

Theorem 1.1. *Let C be a convex subset of a Banach space X, and assume that $0 \in C$. Let $L : C \to C$ be a compact operator, and let*

$$\mathcal{E} = \{x \in C : x = \lambda Lx \text{ for some } \lambda \in (0,1)\}.$$

Then either $\mathcal{E}$ is unbounded or L has a fixed point.

In next section, the main theorem is proved and an example is given to illustrate the main result.

2. Main result

Let $X := X_1 \times X_2$ be a Banach space with norm $\|(u,v)\|_X := \|u\|_1 + \|v\|_2$. Here, for $i = 1, 2$,

$$X_i := \{u \in C[0,\infty) \cap C^1(0,\infty) : \frac{u(t)}{1+\theta_i}, \varphi_p^{-1}(w_i)u' \in C[0,\infty) \cap L^\infty(0,\infty)\}$$

with norm

$$\|u\|_i := \sup_{t\in[0,\infty)} \frac{|u(t)|}{1+\theta_i(t)} + \sup_{t\in[0,\infty)} (\varphi_p^{-1}(w_i)|u'|)(t).$$

Let $Y := L^1(0,\infty)$ with norm $\|h\|_Y := \int_0^\infty |h(s)|ds$.

For $i = 1, 2$, we define $K_i : Y \to X_i$ by, for $h \in Y$ and $t \in [0,\infty)$,

$$K_i(h)(t) := \int_0^t \varphi_p^{-1}\left(\frac{1}{w_i(s)}\left(\varphi_p(l_i) + \int_s^\infty h(\tau)d\tau\right)\right) ds.$$

For $(h_1, h_2) \in Y \times Y$, we define $F_1, F_2 : Y \times Y \to \mathbb{R}$ by

$$F_1(h_1,h_2) := \sum_{j=1}^n (a_j K_1(h_1)(\xi_j) + b_j K_2(h_2)(\xi_j))$$

and

$$F_2(h_1,h_2) := \sum_{j=1}^n (c_j K_1(h_1)(\xi_j) + d_j K_2(h_2)(\xi_j)).$$

Let

$$M := \begin{pmatrix} 1 - \sum_{j=1}^n a_j & -\sum_{j=1}^n b_j \\ -\sum_{j=1}^n c_j & 1 - \sum_{j=1}^n d_j \end{pmatrix}.$$

From (H), $M^{-1} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exists and let

$$\begin{pmatrix} A_1(h_1,h_2) \\ A_2(h_1,h_2) \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_1(h_1,h_2) \\ F_2(h_1,h_2) \end{pmatrix}.$$

For $i = 1, 2$, $T_i : Y \times Y \to X_i$ is defined by, for $(h_1, h_2) \in Y \times Y$,

$$T_i(h_1, h_2)(t) := A_i(h_1, h_2) + K_i(h_i)(t) \text{ for all } t \in (0, \infty).$$

Define $T : Y \times Y \to X$ by, for $(h_1, h_2) \in Y \times Y$,

$$T(h_1, h_2)(t) := (T_1(h_1, h_2)(t), T_2(h_1, h_2)(t)) \text{ for all } t \in (0, \infty).$$

For $(h_1, h_2) \in Y \times Y$, consider the following problem

$$\begin{cases} (w_1\varphi_p(u'))'(t) + h_1(t) = 0, \ \ a.e. \ t \in (0, \infty), \\ (w_2\varphi_p(v'))'(t) + h_2(t) = 0, \ \ a.e. \ t \in (0, \infty), \\ u(0) = \displaystyle\sum_{j=1}^{n} (a_j u(\xi_j) + b_j v(\xi_j)), \ \lim_{t\to\infty} (\varphi_p^{-1}(w_1)u')(t) = l_1, \\ v(0) = \displaystyle\sum_{j=1}^{n} (c_j u(\xi_j) + d_j v(\xi_j)), \ \lim_{t\to\infty} (\varphi_p^{-1}(w_2)v')(t) = l_2. \end{cases} \tag{3}$$

Then we have the following lemma:

Lemma 2.1. *For each $(h_1, h_2) \in Y \times Y$, (3) has a unique solution $(u, v) = T(h_1, h_2)$ in X.*

