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Design and Implementation of Mathematics Textbooks in Support of Effective Teaching for Secondary Schools: A Chinese Case

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Mathematics textbook plays a significant role in shaping students' learning of mathematics. Logic, rigor and abstraction as typical features of the formalization of mathematics, dominate mathematics textbooks around the world, which is regarded as one of the important origins of students' learning difficulties in mathematics. An innovative series of Chinese mathematic textbooks is presented in this paper. Supported by the supplementary materials excerpts from the textbooks, it gives a comprehensive theoretical analysis of the principles of design and implementation of this series of mathematics textbooks. The effectiveness of this series of textbooks is demonstrated by student achievement and secondary research data. It shows that series of Chinese mathematic textbooks has largely decreased students' learning difficulties in mathematics and enhance classroom teaching efficiency. It suggests that prioritizing the essence of mathematics and reducing abstraction is an important notion for mathematics textbook design and implementation.

Keywords: mathematics textbooks, formalization, effective teaching, reducing abstraction
MESC Classification: U20, D70
MSC2010 Classification: 97B50

INTRODUCTION

As a fundamental resource, mathematics textbook shapes the way we teach and learn

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mathematics. Undoubtedly, mathematics textbook plays a fundamental and decisive role in mathematics classroom. Based on his four decades of experience directing studies of instruction and student performance with different mathematics textbooks, Usiskin (2013) identified seven purposes of mathematics textbooks usages including a vehicle for change in the curriculum, exposition of mathematics content, presentation of problems, the centerpiece of a course, the transmission of an ideal curriculum, the definition of mathematics branches as a subject in people's minds, and a vehicle for learning mathematics. And he predicted that mathematics textbook would remain in classrooms even in the digital future (Usiskin, 2013). Mathematics textbook has had played such a significant role in mathematics education since it existed, paradoxically, mathematics textbook research as a field of research is still at an early stage of development (Fan, 2013). Drawing on numbers of literature, Fan (2013) provided a critical analysis of contemporary issues and methods for mathematics textbook research, and he suggested researchers to go beyond textbook analysis, textbook comparison and textbook use to a new and shifted paradigm of research that employs more empirical and experimental methods. Our study presented here aims to make such a contribution by providing rich experimental data from textbook implementation, which concerns an innovative series of Chinese mathematic textbooks in support of effective teaching for secondary schools.

In the following sections, we start with the theoretical background on the nature of mathematics as a formal science, the underlying philosophical ideas in which the series of textbooks resides, and the practical background when the series of textbook is written. Next, we give a detailed analysis of the principles of design and implementation of this series of textbooks. Furthermore, we examine in which way that this series of mathematics textbooks support effective teaching for secondary schools. In the end, we discuss the implications of design and implementation of mathematics textbooks.

BACKGROUND

Mathematics is classified as a formal science because form is more prominent in mathematics than it is in other sciences. According to certain definitions of mathematics it is nothing more than a collection of abstract forms (Byers & Erlwanger, 1984). Mathematical form includes symbolic notation and chains of logical arguments (Byers & Hersovics, 1977). Corresponding to mathematician Leibniz (Van Heijenoort, 1967), the formalization of mathematics includes two aspects: express statement of theorems in a formal language, typically in terms of primitive notions such as set, and write proofs using a fixed set of formal inference rules, whose correct form can be checked algorithmically. Due to the nature of mathematics as a formal subject, typical features of the formalization of mathematics such as logic, rigor and abstraction, dominate mathematics textbooks used in classrooms around the world, which requires students to acquire the *ability to formalize* – to use conventional mathematical form and appropriate formal transformations to elucidate mathematical content (Byers & Erlwanger, 1984), the inability of which is regarded as one of the important origins of students' learning difficulties in mathematics (Ann & Miroslav, 2009; Weinberg & Wiesner, 2011).

