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Hyperspaces and the S-equivariant Complete Invariance Property

Saurabh Chandra Maury
Department of Mathematics, University of Allahabad, Allahabad-211 002, India
e-mail : smaury94@gmail.com

Abstract. In this paper it is investigated as to when a nonempty invariant closed subset A of a S^1-space X containing the set of stationary points (S) can be the fixed point set of an equivariant continuous selfmap on X and such space X is said to possess the S-equivariant complete invariance property (S-ECIP). It is also shown that if X is a metric space and S^1 acts on $X \times S^1$ by the action $(x,\ p) \cdot q = (x,\ p \cdot q)$, where $p,\ q \in S^1$ and $x \in X$, then the hyperspace $2^{X \times S^1}$ of all nonempty compact subsets of $X \times S^1$ has the S-ECIP.

1. Introduction

A topological space X is said to possess the complete invariance property (CIP) if each of its nonempty closed subsets is the fixed point set, for some self continuous map f on X [15]. In case, f can be found to be a homeomorphism, we say that the space enjoys the complete invariance property with respect to homeomorphism ($CIPH$) [7]. A detailed account of spaces possessing CIP or otherwise and various techniques classifying and determining these classes of spaces together with many results and their applications can be found in [5,7,8,9,10,12,13,14,15].

In [2] a space X is defined to have the S-equivariant complete invariance property (S-ECIP), if every nonempty invariant closed set containing the set of stationary points is a fixed point set of an equivariant continuous selfmap on X. In this paper it is shown that a metric space on which S^1 acts freely such that the orbits are equidistant to each other, possesses S-ECIP. This is a more general result than the Proposition 1.8 in [2] which says that if $(X,\ d)$ is a metric space and S^1 is the unit circle group, then the product $X \times S^1$ has S-ECIP.

In the last section of this paper it is investigated that the hyperspace 2^X of nonempty compact subsets of a metric space X enjoys the notion of S-equivariant

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complete invariance property (S-ECIP).

2. Pre-requisites

A. CIP and S-ECIP

By a space we mean a Hausdorff space and by a group, a topological group. A subset F of a space X is called a *fixed point set* of X, if there is a continuous selfmap f on X such that the set fix$f$ of fixed points of f is F. A space X is defined to have the *complete invariance property* (CIP) if every nonempty closed subset of X is the fixed point set of a continuous selfmap on X.

If a topological group G acts continuously on a space X by the action '$\cdot$', then we denote the *orbit* of $x \in X$ by $G_x = \{x \cdot g : g \in G\}$. A point $x \in X$ is called a *stationary point* of X if the orbit G_x is the singleton $\{x\}$. We use the symbol S for the set of all stationary points of X. A subset A of X is called an *invariant set*, if for $g \in G$ and $a \in A$, $a \cdot g \in A$.

Definition 2.1. Let X and Y be G-spaces. A continuous function $f : X \longrightarrow Y$ is said to be *equivariant*, if $f(x \cdot g) = f(x) \cdot g$, for $g \in G$ and $x \in X$.

Definition 2.2.([2]) A G-space X is said to possess the *S-equivariant complete invariance property* (*S-ECIP*), if every nonempty invariant closed set containing the set of stationary points is a fixed point set of an equivariant continuous selfmap on X.

Result 2.3.([7]) *A space X has the CIPH if it satisfies the following conditions:*

(i) *S^1 acts on X freely.*

(ii) *X possesses a bounded metric such that each orbit is (arc length metric) isometric to S^1.*

Result 2.4.([2]) *Let X be a metric space and S^1 act on $X \times S^1$ by the action $(x,\ p) \cdot q = (x,\ p \cdot q)$, where $p,\ q \in S^1$ and $x \in X$. Then $X \times S^1$ has S-ECIP.*

B. Hyperspaces

For a topological spaec X, 2^X denotes the collection of all nonempty compact subsets of X. The set 2^X equipped with some topology is called a *hyperspace* of X. Among various topologies defined on 2^X, the *Vietoris topology* also called the *finite topology* or the *exponential topology* is one of the most well studied topologies on 2^X.

The sets of the form

$$\langle U_1, U_2, \dots, U_n \rangle = \{K \in 2^X : K \subset \cup_{i=1}^n U_i \text{ and for all } 1 \le i \le n,\ K \cap U_i \ne \phi\},$$

where $\{U_1, U_2, \dots, U_n\}$ is a finite collection of open sets of X, form a *base* for the Vietoris topology on 2^X. In case, X is a metric space, the hyperspaces 2^X

of nonempty compact subsets of X can be metrized by the Hausdorff metric d_H, defined by

$$d_H(A,\ B) = \max\{\sup_{a\in A} d(a,\ B),\ \sup_{b\in\ B} d(b,\ A)\}$$

and the Vietoris topology coincides with the one introduced by the Hausdorff metric.

