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New Sufficient Conditions for Starlikeness of Certain Integral Operator

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Abstract. In the present paper, a new analytic function valued integral operator is introduced which is defined on n-copies of a subset of the class of *normalized* analytic functions on the unit disc of the complex plane. This operator, which is denoted here by $\mathcal{J}^{\alpha_i,\ \beta_i}(f_1, ..., f_n)$, unifies and generalizes several integral operators studied earlier. Interesting sufficient conditions are derived for the *univalent starlikeness* of $\mathcal{J}^{\alpha_i,\ \beta_i}(f_1, ..., f_n)$.

1. Introduction

Let $\mathcal{A}$ denote the family of *normalized* functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the *open* unit disc

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

A function $f(z)$ in $\mathcal{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one to one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ [8].

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A function $f(z)$ in $\mathcal{A}$ is said to be *starlike of order* δ $(0 \leq \delta < 1)$ if following is satisfied:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \quad (z \in \mathbb{U}). \tag{1.2}$$

We denote by $\mathcal{S}^*(\delta)$ the class of starlike functions of order δ. Clearly, $\mathcal{S}^*(\delta) \subset \mathcal{S}^*(0) := \mathcal{S}^*$ $(0 < \delta < 1)$ and $\mathcal{S}^* \subset \mathcal{S}$ [8].

Finding sufficient conditions for univalence of integral, derivative and other operators is an important topic of research in Geometric Function Theory. In recent years, several authors have investigated sufficient conditions for the univalence and starlikeness of various integral operators. For example, Breaz and Breaz [2] studied the following integral operator.

$$\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\mu}(f_1,\ldots.,f_n)(z) = \left[\mu\int_0^z t^{\mu-1}\prod_{j=1}^n [f_j'(t)]^{\alpha_j}dt\right]^{\frac{1}{\mu}} \tag{1.3}$$

where the functions $f_j \in \mathcal{A}$ and the parameters α_j, μ $(j = 1,\ldots.,n)$ are so constrained that the integral (1.3) exists. Deniz et al. [6] derived interesting sufficient conditions for univalence by choosing f_j in (1.3) a variety of special functions, namely, generalized Bessel's functions, modified Bessel functions and spherical Bessel's functions and so on. The particular case $\mu = 1$ in (1.3), i.e. the operator

$$\mathcal{H}_{\alpha_1,\ldots,\alpha_n,1}(f_1,\ldots,f_n)(z) := \mathcal{H}_{\alpha_i,n}(z) =: \int_0^z \prod_{j=1}^n [f_j'(t)]^{\alpha_j}dt, \tag{1.4}$$

has been used as an important auxiliary operator in [6]. In another direction Stanciu et al. [19] obtained sufficient conditions for univalence for the following integral operator:

$$\mathcal{H}_{\alpha,\beta}(f,g)(z) = \left[\beta\int_0^z t^{\beta-\alpha-1}(f(t))^\alpha g(t)dt\right]^{\frac{1}{\beta}} \quad (z \in \mathbb{U}) \tag{1.5}$$

where the parameters α, β and the function f and g are suitably chosen. In this paper [19], the special case $\beta = 1$ i.e. the operator

$$\mathcal{H}_\alpha(z) := \mathcal{H}_{\alpha,1}(f,g)(z) =: \int_0^z \left(\frac{f(t)}{t}\right)^\alpha g(t)dt \quad (z \in \mathbb{U}), \tag{1.6}$$

also plays an important role in the proofs of the theorems. The operator

$$\mathcal{F}_{\beta,n}(f_1,\ldots,f_n)(z) := \int_0^z \prod_{j=1}^n \left(\frac{f_j(t)}{t}\right)^\beta dt \quad (f_j \in \mathcal{A}, j = 1,\ldots,n, z \in \mathbb{U}, \beta \geq 1) \tag{1.7}$$

has been used by Breaz et al. [4] for the determination of univalence criteria for a related operator. Some more recent studies on this topic can also be found, for example, in [3],[5],[10],[12],[13],[14],[16],[18].