There is no exception in mathematics classroom in China. Chinese mathematics textbooks emphasized the formalization of mathematics, which resulted in low quality of mathematics teaching and numbers of low-achievers in mathematics who also have heavily learning burden. To tackle these prevalent problems, different series of mathematics textbooks with different features and styles have been written and used in secondary schools; among them there is one that has gained the most far-reaching influence, in a way that its use had greatly improved mathematics teaching quality and alleviated students' learning burden. This series of textbooks aim for Enhancing Classroom Teaching Efficiency (GX for short, which are the first letters of *Gao Xiao*, the Chinese spelling for High Efficiency), and are therefore widely called as GX textbooks. The editors of GX textbooks are two mathematics professors. In their affiliation, there is a large research group focusing on primary and secondary school mathematics textbooks writing and experiment for several decades leaded by them. Since 1985, this research group has written eight series of primary and secondary school mathematics textbooks, among which there are four series of national planning textbooks and other four series examined and approved by the Committee of National Textbooks, and some of them have had been used in many provinces across the country with great reputation. The series of GX textbooks are the ones that have the greatest positive influence in society.

Meanwhile, the teaching reform experiment based on the series of GX textbooks is even regarded as one of the important influential mathematics teaching reforms in China. It has gained positive feedback from both students and teachers, and been highly evaluated by both mathematical and mathematics education community in China. For instance, Yingming Liu, an academician of Chinese Academy of Sciences, Professor at Sichuan University, as an external reviewer for the science and technology achievement of the GX experiment, commented that (Zhang, 2011), "the GX experiment has an advanced theory, achieved good teaching effect…in today when both the education for all-around development is emphasized and students' heavy burden needs to be alleviated, the ideas of GX experiment are still meaningful for cultivating students' creative spirit and their understanding of mathematical ideas". Dianzhou Zhang, Professor at East China Normal University, stated that:

"The slogan *Prioritize the essence of mathematics while de-focalize the presentation*

> *of forms* are really golden words for mathematics curriculum reform in China" (Zhang, 1998).

The GX textbooks had been used for more than eighteen years, although they have not been used in mathematics classrooms today, the principles that guided the design and implementation of this series of textbooks still play influential role in current mathematics education research and practice in China. For instance, they have impacted the establishment of Mathematics Curriculum Standards for Senior High Schools in China (Ministry of Education in China, 2013), in which "focus on essence, pay attention to appropriate mathematical form" is listed as a basic idea for the curriculum reform, and furthermore it is clearly stated that "transfer the academic form of mathematics to its educational form in such a way that be easily accessible for students", "do not limit the formal expressions, pay attention to the essence of mathematics, otherwise, the mathematical thinking might be inundated in the sea of formalization", which are reflections of the ideas advocated in the GX experiment (Zhang, 2011). In this paper, we give a systemic analysis of the principles of design and implementation of this series of textbooks and draw some conclusive suggestions for further mathematics textbook writing and implementation.

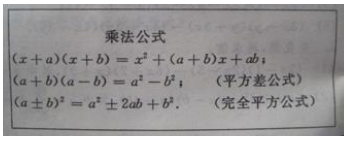
PRINCIPLES OF DESIGN AND IMPLEMENTATION OF GX TEXTBOOKS

The principles of design and implementation of GX textbooks can be concluded by thirty-two Chinese characters (GX principles for short), and they can be described as four set phrases, in English namely, *Progress promptly in contents while review constantly*; *Prioritize the essence of mathematics while de-focalize the presentation of forms*; *Go straight to the point and make connections of relevant issues; Practice before lecturing, teacher and students working together*. Among these principles, *Prioritize the essence of mathematics while de-focalize the presentation of forms* is the core. In this section, these principles are discussed one by one by illustrating mathematical examples from the textbook sequences.

Prioritize the Essence of Mathematics While De-Focalize the Presentation of Forms

The language of mathematics plays a pivotal role in mathematics textbooks (O'Keeffe & O'Donoghue, 2015). An important feature of formalization of mathematics is that its language allows mathematical objects and their relationships to be expressed in a formal way. However, if this formalization is over-emphasized, it has a negative effect on the students' understanding of the essence of mathematical objects, and it easily causes students' cognitive difficulties and brings leaning burden to students. When writing the GX textbooks, the principle of *Prioritize the essence of mathematics while de-focalize the presentation of forms* emerges as an attempt to avoid the over-emphasis of formalization of mathematics. Specifically, it includes three connotations, which are discussed below.