Proof. Let (u, v) be a solution of (3) with a fixed $(h_1, h_2) \in Y \times Y$. Then, for $t \in [0, \infty)$,

$$u(t) = u(0) + K_1(h_1)(t), \ v(t) = v(0) + K_2(h_2)(t).$$

Thus

$$\sum_{j=1}^{n} a_j u(\xi_j) = \sum_{j=1}^{n} a_j u(0) + \sum_{j=1}^{n} a_j K_1(h_1)(\xi_j) \tag{4}$$

and

$$\sum_{j=1}^{n} b_j v(\xi_j) = \sum_{j=1}^{n} b_j v(0) + \sum_{j=1}^{n} b_j K_2(h_2)(\xi_j). \tag{5}$$

Adding (4) and (5),

$$u(0) = \sum_{j=1}^{n} a_j u(0) + \sum_{j=1}^{n} b_j v(0) + F_1(h_1, h_2). \tag{6}$$

In a similar manner,

$$v(0) = \sum_{j=1}^{n} c_j u(0) + \sum_{j=1}^{n} d_j v(0) + F_2(h_1, h_2). \tag{7}$$

By (6) and (7),

$$M \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} F_1(h_1, h_2) \\ F_2(h_1, h_2) \end{pmatrix}.$$

By (H), $(u(0), v(0)) = (A_1(h_1, h_2), A_2(h_1, h_2))$, and thus $(u, v) = T(h_1, h_2)$. $\square$

For convenience, we make use of the following notations:

$$A = \sum_{j=1}^{n} \left((|aa_j| + |bc_j|) \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_1(s)} \right) ds \right),$$

$$B = \sum_{j=1}^{n} \left((|ab_j| + |bd_j|) \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_2(s)} \right) ds \right),$$

$$C = \sum_{j=1}^{n} \left((|ca_j| + |dc_j|) \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_1(s)} \right) ds \right),$$

$$D = \sum_{j=1}^{n} \left((|cb_j| + |dd_j|) \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_2(s)} \right) ds \right).$$

Lemma 2.2. *Assume that* (H) *and* (W) *hold. For* $(h_1, h_2) \in Y \times Y$,

$$\|T_1(h_1, h_2)\|_1 \le (A+2)(|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}} + B(|l_2|^{p-1} + \|h_2\|_Y)^{\frac{1}{p-1}}$$

and

$$\|T_2(h_1, h_2)\|_2 \le C(|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}} + (D+2)(|l_2|^{p-1} + \|h_2\|_Y)^{\frac{1}{p-1}}.$$

Proof. Let $(h_1, h_2) \in Y \times Y$ be given. Then, we have

$$\begin{aligned} &\frac{|aF_1(h_1, h_2)|}{1+\theta_1(t)} \\ &\le |a| \sum_{j=1}^{n} (|a_j K_1(h_1)(\xi_j)| + |b_j K_2(h_2)(\xi_j)|) \\ &\le |a| \sum_{j=1}^{n} \Bigg[\left(|a_j| \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_1(s)} \right) ds \right) \left(|l_1|^{p-1} + \|h_1\|_Y \right)^{\frac{1}{p-1}} \\ &\qquad + \left(|b_j| \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_2(s)} \right) ds \right) \left(|l_2|^{p-1} + \|h_2\|_Y \right)^{\frac{1}{p-1}} \Bigg]. \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned} &\frac{|bF_2(h_1, h_2)|}{1+\theta_1(t)} \\ &\le |b| \sum_{j=1}^{n} \Bigg[\left(|c_j| \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_1(s)} \right) ds \right) \left(|l_1|^{p-1} + \|h_1\|_Y \right)^{\frac{1}{p-1}} \\ &\qquad + \left(|d_j| \int_0^{\xi_j} \varphi_p^{-1} \left(\frac{1}{w_2(s)} \right) ds \right) \left(|l_2|^{p-1} + \|h_2\|_Y \right)^{\frac{1}{p-1}} \Bigg], \end{aligned} \tag{9}$$

$$\frac{|K_1(h_1)(t)|}{1+\theta_1(t)} \le (|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}}, \tag{10}$$

and

$$|(\varphi_p^{-1}(w_1)(K_1(h_1))')(t)| \le (|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}}. \tag{11}$$