Now, let the functions $f_1, ..., f_n$ in $\mathcal{A}$ be such that

$$\frac{f_i'(z) f_i(z)}{z} \neq 0 \tag{1.8}$$

for every $i = 1, ..., n$ and $z \in \mathbb{U}$. On the subset of such functions in $\mathcal{A}^n$ we introduce the following integral operator:

$$\mathcal{J}^{\alpha_i,\ \beta_i}(f_1, ..., f_n)(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt, \tag{1.9}$$

$$(\alpha_i,\ \beta_i \in \mathbb{R}^+ \cup \{0\}, 1 \leq i \leq n;\ z \in \mathbb{U}).$$

We observe that the integral in (1.9) is independent of path. Hence, $\mathcal{J}^{\alpha_i,\ \beta_i}(f_1, ..., f_n) \in \mathcal{A}$. The integral operator $\mathcal{J}^{\alpha_i,\ \beta_i}(f_1, .., f_n)(z)$ unifies and generalizes several previously studied operators as follows:

- Taking $\beta_i = 0$ $(i = 1, ..., n)$ in (1.9), we get the integral operator where

  $$\mathcal{H}_{\alpha_i,n}(z) = \int_0^z \prod_{i=1}^{n} (f_i'(t))^{\alpha_i} dt \tag{1.10}$$

  defined at (1.4). Also see Breaz et al. [5].

- The choices $\alpha_i = 0$ $(i = 1, ..., n)$ give the integral operator

  $$\mathcal{F}_{\beta_i,n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\beta_i} dt \tag{1.11}$$

  introduced and studied by Breaz and Breaz [1]. Furthermore, taking $\beta_i = \beta$ $(i = 1, ..., n)$ we get the operator $\mathcal{F}_{\beta,n}(f_1, ..., f_n)$ defined by (1.7).

- For $n = 1,\ \alpha_1 = \alpha,\ \beta_1 = \beta$ and $f_1 = f$, our integral operator at (1.9) reduces to the operator

  $$\mathcal{F}_{\alpha,\ \beta}(z) = \int_0^z (f'(t))^{\alpha} \left(\frac{f(t)}{t} \right)^{\beta} dt \tag{1.12}$$

  which has been studied in [7]. Furthermore, the choice $\alpha = 0$ gives

  $$\mathcal{F}_{\beta}(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\beta} dt \tag{1.13}$$

  studied in [11].

- Similarly, for $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = 0$ and $f_1 = f$, we obtain the integral operator

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha \, dt \tag{1.14}$$

studied in [15].

In the present paper, we determine sufficient conditions for the integral operator defined in (1.9) to be starlike. Our results unify and generalize several previously studied sufficient conditions for starlikeness.

2. Preliminaries

In order to derive our results, we need the following lemmas.

Lemma 2.1.(see [17]) *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{\delta+1}{2(\delta-1)} \quad (z \in \mathbb{U}) \tag{2.1}$$

for some $2 \leq \delta < 3$, or

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{5\delta-1}{2(\delta+1)} \quad (z \in \mathbb{U}) \tag{2.2}$$

for some $1 < \delta \leq 2$, then $f \in \mathcal{S}^*$.

Lemma 2.2.(see [17]) *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{\delta+1}{2\delta(\delta-1)} \quad (z \in \mathbb{U}) \tag{2.3}$$

for some $\delta \leq -1$, or

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3\delta+1}{2\delta(\delta+1)} \quad (z \in \mathbb{U}) \tag{2.4}$$

for some $\delta > 1$, then $f \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$.

3. Main Results

We shall assume throughout that α_i, β_i $(i = 1, ..., n)$ are non negative real numbers and the functions $f_i \in \mathcal{A}$ satisfy the condition (1.8). Also for the sake of brevity of notation, we shall write $\mathcal{I}(z)$ instead of $\mathcal{J}^{\alpha_i,\beta_i}(f_1, f_2, ..., f_n)(z)$.

Theorem 3.1. *Suppose that for each $i = 1, ..., n$, the functions $f_i \in \mathcal{A}$ satisfy anyone of the following conditions:*

$$\Re\left(\alpha_i \frac{zf_i''(z)}{f_i'(z)} + \beta_i \frac{zf_i'(z)}{f_i(z)}\right) < \beta_i + \frac{3-\delta}{2(\delta-1)n} \quad (z \in \mathbb{U}) \tag{3.1}$$

for some $2 \leq \delta < 3$*, or*

$$\Re\left(\alpha_i \frac{zf_i''(z)}{f_i'(z)} + \beta_i \frac{zf_i'(z)}{f_i(z)}\right) < \beta_i + \frac{3(\delta-1)}{2(\delta+1)n} \quad (z \in \mathbb{U}) \tag{3.2}$$

for some $1 < \delta \leq 2$*. Then* $\mathcal{I}(z) \in \mathcal{S}^*$*.*

Proof. The defining relation (1.9) can be equivalently written as

$$\mathcal{I}'(z) = \prod_{i=1}^{n} \left[\left(\frac{f_i(z)}{z}\right)^{\beta_i} (f_i'(z))^{\alpha_i}\right], \quad \mathcal{I}(0) = 0. \tag{3.3}$$

