Management Science and Financial Engineering
Vol 21, No 1, May 2015, pp.11-17
ISSN 2287-2043 | EISSN 2287-2361 |
http://dx.doi.org/10.7737/MSFE.2015.21.1.011

Min-Max Stochastic Optimization with Applications to the Single-Period Inventory Control Problem

Kyungchul Park*
Department of Business Administration, Myongji University

(Received: February 4, 2015 / Accepted: April 28, 2015)

ABSTRACT

Min-max stochastic optimization is an approach to address the distribution ambiguity of the underlying random variable. We present a unified approach to the problem which utilizes the theory of convex order on the random variables. First, we consider a general framework for the problem and give a condition under which the convex order can be utilized to transform the min-max optimization problem into a simple minimization problem. Then extremal distributions are presented for some interesting classes of distributions. Finally, applications to the single-period inventory control problems are given.

Keywords: Distribution-free Bounds, Convex Order, Inventory Control, Min-Max Stochastic Optimization, Problem of Moments

* Corresponding Author, E-mail: daneel@mju.ac.kr

1. INTRODUCTION

In min-max stochastic programming approach, the distribution of the underlying random variables is not specified but is assumed to be a member of a given class of distributions. Then a solution is found which attains the minimum cost with respect to the worst-case distribution in the class.

In practice, the distribution of the relevant random variable is usually estimated using the historical data. However, the accurate estimation may be very difficult especially when the data are not sufficient or they are inadequate because of the possible significant change of the future state of nature. In this case, the min-max optimization is helpful if we can identify the class of distributions in a reasonable way. Though it can be viewed as pessimistic, it has the advantage of avoiding the decision bias due to the erroneous estimation of the distribution.

This paper presents a unified approach to the min-max optimization problem by utilizing the convex order on the distributions. The extremal distributions (called tight supremum) with respect to the convex order are identified for some interesting classes of distributions. Those distributions are used to transform the min-max optimization problem into a simple minimization problem.

As applications, two important single-period inventory control problems are considered. The first one is the newsvendor problem pioneered by Arrow *et al*. (1951). The newsvendor problem is considered as one of the most important one in the inventory control theory, see Porteus (2002). A good review on the problem can be found in Khouja (1999) and Qin *et al*. (2011). The second one is the lot sizing problem under random yield (Yano and Lee, 1995). In the newsvendor problem, the random variable is the consumers' demand, while in the lot sizing problem, it is the production yield, which is defined as the number of good units produced.

The min-max approach for the newsvendor problem is sometimes called in the literature as a distribution-free newsvendor problem. Scarf (1958), by assuming only the mean and the variance of the demand are known, addressed the problem for the first time. Then Gallego and Moon (1993) disseminated the Scarf's result and extended it to the case when the demand is nonnegative. Alfares and Elmorra (2005) extended the result to the case when the penalty cost for the shortage exists. The results in this paper are general in that they include all the cases given in the previous research. In particular, we will present the optimal control policies for the cases when the mean, the variance, and the support of the demand distribution are given. Other closely related works include Perakis and Roels (2008) and Han *et al*. (2014), which addressed the problem of distribution-ambiguous newsvendor problem in somewhat different contexts. Gallego *et al*. (2001) considered the min-max inventory control problem in the multi-period cases.

The contributions of the paper are summarized as follows.

(1) A general framework for the min-max optimization problem with a single random variable is given. In particular, the condition under which the problem can be transformed into a simple minimization problem is specified.
(2) A general method to find a tight supremum for given class of random variables is discussed with some important theoretical results. It is applied to the problems of finding the tight supremum for some interesting classes of distributions.
(3) The tight supremum is identified for classes of distributions commonly found in the literature, which can be used in solving many interesting problems.
(4) The min-max optimal solutions to the newsvendor problem and the random yield lot-sizing problem are given.

This paper uses the theory of convex orders as a main vehicle to solve the min-max optimization problem. So it seems to be in order to give the brief definition before delving into the details. Müller and Stoyan (2002) is a good reference on the stochastic orders. For a random variable X, let F_X be its distribution. Then for two random variables X and Y, we define the convex order $X \le_{cx} Y$ (or $F_X \le_{cx} F_Y$) if $\mathrm{E}g(X) \le \mathrm{E}g(Y)$ holds for all convex functions g where the expectations exist. Some relevant theory on the convex order will be given when needed in the body of the paper.

