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On the Stability of a Mixed Type Functional Equation

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Abstract. In this paper, we investigate the stability of the functional equation

$$\begin{aligned} &f(-x+y+z+w)+f(x-y+z+w)+f(x+y-z+w)+f(x+y+z-w) \\ &= 3f(x)+f(-x)+3f(y)+f(-y)+3f(z)+f(-z)+3f(w)+f(-w) \end{aligned}$$

by using the direct method in the sense of Hyers.

1. Introduction

In 1940, Ulam [19] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The Ulam's problem for the case of approximately additive functions was solved by Hyers [9] under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x + y) = f(x) + f(y)$, is said to satisfy the Hyers-Ulam stability.

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Rassias [18] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon\big(\|x\|^p+\|y\|^p\big)$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus in [18], a proof of the generalized Hyers-Ulam stability for the linear mapping between Banach spaces was obtained. In 1950, a special case of Rassias' theorem for the stability of the additive mapping was obtained by Aoki [1] (see also [8], [16], [18]).

The stability concept that was introduced by Rassias' theorem provided large influence to a number of mathematicians to develop the notion of what is known today with the term generalized Hyers-Ulam stability of the linear mappings. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see, for example, [7], [10], [11], [12], [14], [15] and the references therein).

Almost all subsequent proofs, in this very active area, have used the Hyers' idea from [9]. Namely, starting from the given mapping f, the solution F of a functional equation is explicitly constructed by

$$F(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^n x) \quad \text{or} \quad F(x)=\lim_{n\to\infty}2^n f\Big(\frac{x}{2^n}\Big).$$

This method of Hyers is called the direct method. There are also other methods for proving the Hyers-Ulam stability of functional equations and differential equations. One of these methods is called the fixed point method that was applied for the first time by Baker (see [2] and also [3], [4], [5], [6], [17]).

Now, we consider the following functional equation

$$\begin{aligned}
&f(-x+y+z+w)+f(x-y+z+w)+f(x+y-z+w)\\
&\quad+f(x+y+z-w)\\
&\quad=3f(x)+f(-x)+3f(y)+f(-y)+3f(z)+f(-z)+3f(w)+f(-w),
\end{aligned} \tag{1.1}$$

which is called the mixed type functional equation. The mapping $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=ax^2+bx$ is a solution of this functional equation, where a, b are real constants. Every solution of (1.1) is called a quadratic-additive mapping.

In 1998, Jung [13] proved the stability of Eq. (1.1) by decomposing f into the odd and even parts. In his proof, it was necessary to let an additive mapping A and a quadratic mapping Q approximate the odd and even parts of f, respectively, and combine A and Q to prove the existence of a quadratic-additive mapping F which is close to the mapping f.

In this paper, we will prove the stability of the quadratic-additive functional equation (1.1) by making use of the direct method. Indeed, we will approximate the given mapping f by a solution F of Eq. (1.1) without decomposing f into its odd and even parts, while in the previous papers [13] the mapping f was decomposed into the odd and even parts and they were separately approximated by the corresponding parts of a solution F of Eq. (1.1), respectively.

2. Main Results

Throughout this paper, let X be a (real or complex) linear space and Y a Banach space. For an arbitrarily fixed $p \in \mathbb{R}$, put $s = \operatorname{sign}(2-p)$ and $t = \operatorname{sign}(1-p)$.

For a given mapping $f : X \to Y$, we use the following abbreviations

$$\begin{aligned}
f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\
f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\
J_n f(x) &:= \frac{16^{-sn}}{2}\left(f(4^{sn}x) + f(-4^{sn}x)\right) + \frac{4^{-tn}}{2}\left(f(4^{tn}x) - f(-4^{tn}x)\right), \\
J'_n f(x) &:= \frac{4^{-sn}}{2}\left(f(2^{sn}x) + f(-2^{sn}x)\right) + \frac{2^{-tn}}{2}\left(f(2^{tn}x) - f(-2^{tn}x)\right), \\
Af(x,y) &:= f(x+y) - f(x) - f(y), \\
Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y), \\
Df(x,y,z,w) &:= f(-x+y+z+w) + f(x-y+z+w) + f(x+y-z+w) \\
&\quad + f(x+y+z-w) - 3f(x) - f(-x) - 3f(y) - f(-y) \\
&\quad - 3f(z) - f(-z) - 3f(w) - f(-w)
\end{aligned}$$

for all $x, y, z, w \in X$. Then we have

$$\begin{aligned}
Df_e(x,y,z,w) &= \frac{1}{2}Df(x,y,z,w) + \frac{1}{2}Df(-x,-y,-z,-w), \\
Df_o(x,y,z,w) &= \frac{1}{2}Df(x,y,z,w) - \frac{1}{2}Df(-x,-y,-z,-w)
\end{aligned}$$

for all $x, y, z, w \in X$.

If f is a solution of the functional equation $Df(x,y,z,w) = 0$ for all $x, y, z, w \in X$, then f is called a quadratic-additive mapping.

Theorem 2.1. *A mapping $f : X \to Y$ is a solution of* (1.1) *if and only if f_e is a quadratic mapping and f_o is an additive mapping.*

Proof. Let $f : X \to Y$ satisfy $Df(x,y,z,w) = 0$ for all $x, y, z, w \in X$. Then we get

$$\begin{aligned}
Qf_e(x,y) &= \frac{3Df_e(x,y,0,0) - Df_e(x,y,0,0)}{4} = 0, \\
Af_o(x,y) &= \frac{Df_o(x,y,0,0)}{2} = 0
\end{aligned}$$

for all $x, y, z, w \in X$, i.e., f_e is a quadratic mapping and f_o is an additive mapping.