Therefore,

$$\Re\left(1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) = 1 - \sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \Re\left[\alpha_i \frac{zf_i''(z)}{f_i'(z)} + \beta_i \frac{zf_i'(z)}{f_i(z)}\right]. \tag{3.4}$$

Now, by using the given condition (3.1) in (3.4) we have

$$\Re\left(1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) < 1 + \frac{3-\delta}{2(\delta-1)} = \frac{\delta+1}{2(\delta-1)} \quad (z \in \mathbb{U}, 2 \leq \delta < 3).$$

Hence by an application of (2.1) of Lemma 2.1, we get $\mathcal{I}(z) \in \mathcal{S}^*$. Similarly, by using the sufficient condition (3.2) in (3.4) and applying (2.2) of Lemma 2.1 we get $\mathcal{I}(z) \in \mathcal{S}^*$. Thus, the proof of Theorem 3.1 is completed. □

Taking $\alpha_i = 0,\ i = 1, ..., n$ in Theorem 3.1, we obtain the following result due to Frasin et al. [9].

Corollary 3.2.(see [9], Theorem 2.1) *Let* $\beta_i > 0$ $(i = 1, ..., n)$*. If* $f_i \in \mathcal{A}$ $(i = 1, ..., n)$ *satisfies*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{3-\delta}{2(\delta-1)n\beta_i} \quad (z \in \mathbb{U})$$

for some $2 \leq \delta < 3$*, or*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{3(\delta-1)}{2(\delta+1)n\beta_i} \quad (z \in \mathbb{U})$$

for some $1 < \delta \leq 2$*, then* $\mathcal{F}_{\beta_i,n}(z) \in \mathcal{S}^*$ *where* $\mathcal{F}_{\beta_i,n}$ *is defined by (1.11).*

Remark 3.3. The particular case $\beta_i = \beta$ for $i = 1, ..., n$ in our Corollary 3.2 yields new sufficient conditions of starlikenss for the integral operator $\mathcal{F}_{\beta,n}(f_1, ..., f_n)(z)$ defined at (1.7). Further, letting $n = 1,\ \beta_1 = \beta$ and $f_1 = f$ in Corollary 3.2 we get a result of Frasin et al. ([9], Corollary 2.2). Similarly, taking $g(z) \equiv 1$ $(z \in \mathbb{U})$ in (1.6) we get new results from our Corollary 3.2 for the operator $\mathcal{F}_{\alpha,1}(f, 1)(z)$ defined at (1.6).

Further, putting $\beta_i = 0$ for all $i = 1, ..., n$ in Theorem 3.1 we get the following result of [9]:

Corollary 3.4.(see [9], Theorem 3.1) *Let* $\alpha_i > 0$ $(i = 1, ..., n)$. *If* $f_i \in \mathcal{A}$ $(i = 1, ..., n)$ *satisfies*

$$\Re\left(\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{3-\delta}{2(\delta-1)n\alpha_i} \quad (z \in \mathbb{U})$$

for some δ $(2 \leq \delta < 3)$, *or*

$$\Re\left(\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{3(\delta-1)}{2(\delta+1)n\alpha_i} \quad (z \in \mathbb{U})$$

for some δ $(1 < \delta \leq 2)$, *then* $H_{\alpha_i,n}(z) \in \mathcal{S}^*$, *where* $H_{\alpha_i,n}$ *is defined by* (1.4).

Remark 3.5. Letting $n = 1, \alpha_1 = \alpha$ and $f_1 = f$ in Corollary 3.4, we obtain one more result of Frasin et al. (see [9], Corollary 3.2).

Theorem 3.6. *Suppose that for each* i $(i = 1, ..., n)$ *the functions* $f_i \in \mathcal{A}$ *satisfy any one of the following conditions:*

$$\Re\left(\alpha_i\frac{zf_i''(z)}{f_i'(z)} + \beta_i\frac{zf_i'(z)}{f_i(z)}\right) > \beta_i + \frac{\delta - 2\delta^2 - 1}{2\delta(\delta-1)n} \quad (z \in \mathbb{U}), \tag{3.5}$$

for some $\delta \leq -1$, *or*

$$\Re\left(\alpha_i\frac{zf_i''(z)}{f_i'(z)} + \beta_i\frac{zf_i'(z)}{f_i(z)}\right) > \beta_i + \frac{\delta - 2\delta^2 + 1}{2\delta(\delta+1)n} \quad (z \in \mathbb{U}), \tag{3.6}$$

for some $\delta > 1$. *Then* $\mathcal{I}(z) \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$.