The rest of the paper is organized as follows. In Section 2, we give the general result on the min-max optimization based on the theory of the convex order. In Section 3, a method to find the extremal distribution (called tight supremum) is presented and is applied to some interesting classes of distributions. The results are used in solving the min-max newsvendor problem and the lot sizing problem in Section 4. Finally, Section 5 closes the paper with concluding remarks.

2. MIN-MAX STOCHASTIC OPTIMIZATION FRAMEWORK

Let Ω, S be two given nonempty subsets of real numbers and let M be a nonempty class of distributions. A cost function $v: S\times\Omega \to R$ is given so that $v(x, \xi)$ is a convex function in $\xi \in \Omega$ for all fixed $x \in S$. The expected cost function $V: S \times M \to R$ is defined as follows:

$$V(x, F) = \int_\Omega v(x, \xi) F(d\xi). \quad (1)$$

For simplicity of presentation, we assume the integral in (1) is well-defined and has a finite value for any $F \in M$ and Ω is suppressed when it is clear from the context.

In the following, if not specified otherwise, let us assume that the distributions in the class M are of the same mean μ. Also $F \in M$ is used to denote both the distribution function and the distribution (the probability measure induced by the distribution function F).

Consider the following min-max optimization problem:

$$\min_{x\in S} \max_{F\in M} V(x, F). \quad (2)$$

The above problem is to seek a solution whose worst expected cost is the minimum among all feasible solutions $x \in S$.

Suppose we have an extremal distribution $F_{\sup}$ which is defined as follows:

$$F \le_{cx} F_{\sup}, \text{ for all } F \in M. \quad (3)$$

Note that the distribution $F_{\sup}$ may not be an element of the class M. Then since $v(x, \xi)$ is a convex function in ξ for each fixed x, it follows that

$$V(x, F) = \int v(x, \xi)F(d\xi) \le \int v(x, \xi)F_{\sup}(d\xi) = V(x, F_{\sup}), \text{ for all } F \in M. \quad (4)$$

Further, suppose that for each $x \in S$, the following holds:

$$\max_{F\in M} V(x, F) = V(x, F_{\sup}). \quad (5)$$

Then it is clear that

$$\min_{x\in S} \max_{F\in M} V(x, F) = \min_{x\in S} V(x, F_{\sup}). \quad (6)$$

Hence in this case the min-max optimization problem reduces to the simple minimization problem, which leads to the following definition.

Definition 1: Let S and M be a subset of the real numbers and a given class of distributions, respectively. A distribution $F_{\sup}$ is a *tight supremum* of the subset S, the

class *M*, and the function *V* if it satisfies the following two conditions:
(1) $F \le_{cx} F_{\sup}$, for all $F \in M$, ,
(2) For each $x \in S$, $\max_{F \in M} V(x, F) = V(x, F_{\sup})$.

When it is clear from the context, we will simply call $F_{\sup}$ a tight supremum. Also when the condition (2) in the above definition 1 does not hold, we call $F_{\sup}$ a *supremum*. The following theorem is a result of the above discussion.

Theorem 1: Let a distribution $F_{\sup}$ be a tight supremum of the subset *S*, the class *M*, and the function *V*. Then $\min_{x \in S} \max_{F \in M} V(x, F) = \min_{x \in S} V(x, F_{\sup})$. When $F_{\sup}$ is not tight (that is, a supremum only), then we have an upper bound on the optimal value, that is, $\min_{x \in S} \max_{F \in M} V(x, F) \le \min_{x \in S} V(x, F_{\sup})$.

Hence for a given class of distributions, if we can find a tight supremum, then the min-max optimal solution can simply be found by minimizing the expected cost with respect to the tight supremum.

2.1 Cost Functions

Assume that we are given a (not necessarily tight) supremum $F_{\sup}$ for the class *M*. For it to be tight, we have to show that $\max_{F \in M} V(x, F) = V(x, F_{\sup})$ holds for all $x \in S$. In the following, we consider a specific family of cost functions and find the conditions under which $F_{\sup}$ is a tight supremum.