By (8)-(11), we have

$$\|T_1(h_1,h_2)\|_1 \le (A+2)(|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}} + B(|l_2|^{p-1} + \|h_2\|_Y)^{\frac{1}{p-1}}.$$

In a similar manner,

$$\|T_2(h_1,h_2)\|_2 \le C(|l_1|^{p-1} + \|h_1\|_Y)^{\frac{1}{p-1}} + (D+2)(|l_2|^{p-1} + \|h_2\|_Y)^{\frac{1}{p-1}},$$

and thus the proof is complete. □

We define the Nemytskii operators $N_f, N_g : X \to Y$ by

$$N_f(u,v)(t) := f(t, u(t), v(t), u'(t), v'(t))$$

and

$$N_g(u,v)(t) := g(t, u(t), v(t), u'(t), v'(t))$$

for almost all $t \in (0,\infty)$, and define $L : X \to X$ by

$$L(u,v) = (L_1(u,v), L_2(u,v)) := T(N_f(u,v), N_g(u,v)) \text{ for } (u,v) \in X.$$

Then L is well defined. By Lemma 2.1, problem (P) has a solution (u,v) if and only if L has a fixed point (u,v) in X.

To show the compactness of the operator L, we use the following compactness criterion.

Theorem 2.3. *([1]) Let Z be the space of all bounded continuous vector-valued functions on $[0,\infty)$ and $S \subset Z$. Then S is relatively compact in Z if the following conditions hold:*
(i) S is bounded in Z;
(ii) the functions from S are equicontinuous on any compact interval of $[0,\infty)$;
(iii) the functions from S are equiconvergent at ∞, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\|\phi(t) - \phi(\infty)\|_{\mathbb{R}^n} < \epsilon$ for all $t > T$ and all $\phi \in S$.

Lemma 2.4. *Assume that $(H), (W)$ and (F) hold. Then the operator $L : X \to X$ is compact.*

Proof. We only prove that $L_1 : X \to X_1$ is compact, since the compactness of $L_2 : X \to X_2$ can be proved in a similar manner, and thus $L : X \to X$ is compact. Recall that $L_1(u,v) = T_1(N_f(u,v), N_g(u,v))$ for $(u,v) \in X$.

Let Σ be bounded in X, i.e., there exists $R_1 > 0$ such that $\|(u,v)\|_X \le R_1$ for all $(u,v) \in \Sigma$. By (F), there exists $h_\Sigma \in Y$ such that

$$|N_f(u,v)(t)| \le h_\Sigma(t) \tag{12}$$

for almost all $t \in [0, \infty)$ and all $(u, v) \in \Sigma$. Indeed, for $(u, v) \in \Sigma$ and for almost all $t \in [0, \infty)$, by (F),

$$\begin{aligned}
& |N_f(u,v)(t)| \\
\leq\ & \alpha_1(t)|u(t)|^{p-1} + \alpha_2(t)|v(t)|^{p-1} + \beta_1(t)|u'(t)|^{p-1} + \beta_2(t)|v'(t)|^{p-1} + \gamma(t) \\
\leq\ & \left(\sum_{i=1}^{2} (1+\theta_i(t))^{p-1}\alpha_i(t) + \frac{\beta_1(t)}{w_1(t)} + \frac{\beta_2(t)}{w_2(t)} \right) \|(u,v)\|_X^{p-1} + \gamma(t).
\end{aligned}$$

Set, for almost all $t \in [0, \infty)$,

$$h_\Sigma(t) := \left(\sum_{i=1}^{2} (1+\theta_i(t))^{p-1}\alpha_i(t) + \frac{\beta_1(t)}{|w_1(t)|} + \frac{\beta_2(t)}{|w_2(t)|} \right) R_1^{p-1} + \gamma(t),$$

then $h_\Sigma \in Y$ and (12) holds. Thus $N_f(\Sigma)$ is bounded in Y. Similarly, we can prove that $N_2(\Sigma)$ is bounded in Y. It follows from Lemma 2.2 that $L_1(\Sigma)$ is bounded in X_1.