Proof. The proof of the theorem follows the same lines as in Theorem 3.1. In fact, using (3.5) in (3.4) we obtain

$$\Re\left(1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) > \frac{-(\delta+1)}{2\delta(\delta-1)} \quad (z \in \mathbb{U}, \delta < -1).$$

Also, from (3.4) and (3.6) we get

$$\Re\left(1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) > \frac{3\delta+1)}{2\delta(\delta+1)} \quad (z \in \mathcal{U}, \delta > -1).$$

Therefore, by application of Lemma 2.2 the result follows. The proof of Theorem 3.6 is completed. □

Taking $\alpha_i = 0$ $(i = 1, ..., n)$ in Theorem 3.6, we get the following result obtained in [9].

Corollary 3.7.(see [9], Theorem 2.3) *Let* $\beta_i > 0$ $(i = 1, ..., n)$. *If* $f_i \in \mathcal{A}$ $(i = 1, ..., n)$ *satisfies*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > 1 + \frac{\delta - 2\delta^2 - 1}{2\delta(\delta-1)n\beta_i} \quad (z \in \mathbb{U})$$

for some $\delta \leq -1$, *or*

$$\Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > 1 + \frac{\delta - 2\delta^2 - 1}{2\delta(\delta+1)n\beta_i} \quad (z \in \mathbb{U})$$

for some $\delta > 1$, *then* $\mathcal{F}_{\beta_i,n}(z) \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$, *where* $\mathcal{F}_{\beta_i,n}(z)$ *is defined at* (1.11).

Remark 3.8. Letting $n = 1$, $\beta_1 = \beta$ and $f_1 = f$ in Corollary 3.7, we obtain the result due to Frasin et al. (see [9], Corollary 2.4).

Taking $\beta_i = 0$ $(i = 1, ..., n)$ in Theorem 3.6, we get the following result.

Corollary 3.9.(see [9], Theorem 3.3) *Let* $\alpha_i > 0$ $(i = 1, ..., n)$. *If* $f_i \in \mathcal{A}$ $(i = 1, ..., n)$ *satisfies*

$$\Re\left(\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{\delta - 2\delta^2 - 1}{2\delta(\delta-1)n\alpha_i}$$

for some $\delta \leq -1$, *or*

$$\Re\left(\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{\delta - 2\delta^2 + 1}{2\delta(\delta+1)n\alpha_i}$$

for some $\delta > 1$, *then* $\mathcal{H}_{\alpha_i,n}(z) \in \mathcal{S}^*(\frac{\delta+1}{2\delta})$, *where* $\mathcal{H}_{\alpha_i,n}(z)$ *is defined at* (1.4).

Remark 3.10. Taking $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Corollary 3.9, we obtain the result of Frasin et al. (see [9], Corollary 3.4).

Theorem 3.11. *Let* $\sum_{i=1}^{n} \alpha_i = 1$. *Suppose that for each* $i = 1, ..., n$ *the functions* $f_i \in \mathcal{A}$ *satisfy anyone of the following conditions:*

$$\frac{zf_i''(z)}{f_i'(z)} < \frac{5 - 3\delta}{4(\delta-1)} \quad \text{and} \quad \beta_i \frac{zf_i'(z)}{f_i(z)} < \beta_i + \frac{\delta+1}{4(\delta-1)n} \quad (z \in \mathbb{U}) \tag{3.7}$$

for some $2 \leq \delta < 3$, *or*

$$\frac{zf_i''(z)}{f_i'(z)} < \frac{\delta - 5}{4(\delta+1)} \quad \text{and} \quad \beta_i \frac{zf_i'(z)}{f_i(z)} < \beta_i + \frac{5\delta - 1}{4(\delta+1)n} \quad (z \in \mathbb{U}) \tag{3.8}$$

for some $1 < \delta \leq 2$. *Then* $\mathcal{I}(z) \in \mathcal{S}^*$.

Proof. We write

$$1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)} = \sum_{i=1}^{n} \alpha_i \left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n} \beta_i \left(\frac{zf_i'(z)}{f_i(z)}\right) - \sum_{i=1}^{n} \beta_i. \tag{3.9}$$

Therefore, using the condition (3.7) in (3.9) we have

$$\Re\left(1 + \frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) < \sum_{i=1}^{n} \alpha_i \left(\frac{\delta+1}{4(\delta-1)}\right) + \frac{\delta+1}{4(\delta-1)} = \frac{\delta+1}{2(\delta-1)} \quad (z \in \mathbb{U}, 2 \leq \delta < 3).$$

Similarly, by using (3.8) we get

$$\Re\left(1+\frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)}\right) < \sum_{i=1}^{n}\alpha_i\left(\frac{5\delta-1}{4(\delta+1)}\right)+\frac{5\delta-1}{4(\delta+1)} = \frac{5\delta-1}{2(\delta+1)} \quad (z\in\mathbb{U}, 1<\delta\le 2).$$

Thus, by application of Lemma 2.1, $\mathcal{I}(z)\in\mathcal{S}^*$. The proof of Theorem 3.11 is completed.