For any distribution *F*, the function $\pi_F(x) = \int (\xi - x)^+ F(d\xi)$ is convex and nonincreasing in $x \in R$. In Müller and Stoyan (2002), it is called as an *integrated survival function*. There, it is shown that $F \le_{cx} F_{\sup}$ holds if and only if $\pi_F(x) \le \pi_{F_{\sup}}(x)$ for all $x \in R$. Also there, it is shown that for any nonincreasing convex function $\pi(x)$ which satisfies $\lim_{x \to \infty} \pi(x) = 0$ and $\lim_{x \to -\infty} [\pi(x) + x] = \mu$, there exists distribution *F* whose integrated survival function is $\pi(x)$. Furthermore, the distribution function *F* is given by $F(x) = 1 + \pi'_+(x)$, where π'_+ is the right-derivative of the function π (since π is convex, it should always exist).

Let the cost function be of the following form:

$$v(x, \xi) = a(x)(\xi - e(x))^+ + b(x)(e(x) - \xi)^+ + c(x)\xi + d(x), \qquad (7)$$

where $a(x) \ge 0, b(x) \ge 0, c(x), d(x), e(x)$ are arbitrary functions of $x \in R$ and for a real number *x*, $x^+ = \max(x, 0)$. Suppose $F_{\sup}$ is given such that $F_{\sup}(x) = \arg\max_{F \in M} \pi_F(x)$ holds for all $x \in R$. Then it is clear that $F \le_{cx} F_{\sup}$ holds for all $F \in M$. Note that we assume that the distributions $F \in M$ are of the same mean and also note that $(e(x) - \xi)^+ = e(x) - \xi + (\xi - e(x))^+$. Hence, if the cost function is of the form (7), then $\max_{F \in M} V(x, F) = V(x, F_{\sup})$, for all $x \in S$.

The result is summarized in the following theorem 2.

Theorem 2: Let a distribution $F_{\sup}$ be given such that $F_{\sup}(x) = \arg\max_{F \in M} \pi_F(x)$ holds for all $x \in R$. Then for a cost function of the form (7), $F_{\sup}$ is a tight supremum.

Remark 1: The form of the cost function can be more general than that given in (7). Especially, when the distributions in *M* have the same moments of higher orders, the cost function can have the corresponding terms.

2.2 The Newsvendor Problem and the Random Yield Lot-Sizing Problem

In the (generalized) newsvendor problem, the cost function is defined as

$$\begin{aligned} v_1(x, \xi) &= c_u(\xi - x)^+ + c_o(x - \xi)^+ \\ &= c_o(x - \xi) + (c_u + c_o)(\xi - x)^+, \end{aligned} \qquad (8)$$

where c_u and c_o be nonnegative underage and overage costs, respectively. For a given class of distributions *M*, the min-max (distribution-free) newsvendor problem is defined as $\min_{x \ge 0} \max_{F \in M} V_1(x, F)$, where $V_1(x, F) = \int v_1(x, \xi) F(d\xi)$. It is clear that the cost function (8) is of the form (7).

When the production yield is a multiplicative random variable and the demand is certain (assumed to be 1, without loss of generality), the cost function of the production lot sizing problem is given by

$$v_2(x, \xi) = (c - \xi)x + (r + h + p)(\xi x - 1)^+, \qquad (9)$$

where *x* is a production lot size and *r*, *c*, *h*, and *p* are unit revenue, production cost, holding cost, and penalty cost, respectively (see Yano and Lee, 1995). Since $(\xi x - 1)^+ = x(\xi - 1/x)^+$, for all $x > 0$, the cost function (9) is also of the form (7).

The above discussions lead to the following result.

Corollary 1: Let a distribution $F_{\sup}$ be given such that $F_{\sup}(x) = \arg\max_{F \in M} \pi_F(x)$ holds for all $x \in R$. Then both for the newsvendor problem and for the lot sizing problem, $F_{\sup}$ is a tight supremum.

3. CHARACTERIZATION OF THE TIGHT SUPREMUM

3.1 Primal and Dual Approaches

This section explores a systematic method to get the tight supremum for the cost function (7). To this end, we should consider the following problem (see Theorem 2):

$$\max_{F \in M} \int_{\Omega} (\xi - x)^+ F(d\xi), \qquad (10)$$

for a given class of distributions *M* and a real number *x*. Simply, the problem is to find the supremum of the class

M with respect to the convex order.