Firstly, it refers that the words of the mathematical language for presenting mathematical objects including mathematical facts, concepts and theorems, should not be taken too seriously and viewed as the only way of representation. Depending on the situation, when the words are suitable to be used to present mathematical facts, concepts and theorems, and when they are helpful for students to understand, and easy to be described, it is good to use words. To the contrary, if the words are too lengthy with much superficial details and information, which can happen in the case of only for ensuring the rigor, completeness and unambiguity of mathematics content, it is good to not use words because otherwise, it will easily distract the mathematical essence. Regarding on this, mathematician Kline has the similar observation: "to say in words what the expression states would not only require more length but would make comprehension more difficult" (Kline, 1974). Therefore, in terms of this principle, in the GX textbooks, for some key mathematical contents such as *multiplication formula*, *basic properties of algebraic fraction*, and *the relationships between the coefficients and roots of quadratic equations with one unknown* (see Figure 1), whose description of using words are long, therefore, their natural words are not presented, and what are presented are their symbolic



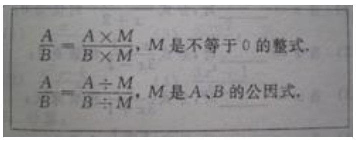
乘法公式

$(x+a)(x+b)=x^2+(a+b)x+ab$；

$(a+b)(a-b)=a^2-b^2$；（平方差公式）

$(a\pm b)^2=a^2\pm 2ab+b^2$.（完全平方公式）

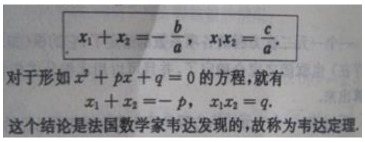
(a) Multiplication formula



$\frac{A}{B}=\frac{A\times M}{B\times M}$，$M$ 是不等于 0 的整式.

$\frac{A}{B}=\frac{A\div M}{B\div M}$，$M$ 是 A、B 的公因式.

(b) Basic properties of algebraic fraction



$$\boxed{x_1+x_2=-\frac{b}{a},\quad x_1x_2=\frac{c}{a}.}$$

对于形如 $x^2+px+q=0$ 的方程，就有

$$x_1+x_2=-p,\quad x_1x_2=q.$$

这个结论是法国数学家韦达发现的，故称为韦达定理.

(c) The relationships between the coefficients and roots of quadratic equations with one unknown

Figure 1. Excerpts from GX textbooks

expressions. The symbolic language is the most obvious property of mathematical text (Österholm, 2006). In the GX textbooks, symbolic expressions are transformed more easily than its counterpart (natural language), and it not only helps to save time and labor for students but also aid their understanding of mathematical contents.

Secondly, it is regarded that it is inappropriate to require secondary students to pursue all the accurate definitions of mathematical concepts; instead, attention should be paid to students' understanding of the essence of mathematical concepts. Rigor and precision are essential to mathematics because they provide firm reasons for believing that ideas and methods are sound (Steen, 1990). The GX principles hold the same view as the traditional emphasis on precise formulation of concepts and rigorous logical proof, which are an essential component of the mathematical endeavor, however, the GX principles further hold that, it is not necessary to give rigorous formal definitions for every mathematical concepts, and it is fine with just some explanations or even without any explanations for some mathematical concepts. Take the mathematical concept *equation* as an example, in the GX textbooks no rigorous formal definition is given to the terminology *equation*. In volume one of Algebra (Chen & Song, 1999a), an *equation* is described as "it is a problem-solving process in which the values of specified letters need to be figured out so as to work for the given equality", and "it also refers to an equality existing in a problem", as well as "an equality with some-unknowns". In fact, if an equation is directly defined as "an equality containing unknowns", which is the classic definition in many mathematics textbooks, it may cause a logical dilemma that could be seen from listed questions below.

In $mx = n$, m is called the known quantity. Why is x called the unknown?

In $x = 5$, why is x called the unknown quantity?

In $x - 1 = x + 1$, x does not exist at all, should we still give a definition to the inexistence?

Therefore, similarly, for some mathematical concepts such as *internal terms of proportion*, *extreme terms of proportion*, *interior angle*, *exterior angle*, *supplementary angle*, etc., there were no accurate definitions for them, and students are only required to understand their key ideas (Chen & Song, 1993). The reason for this is that it aims for better understanding and mastering the essence of mathematical objects, and avoiding focusing on insignificant and negligible information of mathematical concepts.