For any $R > 0$ and $t_1, t_2 \in [0, R]$ with $t_1 < t_2$,

$$\begin{aligned}
& \left| \frac{T_1(N_f(u,v), N_g(u,v))(t_1)}{1+\theta_1(t_1)} - \frac{T_1(N_f(u,v), N_g(u,v))(t_2)}{1+\theta_1(t_2)} \right| \\
\leq\ & |A_1(N_f(u,v), N_g(u,v))| \left| \frac{1}{1+\theta_1(t_1)} - \frac{1}{1+\theta_1(t_2)} \right| \\
& + \left| \frac{K_1(N_f(u,v))(t_1)}{1+\theta_1(t_1)} - \frac{K_1(N_f(u,v))(t_2)}{1+\theta_1(t_2)} \right| \\
\leq\ & \sup_{(u,v)\in\Sigma} \{|A_1(N_f(u,v), N_g(u,v))|\} |\theta_1(t_2) - \theta_1(t_1)| \\
& + \left(\frac{1}{1+\theta_1(t_1)} - \frac{1}{1+\theta_1(t_2)} \right) |K_1(N_f(u,v))(t_1)| \\
& + \frac{1}{1+\theta_1(t_2)} |K_1(N_f(u,v))(t_2) - K_1(N_f(u,v))(t_1)| \\
\leq\ & \Big(\sup_{(u,v)\in\Sigma} \{|A_1(N_f(u,v), N_g(u,v))|\} \\
& + (|l_1|^{p-1} + \|h_\Sigma\|_Y)^{\frac{1}{p-1}} \int_0^R \varphi_p^{-1}\left(\frac{1}{w_1(s)} \right) ds \Big) |\theta_1(t_2) - \theta_1(t_1)| \\
& + (|l_1|^{p-1} + \|h_\Sigma\|_Y)^{\frac{1}{p-1}} \int_{t_1}^{t_2} \varphi_p^{-1}\left(\frac{1}{w_1(s)} \right) ds,
\end{aligned}$$

and

$$
\begin{aligned}
& |(\varphi_p^{-1}(w_1)(T_1(N_f(u,v),N_g(u,v)))')(t_1) \\
& \quad -(\varphi_p^{-1}(w_1)(T_1(N_f(u,v),N_g(u,v)))')(t_2)| \\
= & \left|\varphi_p^{-1}\left(\varphi_p(l_1)+\int_{t_1}^{\infty} N_f(u,v)(s)ds\right)\right. \\
& \left. -\varphi_p^{-1}\left(\varphi_p(l_1)+\int_{t_2}^{\infty} N_f(u,v)(s)ds\right)\right|,
\end{aligned}
$$

which yield that

$$\left\{\frac{T_1(N_f(u,v),N_g(u,v))}{1+\theta_1} : (u,v)\in\Sigma\right\}$$

and

$$\{\varphi_p^{-1}(w_1)(T_1(N_f(u,v),N_g(u,v)))' \ : \ (u,v)\in\Sigma\}$$

are equicontinuous on any finite subinterval of $[0,\infty)$ by (12).

For $(u,v)\in\Sigma$, by L'Hospital's rule,

$$
\begin{aligned}
& \lim_{t\to\infty}\frac{T_1(N_f(u,v),N_g(u,v))(t)}{1+\theta_1(t)} \\
= & \lim_{t\to\infty}\varphi_p^{-1}\left(\varphi_p(l_1)+\int_t^{\infty} N_f(u,v)(\tau)d\tau\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \lim_{t\to\infty}(\varphi_p^{-1}(w_1)(T_1(N_f(u,v),N_g(u,v)))')(t) \\
= & \lim_{t\to\infty}\varphi_p^{-1}\left(\varphi_p(l_1)+\int_t^{\infty} N_f(u,v)(\tau)d\tau\right).
\end{aligned}
$$