To be specific, let us assume that the class M is given by

$$M=\{F\geq 0:\int_{\Omega} g_i(\xi)F(d\xi)=m_i,\ i=0,1,\cdots,K\}, \quad (11)$$

where $K\geq 1$ and $g_i(\xi)=\xi^i$ and $m_0=1,\ m_1=\mu$. The condition $F\geq 0$ means that F is a nonnegative measure, which together with g_0, requires that F should be a probability measure. The class of distributions (11) appears in the well-known problem of the moments.

Then the problem (10) can be restated as follows:

$$\max_{F\geq 0}\{\int_{\Omega}(\xi-x)^+F(d\xi): \quad (12)$$
$$\int_{\Omega} g_i(\xi)F(d\xi)=m_i,\ i=0,1,\cdots,K\}.$$

In the following, we present two approaches to solving the problem (12), that is, the primal and dual approach. The primal approach is based on the result in Rogosinsky (1958), which basically states that if the problem (12) has a finite optimum, there exists an optimal distribution with its support a set of at most $K+1$ different points. In particular, it implies that it is sufficient to consider only the discrete distributions. The dual approach is based on the Lagrangian duality. The Lagrangian dual gives an upper bound on the problem (12). If there is a saddle point, it should be an optimum.

- *Primal Approach*

By using the result in Rogosinsky (1958), the problem (12) is equivalent to the following problem:

$$\max_{\Omega'\subset\Omega} G(\Omega'), \quad (13)$$

where $\Omega'=\{\xi_j\in\Omega: j=1,\cdots,K+1\}$ and

$$G(\Omega')=\max_{\{p_j\geq 0:\ j=1,\cdots,K+1\}}\{\sum_{j=1}^{K+1}p_j(\xi_j-x)^+: \quad (14)$$
$$\sum_{j=1}^{K+1}p_j g_i(\xi_j)=m_i,\ i=0,1,\ \cdots,K\}.$$

Note that the problem in (13) is a linear program to find an optimal discrete distribution with a set of $K+1$ points in Ω as its support.

In some cases, we can find the explicit form of the function $G(\Omega')$, which can be easily maximized. In the next subsection, we give some examples. When this is the case, the primal approach becomes a viable option.

- *Dual Approach*

The dual approach has an advantage of obtaining an upper bound on the optimal value of the problem (12) in a relatively simple way. This property is very useful, especially when there is difficulty in applying the primal approach.

Let $\lambda_i\in R,\ i=0,1,\cdots,K$ be a Lagrangian multiplier with respect to the i-th constraint in (12). Then the Lagrangian dual problem is $\min_{\lambda}\max_{F\geq 0}L(F,\lambda)$, where

$$L(F,\lambda)=\sum_{i=0}^{K}\lambda_i m_i+\int_{\Omega}\{(\xi-x)^+-\sum_{i=0}^{K}\lambda_i g_i(\xi)\}F(d\xi). \quad (15)$$

If for a given $\lambda\in R^{K+1}$, there exists $\xi'\in\Omega$ such that $\sum_{i=0}^{K}\lambda_i g_i(\xi')<(\xi'-x)^+$, then by choosing nonnegative measures $F_\alpha,\ \alpha\geq 0$, where $F(\{\xi'\})=\alpha$ and $F(\Omega\setminus\{\xi'\})=0$, we can see that $\max_{F\geq 0}L(F,\lambda)=\infty$. Also note that if $\sum_{i=0}^{K}\lambda_i g_i(\xi)\geq(\xi-x)^+$ holds for all $\xi\in\Omega$, then $\max_{F\geq 0}L(F,\lambda)=\sum_{i=0}^{K}\lambda_i m_i$ since the maximum of the integral in (15) should be zero. Hence the dual problem to (12) can be stated as follows:

$$\min_{\lambda}\{\sum_{i=0}^{K}\lambda_i m_i:\sum_{i=0}^{K}\lambda_i g_i(\xi)\geq(\xi-x)^+,\ \text{for all }\xi\in\Omega\}. \quad (16)$$

The dual problem (16) can be viewed as finding a function $g(\xi)=\sum_{i=0}^{K}\lambda_i g_i(\xi)$ which gives an upper bound on the function $(\xi-x)^+$ with the minimum possible weighted sum of moments.