Theorem 3.12. *Let* $\sum_{i=1}^{n}\beta_i = 1$. *Suppose that for each* $i = 1, ..., n$ *the functions* $f_i \in \mathcal{A}$ *satisfy any one of the following conditions:*

$$\alpha_i\frac{zf_i''(z)}{f_i'(z)} < \frac{\delta+1}{4(\delta-1)n} \quad \textit{and} \quad \frac{zf_i'(z)}{f_i(z)} < \frac{\delta+1}{4(\delta-1)} \tag{3.10}$$

for some $2\le\delta<3$, *or*

$$\alpha_i\frac{zf_i''(z)}{f_i'(z)} < \frac{5\delta-1}{4(\delta+1)n} \quad \textit{and} \quad \frac{zf_i'(z)}{f_i(z)} < \frac{5\delta-1}{4(\delta+1)}, \tag{3.11}$$

for some $1<\delta\le 2$. *Then* $\mathcal{I}(z)\in\mathcal{S}^*$.

Proof. The proof of this theorem is similar to that of Theorem 3.11. In this case, we have

$$1+\frac{z\mathcal{I}''(z)}{\mathcal{I}'(z)} = \sum_{i=1}^{n}\left[\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\beta_i\frac{zf_i'(z)}{f_i(z)}\right] \quad (z\in\mathbb{U}).$$

Therefore, by using the condition (3.10) or (3.11) and Lemma 2.1 we get that $\mathcal{I}(z)\in\mathcal{S}^*$. The proof of Theorem 3.12 is thus, completed.

The next two theorems can be proved in the manner of Theorem 3.11 and Theorem 3.12 and application of Lemma 2.2. Hence we only state these without proofs.

Theorem 3.13. *Let* $\sum_{i=1}^{n}\alpha_i = 1$. *Suppose that for each* $i = 1, ..., n$ *the functions* $f_i \in \mathcal{A}$ *satisfy anyone of the following conditions:*

$$\frac{zf_i''(z)}{f_i'(z)} < \frac{3\delta-4\delta^2-1}{4\delta(\delta-1)} \quad \textit{and} \quad \beta_i\frac{zf_i'(z)}{f_i(z)} < \beta_i - \frac{\delta+1}{4\delta(\delta-1)n} \quad (z\in\mathbb{U}) \tag{3.12}$$

for some $\delta<-1$, *or*

$$\frac{zf_i''(z)}{f_i'(z)} < \frac{1-\delta-4\delta^2}{4\delta(\delta+1)} \quad \textit{and} \quad \beta_i\frac{zf_i'(z)}{f_i(z)} < \beta_i + \frac{3\delta+1}{4\delta(\delta+1)n} \tag{3.13}$$

for some $\delta>1$. *Then* $\mathcal{I}(z)\in\mathcal{S}^*(\frac{\delta+1}{2\delta})$.

Theorem 3.14. *Let* $\sum_{i=1}^{n}\beta_i = 1$. *Suppose that for each* $i = 1, ..., n$ *the functions* $f_i \in \mathcal{A}$ *satisfy anyone of the following conditions:*

$$\alpha_i\frac{zf_i''(z)}{f_i'(z)} > \frac{-(\delta+1)}{4\delta(\delta+1)n} \quad \textit{and} \quad \frac{zf_i'(z)}{f_i(z)} < \frac{-(\delta+1)}{4\delta(\delta+1)} \quad (z\in\mathbb{U}) \tag{3.14}$$

for some $\delta < -1$, *or*

$$\alpha_i \frac{zf_i''(z)}{f_i'(z)} > \frac{3\delta+1)}{4\delta(\delta+1)n} \quad \textit{and} \quad \frac{zf_i'(z)}{f_i(z)} > \frac{3\delta+1}{4\delta(\delta+1)} \quad (z \in \mathbb{U}) \tag{3.15}$$

for some $\delta > 1$. *Then* $\mathcal{I}(z) \in \mathcal{S}^*(\frac{\delta+1}{2\delta})$.

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