It follows from (12) that, as $t\to\infty$,

$$\frac{T_1(N_f(u,v),N_g(u,v))(t)}{1+\theta_1(t)}\to l_1$$

and

$$(\varphi_p^{-1}(w_1)(T_1(N_f(u,v),N_g(u,v)))')(t)\to l_1$$

uniformly on Σ. Consequently,

$$\left\{\frac{T_1(N_f(u,v),N_g(u,v))}{1+\theta_1} : (u,v)\in\Sigma\right\}$$

and

$$\{\varphi_p^{-1}(w_1)T_1(N_f(u,v),N_g(u,v))' \ : \ (u,v)\in\Sigma\}$$

are equiconvergent at ∞, and thus $T_1(N_f,N_g)$ is compact in view of Theorem 2.3. $\square$

Now we give the main result in this paper.

Theorem 2.5. *Assume that* $(H), (W)$ *and* (F) *hold. Then problem* (P) *has at least one solution* (u, v) *in* X *provided that*

$$\kappa_p((A+1)^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y+C^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y)+\sum_{i=1}^{2}\left\|\frac{\beta_i}{w_i}\right\|_Y < \frac{1}{2} \quad (13)$$

and

$$\kappa_p(B^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y+(D+1)^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y)+\sum_{i=1}^{2}\left\|\frac{\beta_i}{w_i}\right\|_Y < \frac{1}{2} \quad (14)$$

hold. Here $\kappa_p = \max\{1, 2^{p-2}\}$.

Proof. Let $(u, v) \in X$ satisfying

$$(u, v) = \lambda L(u, v)$$

for some $\lambda \in (0, 1)$. Then

$$u = \lambda T_1(N_f(u, v), N_g(u, v))$$

and

$$v = \lambda T_2(N_f(u, v), N_g(u, v)).$$

It is well known that, for $q > 0$ and for any $a, b \in \mathbb{R}$,

$$|a + b|^q \leq \max\{1, 2^{q-1}\}(|a|^q + |b|^q). \quad (15)$$

Using the assumption (F) and the inequality (15), by the same arguments as in the proof of Lemma 2.2, for almost all $t \in (0, \infty)$,

$$
\begin{aligned}
&|N_f(u,v)(t)| \\
\leq\;& \alpha_1(t)|u(t)|^{p-1} + \alpha_2(t)|v(t)|^{p-1} + \beta_1(t)|u'(t)|^{p-1} + \beta_2(t)|v'(t)|^{p-1} + \gamma(t) \\
\leq\;& (1+\theta_1(t))^{p-1}\alpha_1(t)\left(\frac{|T_1(N_f(u,v),N_g(u,v))(t)|}{1+\theta_1(t)}\right)^{p-1} \\
&+(1+\theta_2(t))^{p-1}\alpha_2(t)\left(\frac{|T_2(N_f(u,v),N_g(u,v))(t)|}{1+\theta_2(t)}\right)^{p-1} \\
&+\frac{\beta_1(t)}{w_1(t)}\left|(\varphi_p^{-1}(w_1)T_1(N_f(u,v),N_g(u,v))')(t)\right|^{p-1} \\
&+\frac{\beta_2(t)}{w_2(t)}\left|(\varphi_p^{-1}(w_2)T_2(N_f(u,v),N_g(u,v))')(t)\right|^{p-1} + \gamma(t) \\
\leq\;& (1+\theta_1(t))^{p-1}\alpha_1(t)\Big[(A+1)(|l_1|^{p-1} + \|N_f(u,v)\|_Y)^{\frac{1}{p-1}} \\
&+B(|l_2|^{p-1} + \|N_g(u,v)\|_Y)^{\frac{1}{p-1}}\Big]^{p-1} \\
&+(1+\theta_2(t))^{p-1}\alpha_2(t)\Big[C(|l_1|^{p-1} + \|N_f(u,v)\|_Y)^{\frac{1}{p-1}} \\
&+(D+1)(|l_2|^{p-1} + \|N_g(u,v)\|_Y)^{\frac{1}{p-1}}\Big]^{p-1} \\
&+\frac{\beta_1(t)}{w_1(t)}(|l_1|^{p-1} + \|N_f(u,v)\|_Y) + \frac{\beta_2(t)}{w_2(t)}(|l_2|^{p-1} + \|N_g(u,v)\|_Y) + \gamma(t) \\
\leq\;& \Big[(1+\theta_1(t))^{p-1}\alpha_1(t)\kappa_p(A+1)^{p-1} \\
&+(1+\theta_2(t))^{p-1}\alpha_2(t)\kappa_p C^{p-1} + \frac{\beta_1(t)}{w_1(t)}\Big]\|N_f(u,v)\|_Y \\
&+\Big[(1+\theta_1(t))^{p-1}\alpha_1(t)\kappa_p B^{p-1} \\
&+(1+\theta_2(t))^{p-1}\alpha_2(t)\kappa_p(D+1)^{p-1} + \frac{\beta_2(t)}{w_2(t)}\Big]\|N_g(u,v)\|_Y \\
&+(1+\theta_1(t))^{p-1}\alpha_1(t)\kappa_p((A+1)^{p-1}|l_1|^{p-1} + B^{p-1}|l_2|^{p-1}) \\
&+(1+\theta_2(t))^{p-1}\alpha_2(t)\kappa_p(C^{p-1}|l_1|^{p-1} + (D+1)^{p-1}|l_2|^{p-1}) \\
&+\frac{\beta_1(t)}{w_1(t)}|l_1|^{p-1} + \frac{\beta_2(t)}{w_2(t)}|l_2|^{p-1} + \gamma(t).
\end{aligned}
$$