Now let F^* and λ^* be the optimal solutions to the primal (12) and the dual (16), respectively and assume they are a saddle point. Then, we should have

$$\min_{\lambda}L(F^*,\lambda)=L(F^*,\lambda^*)=\max_{F\in M}L(F,\lambda^*). \quad (17)$$

In particular, we have $\int_{\Omega}(\xi-x)^+F^*(d\xi)=\sum_{i=0}^{K}\lambda_i^* m_i$ and

$$\int_{\Omega}\{\sum_{i=0}^{K}\lambda_i^* g_i(\xi)-(\xi-x)^+\}F^*(d\xi)=0. \quad (18)$$

Since $\sum_{i=0}^{K}\lambda_i^* g_i(\xi)-(\xi-x)^+\geq 0$, for all $\xi\in\Omega$, if F^* has a density with respect to the Lebesgue measure (that is, it is a continuous distribution), then $\sum_{i=0}^{K}\lambda_i^* g_i(\xi)=(\xi-x)^+$, for all $\xi\in\Omega$ (this case is not interesting in general). On the other hand, if it has a density with respect to the counting measure (that is, it is a discrete distribution), then $F^*(\{\xi\})>0$ only if $\sum_{i=0}^{K}\lambda_i^* g_i(\xi)=(\xi-x)^+$, $\xi\in\Omega$. The mixed case can also be treated easily. Hence we can conclude that the support of F^* should be a subset of $\Omega^*=\{\xi\in\Omega:\sum_{i=0}^{K}\lambda_i^* g_i(\xi)=(\xi-x)^+\}$. This result can be viewed as a complementary slackness theorem.

The above discussion is summarized in the following theorem 3.

Theorem 3: Given a distribution F^* and a vector $\lambda^*\in R^{K+1}$, they form a saddle point if and only if

(1) primal feasibility: $\int_{\Omega} g_i(\xi)F^*(d\xi)=m_i$, for all $i=0,1,\cdots,K$,
(2) dual feasibility: $\sum_{i=0}^{K}\lambda_i^* g_i(\xi)\geq(\xi-x)^+$, for all $\xi\in\Omega$, and
(3) complementary slackness: $F^*(\Omega\setminus\Omega^*)=0$, where $\Omega^*=\{\xi\in\Omega:\sum_{i=0}^{K}\lambda_i^* g_i(\xi)=(\xi-x)^+\}$.

It is not clear here when the strong duality holds for the primal and dual pair. Some results for the strong duality can be found in Shapiro and Kleywegt (2002).

However, if we can find a pair of distribution F^* and a vector $\lambda^* \in R^{K+1}$, which satisfies the conditions in Theorem 3, they should be optimal primal and dual solutions, respectively.

The dual approach is first to find an optimal solution to the dual problem (16). Then after finding the set Ω^*, we try to find a distribution in the class M with its support in Ω^*. Examples of the dual approach appear in the next subsection.

In some cases, the set Ω^* can be shown to have specific number of different points, as exemplified in the following corollary.

Corollary 2: Let M be the class of distributions with a finite mean and a positive variance. Also let Ω be one of the followings; $[a, b]$, $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$, where $a<b$. Then the set Ω^* in Theorem 3 should have 2 different points.

Proof: We only consider the case when $\Omega = R$. Let λ^* be an optimal dual solution. Consider the function $g(\xi) = \lambda_2^*\xi^2 + \lambda_1^*\xi + \lambda_0^*$. Then since $g(\xi) \ge (\xi - x)^+$, for all $\xi \in R$, $\lambda_2^* > 0$, so the function $g(\xi)$ is a quadratic strictly convex function. Hence we should have at most one solution to the equation $g(\xi) = 0$. The same is true for the equation $g(\xi) = (\xi - x)^+$. Hence Ω^* has at most 2 points. For a distribution to have a positive variance, its support should have at least two points. □

Remark 2: Under the same assumption in Corollary 2, we can also show that in the primal approach, only two-point distributions are sufficient to consider. This can be seen by considering the dual of the LP problem (14).

3.2 Characterization of the Tight Supremum

In this subsection, we will characterize the tight supremum for some interesting classes of distributions.