Thirdly, it is regarded that mathematical objects could have flexible representation. Actually, standardization of representation of mathematical objects is a manifestation of formalization of mathematics, because mathematical notations are seldom arbitrary and they are designed to express mathematical concepts (Byers & Erlwanger, 1984), and it is understandable to require students to conform to certain rules of presentation of mathematical objects. However, if the standardization of representation of mathematical objects is overemphasized, and "if textbook is the only one standard, in other words, if all the mathematical objects consisting of mathematical facts, concepts and theorems should follow the textbooks without any changes, it is too serious, and it will restrain students' cognitive development " (Chen & Song, 1993). Therefore, in the GX textbooks, there are no strict requirements for representation of mathematical objects. For instance, the mathematical signs for "deduction" that is widely used in describing reasoning process, both

"$\because$, $\therefore$" and "$\Rightarrow$"

are applicable. A further example is that, it is not required for students to differentiate the multiplier and multiplicand of multiplication in the GX textbooks, because it is regarded that the different expression will not bring much essential impact on students' understanding of the essence of multiplication. In this sense, there is no main difference between the multiplication of 3×4 and 4×3.

As we can see from above, under the principle of *Prioritize the essence of mathematics while de-focalize the presentation of forms*, mathematical logic and rigor are not over-emphasized, hence, teachers and students could save their time and energy from strictly adhering to formalization of mathematics, and spend more time on understanding the essence of mathematics.

Of course, there might be some philosophical debates on the question whether we should de-focalize the presentation of forms. People may argue that it might lead to remove the soul of mathematics if the presentation of forms is de-focalized; this is because that the abstraction of mathematics appears to be consequence of its formal character, and the rigor of mathematics derives from its formal deductive rules. Furthermore, people might argue that if the presentation of forms is de-focalized, mathematics education will lose its great value as a tool for training and development of logical thinking (Kounine, Marks & Truss, 2008), given the fact that mathematics has a fundamental and irreplaceable function due to its nature as a formal science. The GX principle doesn't object those arguments, however, what we are reminded is that, the GX principle is directed against the inappropriate formalization of mathematics at the basic education level (Song & Chen, 1996). Therefore, even for some important mathematical concepts in this educational phrase, the exact mathematical definitions are not required to be presented in the GX textbooks, and attention is expected to be paid to the understanding and comprehension of their essence (The GX experiment Group, 1998).

Progress Promptly in Contents while Review Constantly

Oriented by the longstanding examination culture in China, there are popular teaching beliefs on high efficiency in secondary mathematics classroom such as "no half-cooked rice" and "one step at a time". More importantly, the teaching principle of gradual improvement is viewed as a golden rule. Therefore, mathematical contents are reviewed repeatedly in classroom, which resulted in the waste of teaching time and low teaching efficiency, meanwhile, it restrains teachers' subjective initiative and lower down students' leaning interests in mathematics, both of which finally lead to the low quality in mathematics education. To cope with this situation, the principle *Progress promptly in contents while review constantly* is proposed for writing GX textbooks.

Literally speaking, *Progress promptly in contents* refers to keep fast teaching speed to enable students to keep feeling of novelty, success and leaning enthusiasm. For instance, in the chapter Circle in the third volume of Geometry (Chen & Song, 1999b), it includes numbers of mathematical concepts including *the centre of a circle*, *radius*, *tangency*, *secant line*, *tangent line*, *arc*, *chord*, *central angle*, *angle of circumference* and *angle of osculation*, which are designed collectively for one section in the GX textbook.

Table 1. Comparison of arrangement of Factorization between the GX textbook and the textbook published by PEP

GX textbooks		Textbook published by PEP
Lesson 1	Significance of factorization	8.1 Method of extracting common factor (3 lessons)
Lesson 2	Method of extracting common factor	8.2 Method of applying formula (7 lessons)
Lesson 3	Method of decomposing a formula	8.3 Method of grouping and decomposition (4 lessons)
Lesson 4	Comprehensive exercises	8.4 Cross-multiplication method (5 lessons)
Lesson 5	Formula of difference of cubes and cubic sum	Reading: quadratic trinomial decomposition by using method of completing square
Lesson 6	Method of grouping and decomposition	Summary and review (3 lessons)
Lesson 7	Cross-multiplication method	Review
Lesson 8	Method of completing square sum	Self-test
Lesson 9	Method of lowering the unknown	
Lesson 10	comprehensive exercises	
Lesson 11	solve quadratic equations with one unknown by using factorization	
Lesson 12	Summary	
Self-test		
Review		

Reviewing constantly is based on *Progressing promptly in contents*. It refers to proceed under the connection between new and prior knowledge, and enhance the understanding of mathematics in a continuous way. For instance, in the same chapter Circle, there are repetitions after having moved forward to both the forth and fourteenth topics. In all, independent on the mathematical contents, this principle is applied to all the GX textbooks. The arrangement of Factorization is another typical example on algebraic contents: as seen from Table 1 on the comparison of the arrangement of Factorization, compared to 53 lessons in total designed in the textbook published by People's Education Press (PEP), there are only 15 lessons designed in the GX textbook, not over 40 lessons in total plus review lessons designed by teacher in terms of the situation of the class.