Thus, we have

$$\begin{aligned}
\|N_f(u,v)\|_Y \quad \leq \quad & \Big[\kappa_p((A+1)^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +C^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y) + \left\|\frac{\beta_1}{w_1}\right\|_Y\Big]\|N_f(u,v)\|_Y \\
& +\Big[\kappa_p(B^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +(D+1)^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y) + \left\|\frac{\beta_2}{w_2}\right\|_Y\Big]\|N_g(u,v)\|_Y \\
& +\kappa_p((A+1)^{p-1}|l_1|^{p-1} + B^{p-1}|l_2|^{p-1})\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +\kappa_p(C^{p-1}|l_1|^{p-1} + (D+1)^{p-1}|l_2|^{p-1})\|(1+\theta_2)^{p-1}\alpha_2\|_Y \\
& +|l_1|^{p-1}\left\|\frac{\beta_1}{w_1}\right\|_Y + |l_2|^{p-1}\left\|\frac{\beta_2}{w_2}\right\|_Y + \|\gamma\|_Y. \qquad (16)
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
\|N_g(u,v)\|_Y \quad \leq \quad & \Big[\kappa_p((A+1)^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +C^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y) + \left\|\frac{\beta_1}{w_1}\right\|_Y\Big]\|N_f(u,v)\|_Y \\
& +\Big[\kappa_p(B^{p-1}\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +(D+1)^{p-1}\|(1+\theta_2)^{p-1}\alpha_2\|_Y) + \left\|\frac{\beta_2}{w_2}\right\|_Y\Big]\|N_g(u,v)\|_Y \\
& +\kappa_p((A+1)^{p-1}|l_1|^{p-1} + B^{p-1}|l_2|^{p-1})\|(1+\theta_1)^{p-1}\alpha_1\|_Y \\
& +\kappa_p(C^{p-1}|l_1|^{p-1} + (D+1)^{p-1}|l_2|^{p-1})\|(1+\theta_2)^{p-1}\alpha_2\|_Y \\
& +|l_1|^{p-1}\left\|\frac{\beta_1}{w_1}\right\|_Y + |l_2|^{p-1}\left\|\frac{\beta_2}{w_2}\right\|_Y + \|\gamma\|_Y. \qquad (17)
\end{aligned}$$

Adding (16) and (17), we can conclude that there exists a constant $C > 0$ such that

$$\|N_f(u,v)\|_Y + \|N_g(u,v)\|_Y \leq C$$

provided (13) and (14) hold. It follows from Lemma 2.2 that there exists $R > 0$ such that $\|(u,v)\|_X < R$ for all (u,v) satisfying $(u,v) = \lambda L(u,v)$ for some $\lambda \in (0,1)$. Thus problem (P) has at least one solution (u,v) in X in view of Theorem 1.1. □

Finally, we give an example to illustrate the main result.