• *Fixed Mean and Support*

For given three real numbers μ, a, and b with $a < \mu < b$, let $M_\mu^{[a,b]}$ be the set of all distributions on the interval $[a, b]$ with the finite mean μ. The dual problem (16) is the following:

$$\min_\lambda\{\lambda_0 + \mu\lambda_1 : \lambda_0 + \lambda_1\xi \ge (\xi - x)^+, \text{ for all } \xi \in [a, b]\}, \quad (19)$$

where $x \in S = [a, b]$. Since the function $\lambda_0 + \lambda_1\xi$ is linear, it is easy to see that the minimum is attained by the line connecting two points $(a, 0)$ and $(b, b-x)$. Thus the optimal dual solution is

$$\lambda_0^* = -(b-x)a/(b-a),\ \lambda_1^* = (b-x)/(b-a). \quad (20)$$

Hence the tight supremum has the integrated survival function

$$\pi(x) = (b-x)(\mu-a)/(b-a), \quad (21)$$

which corresponds to the distribution

$$F_1 = ((b-\mu)/(b-a))\delta_a + ((\mu-a)/(b-a))\delta_b, \quad (22)$$

where δ_t is a (degenerate) random variable with its whole mass on the point $t \in R$. Note that the tight supremum F_1 has its support $\{a, b\}$.

Remark 3: The above result shows that for any random variable X with its distribution $F_X \in M_\mu^{[a,b]}$, $\mathrm{Var}(X) \le (a-\mu)(\mu-b)$.

• *Fixed Mean and Variance*

Let $M_{\mu,\sigma}$ be the set of all distributions with a finite mean μ and a standard deviation $\sigma > 0$. First, let us consider the case $\mu = 0$ and $\sigma = 1$. The corresponding dual problem is

$$\min_\lambda\{\lambda_0 + \lambda_2 : \lambda_0 + \lambda_1\xi + \lambda_2\xi^2 \ge (\xi - x)^+,\ \xi \in R\}. \quad (23)$$

For $\lambda \in R^3$ to be dual feasible, we should have $\lambda_2 > 0$. Also by inspection of the dual problem, it can be seen that it is sufficient to consider the case

$$\lambda_0 + \lambda_1\xi + \lambda_2\xi^2 = \alpha(\xi - \beta)^2, \quad (24)$$

where $\alpha > 0$ and $\beta < x$. Then by following the same proof in p. 58 in Müller and Stoyan (2002), we can show that the tight supremum is given by the distribution

$$F_0(x) = (1/2)(1 + x(x^2+1)^{-1/2}). \quad (25)$$

Now since $X \le_{cx} Y$ if and only if $(X-\mu)/\sigma \le_{cx} (Y-\mu)/\sigma$ (see Müller and Stoyan 2002), the tight supremum in the general case $M_{\mu,\sigma}$ is given by

$$F_2(x) = F_0((x-\mu)/\sigma). \quad (26)$$

We can also use the primal approach as follows. By Corollary 2 and Remark 2, we know that it is sufficient to consider only two-point distributions. For simplicity, let us consider the case $M_{0,1}$. Let $\xi_1 < 0 < \xi_2$ be given. From the result of the dual approach, we can safely assume $\xi_1 < x$. Then the corresponding primal problem is

$$\max_{p \ge 0}\{p_2(\xi_2 - x)^+ : p_1 + p_2 = 1, \quad (27)$$
$$p_1\xi_1 + p_2\xi_2 = 0,\ p_1\xi_1^2 + p_2\xi_2^2 = 1\}.$$

By rearranging the constraints in (27), we can conclude that $\xi_1 = t$, $\xi_2 = -1/t$, for some real number $t < 0$ and $p_2 = t^2/(1+t^2)$, which constitutes feasibility conditions of the problem (27). Now by maximizing $p_2(\xi_2 - x)^+$ with substitution of the terms, we can show that the optimal solution is obtained when $t = x - (x^2+1)^{1/2}$, which results in the integrated survival function (the optimal primal value)

$$\pi_0(x) = (1/2)(-x + (x^2+1)^{1/2}), \quad (28)$$

of which distribution function is given by F_0 in (25).

Remark 4: For the class $M_{0,1}$, the optimal distribution of the primal problem is given by $F_x = p\delta_t + (1-p)\delta_{-1/t}$, where $p = 1/(t^2+1)$ and $t = x - (x^2+1)^{1/2}$.

• *Fixed Mean, Variance, and Lower Bound*

Let $M_{\mu,\sigma}^{[a,\infty)}$ be the set of all distributions defined on $[a, \infty)$ with a finite mean $\mu > a$ and a standard deviation $\sigma > 0$. As in the case of the fixed mean and variance, we can use either the primal or dual approach. Here, we consider the primal approach.