Result 2.5. The hyperspace 2^X *is a compact metric space if and only if* X *is a compact metric space.*

3. S-ECIP on Metric Spaces

Result 2.3 gives the information about the CIPH over a metric space X. After removing the condition (ii) from this result we get the following theorem.

Theorem 3.1. *A metric space on which* S^1 *acts freely has the CIP.*

Proof. Let $(X,\ d)$ be a metric space with $d \leq 2\pi$ and let the map $\cdot : X \times S^1 \longrightarrow X$ be a free action.

For a nonempty closed subset A of X, define the map $f_A : X \longrightarrow X$ by

$$f_A(x) = x \cdot e^{\frac{i}{2} d(x,\ A)}.$$

Since the action '$\cdot$' is free and $0 < \frac{1}{2} d(x,\ A) < 2\pi$ if $x \notin A$, we get fix$f_A = A$. □

Theorem 3.2. *Let* X *be a metric space on which* S^1 *act freely such that the orbits are equidistant to each other. Then* X *has S-ECIP.*

Proof. Let $(X,\ d)$ be a metric space with $d \leq 2\pi$ and let the map $\cdot : X \times S^1 \longrightarrow X$ be a free action. By equidistant orbits to each other we mean, if O_1 and O_2 are two orbits in X then $d(x, O_2)$ is fixed for all $x \in O_1$ and equal to $d(x, O_1)$ for all $x \in O_2$. The set of stationary points of X is empty and orbits are homeomorphic to S^1. If A is nonempty invariant closed subset of X, then A is union of some orbits. Consider the map $f_A : X \longrightarrow X$ as defined in the previous theorem. We have fix$f_A = A$.

Now, to prove that f_A is an equivariant map we show that

$$f_A(x \cdot p) = f_A(x) \cdot p, \qquad p \in S^1$$

or, $$x \cdot p \cdot e^{\frac{i}{2} d(x \cdot p,\ A)} = x \cdot e^{\frac{i}{2} d(x,\ A)} \cdot p$$

or, $$d(x \cdot p,\ A) = d(x,\ A).$$

Since $x \cdot p$ and x are in the same orbit and the orbits are equidistant to each other we get the result.

This proves that for any nonempty invariant closed subset A of X there exists an equivariant selfmap f_A on X whose fixed point set is A. □

4. S-ECIP on Hyperspaces

Let G be a compact group, X a metrizable space and 2^X is a hyperspace of all nonempty compact subsets of X.

If $\cdot : X \times G \longrightarrow X$ is an action, then the map $* : 2^X \times G \longrightarrow 2^X$ defined by $*(A, g) = A * g = \{a \cdot g : a \in A\}$, where $g \in G$, $A \in 2^X$ is a continuous action[1]. That is if X is G-space, then 2^X is a G-space. This led to the following:

Theorem 4.1. *Let X be a metric space and S^1 act on $X \times S^1$ by the action $(x,\ p) \cdot q = (x,\ p \cdot q)$, where $p,\ q \in S^1$ and $x \in X$. Then the hyperspace $2^{X \times S^1}$ of all nonempty compact subsets of $X \times S^1$ has S-ECIP.*

Proof. Consider an equivalent metric $d_1 (\leq 1)$ on X and the arc length metric d_2 on S^1. Then $X \times S^1$ is a metric space with the metric d, defined by

$$d^2((x,\ p),\ (y,\ q)) = d_1^2(x,\ y) + d_2^2(p,\ q),$$

where $(x,\ p)$ and $(y,\ q)$ are elements of $X \times S^1$.

Since S^1 is a compact group acting on the metric space $(X \times S^1,\ d)$, then the map $* : 2^{X \times S^1} \times S^1 \longrightarrow 2^{X \times S^1}$ defined by $*(A,\ q) = A * q = \{a \cdot g : a \in A\}$ is a continuous action.

The hyperspace $2^{X \times S^1}$ is a metric space, with the Hausdorff metric d_H. Metric d_H is bounded follows from below:

Since

$$d(a, B) = \inf_{(b_1, b_2) \in B} \sqrt{d_1^2(a_1, b_1) + d_2^2(a_2, b_2)} \leq \sqrt{1 + (2\pi)^2}$$

where $a = (a_1, a_2) \in X \times S^1, (b_1, b_2) \in B$ and $B \in 2^{X \times S^1}$. We have

$$d_H(A,\ B) = \max\{\sup_{a \in A} d(a,\ B),\ \sup_{b \in\ B} d(b,\ A)\} \leq \sqrt{1 + (2\pi)^2}.$$

The set of stationary points of the S^1-space $2^{X \times S^1}$ is

$$S = \{C \times S^1 : C \text{ is a compact set in } X\}.$$

Let K be an invariant closed subset of $2^{X \times S^1}$ containing S. Define a map

$$f : 2^{X \times S^1} \longrightarrow 2^{X \times S^1}$$

by $f(A) = A * e^{ia(A)}$, where $A \in 2^{X \times S^1}$ and $a(A) = \frac{1}{2} d_H(A,\ K)$.

