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Hyers-Ulam Stability of Pompeiu's Point

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Abstract. In this paper, we investigate the stability of Pompeiu's points in the sense of Hyers-Ulam.

1. Introduction

In 1946, Pompeiu [8] derived a variant of Lagrange's mean value theorem, now known as Pompeiu's mean value theorem.

Definition 1.1. For every real valued function f differentiable on an interval [a,b] not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ in (x_1, x_2) such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

Such an intermediate point ξ will be called Pompeiu's point of the function f. The geometric meaning of this is that the tangent at the point $(\xi, f(\xi))$ intersects on the y-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In 1954, Hyers and Ulam [4] considered the stability of differential expressions and proved the following theorem, by which many mathematicians have obtained some interesting theorems.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighborhood N of the point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at η. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for each function $h : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|h(x) - f(x)| < \delta$ in N, there exists a point ξ in N such that $h^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

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Let $[a,b] \subset \mathbb{R}$ be a closed interval and

$$\phi = \{f : [a,b] \to \mathbb{R} \mid \text{f is continuously differentiable, } f'(a) = f'(b)\}.$$

In 2003, M. Das, T. Riedel and P. K. Sahoo [1] gave a Hyers-Ulam type stability result for Flett's points.

Theorem 1.3. *Let $f \in \phi$ and η be a Flett's point of f in (a,b). Assume that there is a neighborhood N of η in (a,b) such that η is the unique Flett's point of f in N. Then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $h \in \phi$ satisfying $h(a) = f(a)$ and $|h(x) - f(x)| < \delta$ for all x in N, there exists a point $\xi \in N$ such that ξ is a Flett's point of h and $|\xi - \eta| < \delta$.*

Unfortunately, there are some errors in the proof of M. Das et al.. In 2009, W. Lee, S. Xu and F. Ye [6] constructed a counter example to show that theorem is incorrect, then they proved the Hyers-Ulam stability of the Sahoo-Riedel's point, and as a corollary they got the correct theorem of the stability of Flett's point.

Theorem 1.4. *Let $f, h : [a,b] \to \mathbb{R}$ be differentiable and η be a Sahoo-Riedel's point of f in (a,b). If f has 2nd derivative at η and*

$$f''(\eta)(\eta - a) - 2f'(\eta) + \frac{2(f(\eta) - f(a))}{\eta - a} \neq 0,$$

then corresponding to any $\varepsilon > 0$ and any neighborhood $N \subset (a,b)$ of η, there exists a $\delta > 0$ such that for every h satisfying $|h(x) - h(a) - (f(x) - f(a))| < \delta$ for x in N and $h'(b) - h'(a) = f'(b) - f'(a)$, there exists a point $\xi \in N$ such that ξ is a Sahoo-Riedel's point of h and $|\xi - \eta| < \varepsilon$.

In 2010, P. Găvruţă, S.-M. Jung and Y. Li [3] investigated the stability of the Lagrange's mean value points.

Theorem 1.5. *Let a, b, η be real numbers satisfying $a < \eta < b$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function and η is the unique Lagrange's mean value point of f in an open interval (a,b) and moreover that $f''(\eta) \neq 0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then, for a given $\varepsilon > 0$, there exists a $\eta > 0$ such that if $|f(x) - g(x)| < \eta$ for all $x \in [a,b]$, then there is a Lagrange's mean value point $\xi \in (a,b)$ of g with $|\xi - \eta| < \varepsilon$.*

In this paper, we prove the Hyers-Ulam stability of Pompeiu's point by employing the ideas of theorem 1.3, 1.4 and 1.5.

2. Hyers-Ulam Stability of Pompeiu's Point

In this section, we investigate the stability of the Pompeiu's point.

Theorem 2.1. *Let $f, h : [a,b] \to \mathbb{R}$ be differentiable and η be a Pompeiu's point of f. If f has 2nd derivative at η with*

$$f''(\eta) \neq 0,$$

then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that for every h satisfying $|h(t) - f(t)| < \delta$ for all $t \in [a,b]$, there exists a point $\xi \in (a,b)$ such that ξ is a Pompeiu's point of h with $|\xi - \eta| < \varepsilon$.

Proof. Without loss of generality, we shall assume that $a, b > 0$. Define a real valued function F on the interval $[\frac{1}{b}, \frac{1}{a}]$ by

$$F(t) = tf\Big(\frac{1}{t}\Big).$$

Since f is differentiable on $[a,b]$ and 0 is not in $[a,b]$, we see that F is differentiable on $(\frac{1}{b}, \frac{1}{a})$ and

$$F'(t) = f\Big(\frac{1}{t}\Big) - \frac{1}{t}f'\Big(\frac{1}{t}\Big).$$

Consider the auxiliary function $G_F(t) : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ corresponding to F defined by

$$G_F(t) = F(t) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}).$$

Since η is a Pompeiu's point, we have

$$G'_F(\frac{1}{\eta}) = F'(\frac{1}{\eta}) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}} = f(\eta) - \eta f'(\eta) - \frac{af(b) - bf(a)}{a - b} = 0.$$

Moreover, by the assumption that $f''(\eta) \neq 0$, we obtain that

$$G''_F(\frac{1}{\eta}) = F''(\frac{1}{\eta}) = \eta^3 f''(\eta) \neq 0,$$

which implies $G'_F(t)$ changes sign at $\frac{1}{\eta}$.