As in the previous case, let us assume $\mu = 0$ and $\sigma = 1$. Using the previous results, we can see that if $t = x - (x^2+1)^{1/2} \ge a$, then the optimal primal value is $\pi_0(x)$ given in (28). So consider the case $x - (x^2+1)^{1/2} \le a$, which is equivalent to $x \le (a^2-1)/2a$. Then the maximum in (27) with $\xi_1 = t$, $\xi_2 = -1/t$, for some $t < 0$ is obtained when $t = a$. So in this case, the optimal primal value is

$$\max_{F\in M_{0,1}^{[a,\infty)}} \int (\xi - x)^+ F(d\xi) = -(a^2x + a)/(1+a^2), \quad (29)$$

where $a \le x \le (a^2-1)/2a$. Now also using the result that $X \le_{cx} Y$ if and only if $(X-\mu)/\sigma \le_{cx} (Y-\mu)/\sigma$, we can get tight supremum for $M_{\mu,\sigma}^{[a,\infty)}$ as given by

$$F_3(x) = \begin{cases} \sigma^2/\{(a-\mu)^2 + \sigma^2\}, \\ \text{if } x \in [a, \mu + \{(a-\mu)^2 - \sigma^2\}/2(a-\mu)] \\ F_0((x-\mu)/\sigma), \text{ otherwise.} \end{cases} \quad (30)$$

• *Fixed Mean, Variance, and Support*

Let $M_{\mu,\sigma}^{[a,b]}$ be the set of all distributions defined on $[a, b]$ with a finite mean $a < \mu < b$ and a standard deviation $\sigma > 0$ and $\sigma^2 \le (\mu - a)(b - \mu)$ (see Remark 3). In this case, by using a similar approach as in the case with lower bound, we can show that the tight supremum is given by the distribution

$$F_4(x) = \begin{cases} \sigma^2/\{(a-\mu)^2 + \sigma^2\}, \text{ if } x \in [a, \mu + \{(a-\mu)^2 - \sigma^2\}/2(a-\mu)] \\ F_0((x-\mu)/\sigma), \\ \text{if } x \in [(\mu^2+\sigma^2)/2\mu, \mu + ((b-\mu)^2 - \sigma^2)/2(b-\mu)]. \\ (b-\mu)^2/((b-\mu)^2 + \sigma^2), \text{ if } x \in [\mu + ((b-\mu)^2 - \sigma^2)/2(b-\mu), b) \end{cases} \quad (31)$$

4. MIN-MAX OPTIMAL SOLUTION OF THE SINGLE PERIOD INVENTORY CONTROL PROBLEM

4.1 The Newsvendor Problem

It is well known that for a given distribution F, the optimal solution of the newsvendor problem is given by Q which satisfies

$$Q = \inf\{x : F(x) \ge \zeta\}, \quad (32)$$

where $1 - \zeta = c_o/(c_u + c_o)$ is the optimal stock-out ratio (Porteus, 2002). Hence the min-max optimal solution for a given class of distributions M is

$$Q^* = \inf\{x : F_{\sup}(x) \ge 1 - \zeta\}, \quad (33)$$

where $F_{\sup}$ is a tight supremum of the class M.

Now using the results given in Section 3, we can get the following results (the results are stated for non-negative random variables when there is lower bound specified on the support).

Corollary 3:
1. For $M_{\mu}^{[0,b]}$, $Q^* = 0$, if $\zeta \le (b-\mu)/b$ and $Q^* = b$, otherwise.
2. For $M_{\mu,\sigma}$, $Q^* = \mu + (\sigma/2)[(c_u/c_o)^{1/2} - (c_o/c_u)^{1/2}]$.
3. For $M_{\mu,\sigma}^{[0,\infty)}$, $Q^* = \mu + (\sigma/2)[(c_u/c_o)^{1/2} - (c_o/c_u)^{1/2}]$, if $\zeta \ge \sigma^2/(\mu^2+\sigma^2)$ and $Q^* = 0$, otherwise.
4. For $M_{\mu,\sigma}^{[0,b]}$,

$$Q^* = \begin{cases} 0, \text{ if } \zeta \le \sigma^2/(\mu^2+\sigma^2) \\ \mu + (\sigma/2)[(c_u/c_o)^{1/2} - (c_o/c_u)^{1/2}], \\ \text{if } \zeta \in [\sigma^2/(\mu^2+\sigma^2), (b-\mu)^2/((b-\mu)^2+\sigma^2)). \\ b, \text{ if } \zeta \ge (b-\mu)^2/((b-\mu)^2+\sigma^2) \end{cases}$$

Remark 5: The solution given in Corollary 3-2 corresponds to a generalization of the Scarf's ordering rule (Scarf 1598, Gallego and Moon 1993, Alfares and Elmorra, 2005).