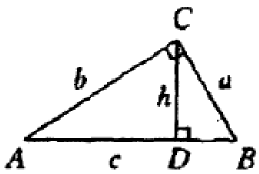
In summary, with the principle *Progress promptly in contents while review constantly*, the GX textbooks are designed from a whole perspective that enhances inherent connections between different knowledge; it is in this way that, the integral structure of mathematical contents and the connections between prior and present knowledge in the textbooks are emphasized, accordingly, it helps to solve the contradiction between teaching speed and leaning efficiency.

Go straight to the Point and Make Connections of Relevant Issues

Mathematical knowledge has the structure of a network. Mathematical concepts, definitions, theorems, proofs, algorithms, rules, theories, are manifold interrelated but also connected with components of the external world (Brinkmann, 2005). In this mathematical network, two types of mathematical knowledge could be identified. One is those basic mathematical concepts and fundamental theorems, which are the most abstract and general ones in the structure of mathematics, and they can be viewed as backbone knowledge (knowledge source) or knowledge roots. The other is the supplementary knowledge or knowledge flow, which is derived from basic concepts and fundamental theorems. There is a close relationship between these two types of knowledge. The backbone knowledge plays the role of explanation, organization, and collection on the supplementary knowledge. Once the backbone knowledge is mastered, the supplementary knowledge will be easily mastered. Meanwhile, these two types of knowledge form an organic structure with interrelationship.

Under this theoretical perspective, the principle *Go straight to the Point and Make Connections of Relevant Issues* is proposed to write and implement the GX textbooks. *Go straight to the point* refers to directly touch the essential concepts and fundamental theorems in mathematics and quickly get to know the point of knowledge structure, while *make connections of relevant issues* refers to follow the inherent logic connections between essential mathematical concepts and fundamental theorems, and construct the lines existing in the structure. The principle *Go straight to the point and make connections of relevant issues* reflects the idea of designing textbooks by using an integrated-connected perspective. Based on certain knowledge points, the textbooks form knowledge lines, planes, then volumes, and finally a dynamic knowledge network structure. Written according to this principle, the textbooks are helpful for students to build mathematical knowledge network, and consequently, help to improve students' mathematics literacy and enhance students' learning efficiency. Taking the chapter *Similarity* in the third volume in Geometry as an example (Chen & Song, 1999b), it starts with *Proportion and Its Properties* which acts as *coming straight to the point*; based on it, *Common Height Theorem* and *Common Angle Theorem* are then presented, followed by *Proportional Segments in a Triangle and Their Relevant Theorems*, as well as *Similar Triangle and Their Properties*, and it ends with *Common Side Theorem*, *Similar Polygon* and *Homothetic Construction*. With such a textbook design, the mathematical content is concentrated, and mathematical knowledge is organized in a systematic and complete way. The proof of the Projective Theorem in a right triangle below is an example showing how the mathematical knowledge could be effectively organized.

Projective Theorem. As shown in the following figure, in a right triangle ABC with angle C equal to 90°, CD⊥AB, the lengths of the three edges BC, AC and AB of the right triangle ABC are a, b and c, respectively, and the height CD = *h*, prove that



$$h^2 = AD \bullet DB$$

Proof.

$$\frac{S_{\triangle ACD}}{S_{\triangle CBD}} = \frac{bh}{a \bullet DB} = \frac{b \bullet AD}{ah} \Rightarrow h^2 = AD \bullet DB$$

As we can see from the example above, by using the relationships between the ratio of the areas of triangles and ratios of segments, which lies in a systemic mathematical network, the Projective Theorem in a right triangle is easily proved (The GX experiment group, 1998). This structural view of mathematical knowledge reflected in the GX principle was highly echoed by the great educationalist Bruner (1960) who firmly advocated the importance of structure. In Bruner's view, "emphasis should shift to teaching basic principle, underlying axioms, pervasive themes" and "it is still the only one that makes much sense, if the object is to transmit knowledge and to create intellectual skill" (Bruner, 1960).