Example 2.6. Consider the following problem

$$\begin{cases} (t|u'(t)|u'(t))' + f(t,u(t),v(t),u'(t),v'(t)) = 0,\ t \in (0,\infty), \\ (|v'(t)|v'(t))' + g(t,u(t),v(t),u'(t),v'(t)) = 0,\ t \in (0,\infty), \\ u(0) = -u(1) - \frac{1}{2}v(1) + 3u(4) + \frac{3}{2}v(4),\ \lim_{t\to\infty} t^{\frac{1}{2}}u'(t) = l_1 \in \mathbb{R}, \\ v(0) = \frac{3}{4}u(1) - 2v(1) - \frac{1}{4}u(4) + 4v(4),\ \lim_{t\to\infty} v'(t) = l_2 \in \mathbb{R}. \end{cases} \tag{18}$$

Corresponding to the problem (P), $p = 3$, $n = 2$, $\xi_1 = 1$, $\xi_2 = 4$, $a_1 = -1$, $a_2 = 3$, $b_1 = -1/2$, $b_2 = 3/2$, $c_1 = 3/4$, $c_2 = -1/4$, $d_1 = -2$, $d_2 = 4$, $w_1(t) = t$, and $w_2(t) = 1$. Then $\theta_1(t) = 2t^{\frac{1}{2}}$ and $\theta_2(t) = t$, and thus (H) and (W) hold. Let

$$f(t,u,v,y,z) = \alpha_1(t)u\sin(tv) + \alpha_2(t)v^2\left(\frac{y^2}{1+y^2}\right) + \beta_1(t)y^2 + \beta_2(t)z\cos(yz) + \gamma_1(t)$$

and

$$g(t,u,v,y,z) = \alpha_1(t)u^2 + \alpha_2(t)v + \beta_1(t)y^2\left(\frac{y^2+z^2}{1+y^2+z^2}\right) + \beta_2(t)z^2 + \gamma_2(t),$$

where $\alpha_1(t) = 10^{-5}e^{-t}(1+2t^{\frac{1}{2}})^{-2}$, $\alpha_2(t) = 10^{-5}e^{-t}(1+t)^{-2}$,

$$\beta_1(t) = \begin{cases} 10^{-2}t^{\frac{1}{2}}, & t \in (0,1) \\ 10^{-2}t^{-1}, & t \in [1,\infty) \end{cases},\quad \beta_2(t) = \begin{cases} 10^{-2}t^{-\frac{1}{2}}, & t \in (0,1) \\ 10^{-2}t^{-2}, & t \in [1,\infty) \end{cases},$$

and γ_1, γ_2 are any functions in Y. Then

$$|f(t,u,v,y,z)| \le \alpha_1(t)u^2 + \alpha_2(t)v^2 + \beta_1(t)y^2 + \beta_2(t)z^2 + \alpha_1(t) + \beta_2(t) + |\gamma_1(t)|$$

and

$$|g(t,u,v,y,z)| \le \alpha_1(t)u^2 + \alpha_2(t)v^2 + \beta_1(t)y^2 + \beta_2(t)z^2 + \alpha_2(t) + |\gamma_2(t)|.$$

Taking $\gamma(t) = \alpha_1(t) + \alpha_2(t) + \beta_2(t) + |\gamma_1(t)| + |\gamma_2(t)|$, then (F) holds, and

$$\|(1+\theta_1)^2\alpha_1\|_Y = \|(1+\theta_2)^2\alpha_2\|_Y = 10^{-5} \text{ and } \|\frac{\beta_1}{w_1}\|_Y = \|\frac{\beta_2}{w_2}\|_Y = \frac{3}{100}.$$

By direct calculation, $\kappa_3 = 2$,

$$M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix},$$

$A = 33$, $B = 49$, $C = 19$, and $D = 85/2$. Consequently, (13) and (14) hold. By Theorem 2.5, the problem (18) has at least one solution for any $l_1, l_2 \in \mathbb{R}$.

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