4.2 The Random Yield Problem

For a given distribution F, the optimal solution of the production lot sizing problem is given by Q which satisfies

$$Q = \inf\{x : \int_{-\infty}^{1/x} \xi F(d\xi) \le \tau\}, \quad (34)$$

where $\tau = (c + h\mu)/(r + h + p)$. Hence the min-max optimal solution for a given class of distributions M is

$$Q^* = \inf\{x : \int_{-\infty}^{1/x} \xi F_{\sup}(d\xi) \le \tau\}, \quad (35)$$

where $F_{\sup}$ is a tight supremum of the class M. Similar to the newsvendor problem, we can get the optimal lot size by using the results of Section 3. However, unlike the newsvendor problem, the explicit form of the solution is complicated in general, so it is omitted. We only mention that the solution can be obtained by solving the convex function minimization problem, that is, $\min_{x\ge 0} V(x, F_{\sup})$, where

$$V(x, F) = (c - \mu)x + (r + h + p)\int_{1/x}^{\infty}(\xi x - 1)F(d\xi).$$

5. CONCLUDING REMARKS

This paper gives a unified approach to the min-max stochastic optimization problem and characterizes the tight supremum for some interesting classes of distributions. Besides the newsvendor problem and the lot sizing problem considered in this paper, the results will be useful in other contexts.

The tight supremum given in this paper is derived under the assumption that strong duality holds for the primal and dual problems. General conditions for the strong duality call for more research. Also more systematic method to derive the tight supremum is needed for more general cost functions. Other interesting classes of distributions (for example, symmetric distributions) require more investigation.

REFERENCES

Alfares, H. K. and H. H. Elmorra, "The distribution-free newsboy problem: Extensions to the shortage penalty case," *International Journal of Production Economics* 93/94 (2005), 465-477.

Arrow, K. A., T. E. Harris, and J. Marshak, "Optimal inventory policy," *Econometrica* 19 (1951), 250-272.

Gallego, G. and I. Moon, "The distribution free newsboy problem: Review and extensions," *Journal of the Operational Research Society* 44 (1993), 825-834.

Gallego, G., J. K. Ryan, and D. Simchi-Levy, "Minimax analysis for finite horizon inventory models," *IIE Transactions* 33 (2001), 861-874.

Han, Q., D. Du, and L. F. Zuluaga, "Technical Note-A Risk- and Ambiguity-Averse Extension of the Max-Min Newsvendor Order Formula," *Operations Research* 62, 3 (2014), 535-542.

Khouja, M., "The single period (news-vendor) problem: Literature review and suggestions for future research," *Omega*, *International Journal of Management Science* 27 (1999), 537-542.

Müller, A. and D. Stoyan, "Comparison methods for stochastic models and risks," Wiley, NY 2002.

Perakis G. and G. Roels, "Regret in the newsvendor model with partial information," *Operations Research* 56, 1 (2008), 188-203.

Porteus, E. L., *Foundations of stochastic inventory theory*, Stanford University Press, 2002.

Rogosinsky, W. W., "Moments of non-negative mass," Proc. Roy. Soc. London Ser. A 245 1958.

Scarf, H., "A min-max solution of an inventory problem," K. Arrow, S. Karlin, H. Scarf (Eds.), Studies in the mathematical theory of inventory and production, Stanford University Press, Palo Alto, CA 1958.

Shapiro, A. and A. Kleywegt, "Minimax analysis of stochastic problems," *Optimization Methods and Software* 17, 3 (2002), 523-542.

Qin, Y., R. Wang, A. J. Vakharia, Y. Chen, and M. M. H. Seref, "The newsvendor problem: Review and directions for future research," *European Journal of Operational Research* 213 (2011), 361-374.

Yano, C. and H. L. Lee, "Lot sizing with random yields: a review," *Operations Research* 43 (1995), 311-334.