Practice before Lecturing, Teacher and Students Working Together

The principle *Practice before lecturing, teacher and students working together* is for both writing and implementing GX textbooks. *Practice before lecturing* refers that, for teaching any mathematical objects, it is good to present a relevant problem at the beginning of the lesson, followed by specific examples that guide students to observe, to think, to talk, and do. Once students have perceptual recognitions of the mathematical object, then, mathematical definitions, algorithms and methods are introduced.

Teacher and students work together puts attention to the importance of both teacher and students in mathematics classroom. On the one hand, teacher plays a significant role in guiding students to participate in the learning, and on the other hand, students solve problems by themselves under teacher's guidance. In terms of this principle, mathematical contents in GX textbooks are designed in such a way to create a classroom environment where both teacher and students' roles are emphasized. Taking the mathematical content Multiplication of Monomial Expression as an example, a problem "what is the result of $3a^2x^2$ multiplying $5ax^3y^2$?" is presented in the beginning of the chapter (Chen & Song, 1999a). And it is further designed for teacher and students to work together, and for further guidance for students to derive algorithm for multiplication of monomial expression. Below is an exemplary teaching design on how teacher guide students to learn Multiplication of Monomial Expression by using GX textbooks, where T stands for teacher, and S means students.

T: If the product of the multiplication of two monomial expressions is $3a^2x^2 \cdot 5ax^3y^2$. Is this product a monomial expression?

T: Hi, guys, let's work together through an example below.

$$3a^2x^2 \cdot 5ax^3y^2 = (3\times 5)(a^3 \cdot a)(x^2 \cdot x^3)y^2 = 15a^4x^5y^2$$

T: Think about it, can this monomial expression be simplified?

T: As we can see, normally, the product is a standard form after having been simplified. The question is that, how to get the standard form of a monomial expression? Please think about it, and then compare the following steps: first, calculate the product of the multiplication of number factors and write it ahead of letter factors; second, put together the same letters and write them in the form of power; third, order the letter factors according to the order of alphabet.

T: Now let's work together to try another example.

T: Calculate

$$\frac{2}{3}xy^2 \cdot (-\frac{3}{4}x^2y^4)\ .$$

T: The algorithm (or rule) for multiplication of monomial expression is that, first write them together, and then sort it out as a standard form.

The teaching design above provides a hypothetical learning trajectory that offered a description of key aspects of a planned mathematics lesson (Simon & Tzur, 2004), which is underpinned by the GX principle *Practice before lecturing, teacher and students working together*. This teaching design reflects teacher's important role in guiding students to learn the algorithm for multiplication of monomial expression. Although the algorithm is discovered and summarized during students' "doing", teacher play a pivotal role in the process; meanwhile, the teacher also participates the process of "doing" by providing appropriate examples and necessary hints, and finally summarizing the steps together with students. Below we present a classroom teaching episode on Solving System of Equations by using Method of Substitution (The GX experiment group, 1998), which reflects how teacher and students actually "do" in the classroom.

A system of equations is given on the blackboard

$$\begin{cases} 3x + 2y = 13 & (1) \\ y = x - 1 & (2) \end{cases}$$

The teacher asked one student to work on it on the blackboard, and other students remain sitting to work.

T: Please have a look. Without the method of addition and subtraction, how to solve it?

S: Insert (2) into (1).

T: Yes. Why can (2) be inserted into (1)?

S: This is because y is the same in the two equations.

T: That is right. All equivalences can be substituted. Have you worked on this type of substituting problems?

S: Yes. We did in the last lesson by substituting an un-known in an equation by using a known.

T: Yes. Previously it is to substitute a number; today what we study here is to substitute an algebraic expression, which we will often use in the later.

T: Now we will invite a student to solve this system of equations on the blackboard, and other students work in their seats.

T: Let's look at this step: substitute (2) by using (1), we get $3x + 2(x - 1) = 13$. By equivalence substitution, we eliminate the unknown y; this is what we are going to learn—method of substitution.

In the end, taking this task as an example, the students are asked to make a summary about the steps on how to solve the system of linear equations of two unknowns using method of substitution.