If $A \notin K$, then A is not an stationary point and $0 < a(A) < 2\pi$. Thus we get that $\text{fix} f = K$.

Now we show that the map f is equivariant. If $p \in S^1$, then

$$f(A) * p = A * e^{ia(A)} * p,$$

and

$$f(A * p) = A * p * e^{ia(A * p)}.$$

Since the invariant set K is a union of some orbits we have

$$d_H(A,\ K) = \inf_{S^1_B \in K} d_H(A,\ S^1_B).$$

By noting that

$$d_H(A,\ S^1_B) = \inf_{q\in S^1} d_H(A,\ B*q) = \inf_{q\in S^1} d_H(A*p,\ B*q) = d_H(A*p,\ S^1_B)$$

we have

$$d_H(A,\ K) = \inf_{S^1_B\in K} d_H(A,\ S^1_B) = \inf_{S^1_B\in K} d_H(A*p,\ S^1_B) = d_H(A*p,\ K),$$

where S^1_B is an orbit of B in the space $2^{X\times S^1}$.

Thus

$$a(A*p) = \frac{1}{2}d_H(A*p,\ K) = \frac{1}{2}d_H(A,\ K) = a(A)$$

shows that

$$f(A)*p = f(A*p), \qquad \text{for all } p\in S^1.$$

This proves that for each nonempty invariant closed subset K in $2^{X\times S^1}$ containing the set S of stationary points there exists an equivariant map f from $2^{X\times S^1}$ to $2^{X\times S^1}$ whose fixed point set is K. □

Remark 4.2. We remark that in the above proof, $X\times S^1$ could be replaced by any space Y on which S^1 acts freely such that the orbits are equidistant to each other.

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References

[1] S. Antonyan, *West's problem on equivariant hyperspaces and Banach-Mazur compacta*, Trans. Amer. Math. Soc., **355**(2003), 3379–3404.

[2] K. K. Azad and K. Srivastava, *On S-equivariant complete invariance property*, Journal of the Indian Math. Soc., **62**(1996), 2005–2009.

[3] T. Banakh, R. Voytsitskyy, *Characterizing metric spaces whose hyperspaces are absolute neighborhood retracts*, Topology Appl., **154**(2007), 2009–2025.

[4] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, London, 1972.

[5] A. Chigogidze, K. H. Hofmann and J. R. Martin, *Compact groups and fixed point sets*, Trans. Amer. Math. Soc., **349**(1997), 4537–4554.

[6] D. W. Curtis, *Hyperspaces of noncompact metric spaces*, Compositio Mathematica, **40**(1980), 126–130.

[7] J. R. Martin, *Fixed point sets of homeomorphisms of metric products*, Proc. Amer. Math. Soc., **103**(1988), 1293–1298.

[8] J. R. Martin and S. B. Nadler Jr., *Examples and questions in the theory of fixed point sets*, Canad. J. Math., **31**(1997), 1017–1032.

[9] J. R. Martin, L. G. Oversteegen and E. D. Tymchatyn, *Fixed point sets of products and cones*, Pacific J. Math., **101**(1982), 133–139.

[10] J. R. Martin and W. A. R. Weiss, *Fixed point sets of metric and nonmetric spaces*, Trans. Amer. Math. Soc., **284**(1984), 337–353.

[11] S. B. Nadler Jr., *Hyperspaces of sets: A text with research questions*, Monographs and Textbooks, Pure Appl. Math. 49, Marcel Dekker, New York and Basel, 1978.

[12] H. Schirmer, *Fixed point sets of continuous selfmaps*, in: Fixed Point Theory, Conf. Proc., Sherbrooke, 1980, Lecture Notes in Math., **866**(1981), 417–428.

[13] H. Schirmer, *Fixed point sets of homeomorphisms of compact surfaces*, Israel J. Math., **10**(1971), 373–378.

[14] H. Schirmer, *On fixed point sets of homeomorphisms of the n-ball*, Israel J. Math., **7**(1969), 46–50.

[15] L. E. Ward Jr., *Fixed point sets*, Pacific J. Math., **47**(1973), 553–565.