According to theorem 1.2, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for each function $\phi : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ which is differentiable in $(\frac{1}{b}, \frac{1}{a})$ and satisfies $|\phi(t) - G_F(t)| < \frac{3\delta}{a}$ in $[\frac{1}{b}, \frac{1}{a}]$, there exists a point ξ_0 in $(\frac{1}{b}, \frac{1}{a})$ such that $\phi'(\xi_0) = 0$ and $|\xi_0 - \frac{1}{\eta}| < \frac{1}{b^2}\varepsilon$.

Now, let us define differentiable functions H and G_H by

$$H(t) = th\Big(\frac{1}{t}\Big)$$

and

$$G_H(t) = H(t) - \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}).$$

Recall $F(t) = tf(\frac{1}{t})$, we have

$$|H(t) - F(t)| = \Big|th\Big(\frac{1}{t}\Big) - tf\Big(\frac{1}{t}\Big)\Big| \leq \frac{1}{a}\Big|h\Big(\frac{1}{t}\Big) - f\Big(\frac{1}{t}\Big)\Big|$$

for all $t \in [\frac{1}{b}, \frac{1}{a}]$. On the other hand,

$$\begin{aligned} |G_H(t) - G_F(t)| &\le \Big|H(t) - \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}) - \Big(F(t) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a})\Big)\Big| \\ &\le |H(t) - F(t)| + \Big|(t - \frac{1}{a})\Big(\frac{H(\frac{1}{b}) - F(\frac{1}{b})}{\frac{1}{b} - \frac{1}{a}}\Big)\Big| + \Big|(t - \frac{1}{a})\Big(\frac{H(\frac{1}{a}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}\Big)\Big| \\ &\le |H(t) - F(t)| + \Big|(\frac{1}{b} - \frac{1}{a})\Big(\frac{H(\frac{1}{b}) - F(\frac{1}{b})}{\frac{1}{b} - \frac{1}{a}}\Big)\Big| + \Big|(\frac{1}{b} - \frac{1}{a})\Big(\frac{H(\frac{1}{a}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}\Big)\Big| \\ &\le |H(t) - F(t)| + |H(\frac{1}{b}) - F(\frac{1}{b})| + |H(\frac{1}{a}) - F(\frac{1}{a})| \end{aligned}$$

for all $t \in [\frac{1}{b}, \frac{1}{a}]$. Let $|h(t) - f(t)| < \delta$ for all $t \in [a, b]$, we have $|G_H(t) - G_F(t)| \le \frac{3\delta}{a}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$, which implies that there exists a point ξ_0 in $(\frac{1}{b}, \frac{1}{a})$ such that $G'_H(\xi_0) = 0$ and $|\xi_0 - \frac{1}{\eta}| \le \frac{1}{b^2}\varepsilon$.

Define $\xi = \frac{1}{\xi_0}$. Recall $G'_H(\xi_0) = 0$, which implies $H'(\xi_0) = \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}$, we obtain that

$$h(\xi) - \xi h'(\xi) = \frac{\frac{1}{b}h(b) - \frac{1}{a}h(a)}{\frac{1}{b} - \frac{1}{a}} = \frac{ah(b) - bh(a)}{a - b},$$

from which it follows that ξ is a Pompeiu's point of h. Moreover,

$$|\xi - \eta| = \Big|\frac{1}{\xi_0} - \eta\Big| = \Big|\frac{\xi_0 - \frac{1}{\eta}}{\xi_0 \cdot \frac{1}{\eta}}\Big| \le b^2\Big|\xi_0 - \frac{1}{\eta}\Big| \le \varepsilon.$$

The proof is completed. □

Corollary 2.2. *Let $f, h : [a, b] \to \mathbb{R}$ be differentiable and η be a Pompeiu's point of f. If f has 2nd derivative at η and*

$$f''(\eta) \neq 0,$$

then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that for every h satisfying $|h(t) - f(t) - c| < \delta$ for t in $[a, b]$, where c is a constant, there exists a point $\xi \in (a, b)$ such that ξ is a Pompeiu's point of h and $|\xi - \eta| < \varepsilon$.

The following counter example shows that Theorem 2.1 will be incorrect if we remove the condition $f''(\eta) \neq 0$.

Example 2.3. Let $[a, b] = [1, 2]$,

$$f(x) = 0.$$

Then, we can see that $f(x)$ is twice differentiable on $(1, 2)$ and $f''(x) = 0$ for all $x \in (1, 2)$. What's more, every $x \in (1, 2)$ is a Pompeiu's point of $f(x)$. Let $\eta = \frac{7}{4}$, which is a Pompeiu's point of f.

For sufficiently small $\delta > 0$, define

$$h(x) = \delta[4(x - \frac{3}{2})^2 - 1]$$

for $x \in [1, 2]$. Then, we can know from the geometric meaning of Pompeiu's point that the Pompeiu's point ξ of $h(x)$ is in $(1, \frac{3}{2})$, or rather, $\xi = \sqrt{2}$ is the unique Pompeiu's point of h in $[1, 2]$.

Finally, we have

$$|\xi - \eta| = |\sqrt{2} - \frac{7}{4}| > \frac{1}{4}.$$

In other words, for all $\delta > 0$, there exists a twice differentiable function h satisfying $|f - h| < \delta$, but there is no Pompeiu's point of h in the neighborhood of η which is a Pompeiu's point of f.

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