The classroom teaching episode apparently implies that, the GX principle *Practice before lecturing, teacher and students working together*, emphasizes the co-work between teacher and students in the teaching process. The consequent dialogs between teacher and students around how to solving the system of equations finally leads to the birth of Method of Substitution, which shows that the teaching of mathematics is conducted based on both teacher and students, neither teacher nor students should be overemphasized at the expense of the other.

In summary, from the analysis above, we understand how the principles of design and implementation of GX textbooks help to enhance mathematics classroom teaching efficiency from a theoretical perspective. It is *Progress promptly in contents while review constantly* that enhances the efficiency from a macro perspective; it is *Prioritize the essence of mathematics while de-focalize the presentation of forms* that puts time on the crucial point from an integrated perspective; it is *Go straight to the point and make connections of relevant issues* that puts time on the right spot from a technical perspective; it is *Practice before lecturing, teacher and students working together* that enhances the efficiency from a micro perspective.

EFFECTIVENESS OF IMPLEMENTATION OF GX TEXTBOOKS

In this section, the effectiveness of GX textbooks implementation will be demonstrated by rich experimental data from student achievement and research literature. They are summarized from the aspects of student achievement in mathematics, student learning efficiency, and the application of GX principles, all of which shows that the GX textbooks are greatly in support of effective mathematics teaching for secondary schools in China.

Improve Student Achievement in Mathematics

GX textbooks were initially implemented in 1992 in six schools in Chongqing, a city located in southwest China. In the autumn of 1997, there were more than 120 classes that adopted GX textbooks. Due to the unexpected effectiveness, the regions that adopted GX textbooks expanded quickly, from one city to ten provinces, in total involving more than hundreds of schools. Numbers of data showed that GX textbooks benefited a lot for students. From 1996 to 1998, the GX project team members conducted three sampling in Yunnan, Guizhou, Sichuan, and Chongqing, in total 9431 samples including 4905 students in the classes having adopted GX textbooks (the experimental classes) and 4526 students in the classes that hadn't adopted GX textbooks (non-experimental classes). The results showed that, the experimental classes had higher average achievement, pass rate, distinguished achiever ratio, and lower low achiever ratio than non-experimental classes. Table 2 and Table 3 show a comparison of student achievement in mathematics in the experimental classes and non-experimental classes from Chongqing, where ten experimental classes and eight non-experimental classes were chosen by random (the GX experiment group, 1998).

Table 2. Comparison of student achievement in mathematics between the GX experimental classes and non-experimental classes (at the good level)

	Average	Pass rate	DAR	LAR
Experimental classes	85.5	91.2%	73.2%	1.4%
Non-experimental classes	68.0	71.6%	30.7%	10.5%

Note. DAR= distinguished achiever ratio, LAR= low achiever ratio.

Table 3. Comparison of student achievement in mathematics between the GX experimental classes and non-experimental classes (at the general level)

	Average	Pass rate	DAR	LAR
Experimental classes	68.7	69.9%	43.0%	17.3%
Non-experimental classes	53.9	46.7%	26.6%	41.0%

Note. DAR= distinguished achiever ratio, LAR= low achiever ratio.

As shown in Tables 2 and 3, whatever the classes were at a good level or general level, the average scores in the GX experimental classes were higher than in the non-experimental classes, but this difference was more obvious in classes at a good level, in which the pass rate and distinguished achiever ratio in the experimental classes were much higher than the non-experimental classes, whereas their low achiever ratio was obviously lower than those in the non-experimental classes.

Enhance Student Leaning Efficiency

As we see from the analysis above that, student achievement in mathematics in the GX experimental classes were significantly higher than those in non-experimental classes. However, paradoxically, students in the GX experimental classes spent less time on regular learning. The experimental data showed that, students in the GX experimental classes did more practice in classrooms, which greatly alleviated the burden of both teacher and students in a way that they do not have to work more after having left the classroom. Take one of the experimental schools as an example, in which the GX experimental class had a significant achievement that is featured as *More, Fast, Good, and Timesaving* (The GX experiment group, 1998). The feature *More* means that, students in the GX experimental class learnt more mathematical knowledge than their counterparts in the same school. The feature *Fast* means that, the GX experimental class ended the new lessons two weeks earlier than the non-experimental classes. The feature *Good* refers that, the experimental class had higher average score at the district level than other classes, and it also had higher rankings and more awards in competitions. The feature *Timesaving* means that, compared with the non-experimental classes, the experimental class spent less time in teaching and learning of mathematics, and both the teacher and students in the experimental class felt relax.

That the GX experiment alleviated learning burden of students, not only helped to improve their leaning of other subjects, but also saved time for students to attend activities after having left the classroom to improve their all-round development. For example, a study conducted in a middle school showed that (Zeng, J., 1997), during their learning when having adopted GX textbooks for three years, students always actively engaged in different mathematical leaning activities and experienced the happiness of success of mathematics, hence students in the class were very interested in mathematics. Meanwhile, students developed their ability of self-study, which in turn improved their learning of other subjects. Compared with the same grade classes in the school, leaning atmosphere and learning achievement of the GX experimental class were on the top, and students' overall competence had been developed.

Application of the Principles of Design and Implementation of GX Textbooks

Although GX textbooks were initially designed for middle school students, the principles of design and implementation of GX textbooks had been applied successfully in mathematics textbooks for high school students, in other subjects, and in ethnic minority regions.

Zhang (2005) conducted a study by transferring the GX principles from middle schools to high schools. Her experimental results showed that, the GX experiment at the high school greatly enhanced student achievement in mathematics. It positively transferred to the learning of other subjects, and effectively aroused students' leaning interests and improved learning methods, hence apparently alleviated students' leaning burden.

Due to the success of the GX experiment, teachers in other subjects also applied the GX principles to guide their teaching and consequently gained good effect. For example, Zeng, T. (1997) conducted an experimental study by adopting the GX principles in her teaching in history. Her study showed that, "the transfer of the principles of design and implementation of GX textbooks to the teaching of history is feasible" (Zeng, T., 1997).

China is a multinational country composed of 56 official ethnic minority groups (55 minorities plus the dominant Han). Not only in the Han regions did the GX experiment achieve good results, but also in ethnic minority regions. For example, Chen, Yan and Li (2003) conducted a study by using the GX textbooks in Yi minority middle schools. Their research results showed that, "by using the GX textbooks and applying its principles in teaching, it can overcome Yi middle school students' language obstacles in their mathematics learning; the GX textbook laid a good foundation for Yi students, which can't be found in other textbooks" (Chen et al., 2003).

CONCLUSION AND DISCUSSION

Mathematics is often understood as an exact science because of its prominent form, which allows scientific theories to be presented by the principles, axioms, or propositions. However, it is also the mathematical form that causes difficulties for numbers of students to learning mathematics. Internationally, the notion of abstraction derived from the mathematical form, has recently received a lot of attention within the mathematics research community (Dreyfus & Gray, 2002; Hazzan & Zazkis, 2005; Mitchelmore & White, 2004; Raychaudhuri, 2014), by which it is recognized that the ability to abstract is imperative to learning and doing meaningful mathematics, and accordingly, a notion of reducing abstraction is proposed in order to better understand how students reduce abstraction while leaning mathematical concepts. The principles of design and implementation of GX textbooks are consistent in this line of research, but it moves step further towards this than the relevant studies. Firstly, the notion of reducing abstraction so far has only be applied to analyze students' mathematical thinking, never has yet been applied to the writing and implementation of school mathematics textbooks. Secondly, the GX principles offered a broader explanation to the origins that cause students' learning difficulties in mathematics by extending abstraction to other faces of mathematical form including rigor and logic.

Textbooks form the backbone as well as the Achilles' heel of the school experience in mathematics (Ann & Miroslav, 2009). Seeking explanations for the phenomenon of unsuccessful mathematics secondary school students, until recently researchers began to draw attentions to features of textbooks. And research results have continuously shown that, textbooks affect the ways students read them, and consequently impact their mathematical learning (O'Keeffe & O'Donoghue, 2015; Weinberg & Weisner, 2011). For instance, the use of symbols in the mathematics textbooks is a rather relevant factor for students' comprehension of mathematical text (Österholm, 2006); tasks in textbooks designed in informal, context-based and investigative mathematics have the potential to promote low-achievers' successful solutions of mathematical tasks (Friedlander, Robinson & Koren, 2011). These studies supports that the GX principle de-focalize the presentation of forms is an important way to help students to overcome their learning difficulties in mathematics. Meanwhile, GX principles offer an overall perspective on textbook design and implementation by considering the different factors in teaching and learning of mathematics. They offer a comprehensive and applicable framework for design and implementation of mathematics textbooks in support of effective mathematics teaching for secondary schools.

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