Conversely, assume that f_e is a quadratic mapping and f_o is an additive mapping. Then we get

$$\begin{aligned} Df(x,y,z,w) &= Df_e(x,y,z,w) + Df_o(x,y,z,w) \\ &= Qf_e(x+y,z-w) + Qf_e(x-y,z+w) + 2Qf_e(x,y) \\ &\quad + 2Qf_e(z,w) + Af_o(x+y,z-w) \\ &\quad + Af_o(z+w,x-y) + 2Af_o(x,y) + 2Af_o(z,w) \\ &= 0 \end{aligned}$$

for all $x, y, z, w \in X$, i.e., f is a solution of (1.1). □

In the following lemma, we will prove that a mapping f is also quadratic-additive provided $Df(x,y,z,w) = 0$ holds for x, y, z, w except zero.

Lemma 2.2. *If a mapping $f : X \to Y$ satisfies $Df(x,y,z,w) = 0$ for all $x, y, z, w \in X\backslash\{0\}$, then f is a quadratic-additive mapping.*

Proof. It is enough to show that $Df(x,y,z,w) = 0$ for all $x, y, z, w \in X$. For an arbitrarily fixed $x \in X\backslash\{0\}$, we get

$$\begin{aligned} f(0) &= \frac{1}{3}\big(Df(x,x,x,-x) - \frac{1}{4}Df(2x,2x,2x,2x) \\ &\quad - \frac{3}{4}Df(x,x,x,x) - \frac{1}{4}Df(-x,-x,-x,-x)\big) \\ &= 0. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &Df(x,y,z,0) \\ &= Df\left(\frac{x}{2}, y, z, -\frac{x}{2}\right) + Df\left(\frac{x}{2}, y, -x, \frac{x}{2}\right) - Df\left(\frac{y}{2}, \frac{y}{2}, \frac{z}{2}, -\frac{z}{2}\right) \\ &\quad - Df\left(-\frac{y}{2}, \frac{y}{2}, \frac{z}{2}, \frac{z}{2}\right) - \frac{3}{4}Df\left(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}\right) - \frac{1}{4}Df\left(-\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}, -\frac{x}{2}\right) \\ &\quad + \frac{3}{4}Df\left(\frac{y}{2}, \frac{y}{2}, \frac{y}{2}, \frac{y}{2}\right) + \frac{1}{4}Df\left(-\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}, -\frac{y}{2}\right) \\ &\quad + \frac{3}{4}Df\left(\frac{z}{2}, \frac{z}{2}, \frac{z}{2}, \frac{z}{2}\right) + \frac{1}{4}Df\left(-\frac{z}{2}, -\frac{z}{2}, -\frac{z}{2}, -\frac{z}{2}\right) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X\backslash\{0\}$.

Furthermore, it is easy to prove that

$$\begin{aligned} Df(x,y,0,0) = & Df\Big(\frac{y}{2},\frac{y}{2},\frac{x}{2},-\frac{x}{2}\Big) + Df\Big(-\frac{y}{2},\frac{y}{2},\frac{x}{2},\frac{x}{2}\Big) \\ & -\frac{3}{4}Df\Big(\frac{x}{2},\frac{x}{2},\frac{x}{2},\frac{x}{2}\Big) - \frac{1}{4}Df\Big(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2},-\frac{x}{2}\Big) \\ & -\frac{3}{4}Df\Big(\frac{y}{2},\frac{y}{2},\frac{y}{2},\frac{y}{2}\Big) - \frac{1}{4}Df\Big(-\frac{y}{2},-\frac{y}{2},-\frac{y}{2},-\frac{y}{2}\Big) \\ = & \ 0 \end{aligned}$$

for all $x, y \in X\backslash\{0\}$. By considering the symmetry in the variables x, y, z, w, it completes the proof. □

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1.1) by making use of the direct method.

Theorem 2.3. *Let θ be a nonnegative constant. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z,w)\| \le \theta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\big) \tag{2.1}$$

for all $x, y, z, w \in X\backslash\{0\}$ with a real constant $p \notin \{1,2\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{4\theta\|x\|^p}{4^p-16} & (\text{for } \ p>2), \\ \Big(\frac{4\theta}{16-4^p} + \frac{4\theta}{4^p-4}\Big)\|x\|^p & (\text{for } \ 1<p<2), \\ \frac{4\theta\|x\|^p}{4-4^p} & (\text{for } \ 0<p<1), \\ \frac{29}{15}\theta & (\text{for } \ p=0) \end{cases} \tag{2.2}$$

for all $x \in X\backslash\{0\}$ and f is itself a quadratic-additive mapping provided $p < 0$.

Proof. By a tedious calculation, we have

$$\begin{aligned} \|f(0)\| = & \frac{1}{3}\Big\|Df\big(2^n x, 2^n x, 2^n x, -2^n x\big) - \frac{1}{4}Df\big(2^{n+1}x, 2^{n+1}x, 2^{n+1}x, 2^{n+1}x\big) \\ & -\frac{3}{4}Df\big(2^n x, 2^n x, 2^n x, 2^n x\big) - \frac{1}{4}Df\big(-2^n x, -2^n x, -2^n x, -2^n x\big)\Big\| \\ \le & \frac{(8+2^p)2^{np}\theta\|x\|^p}{3} \end{aligned}$$

holds for any fixed $x \in X\backslash\{0\}$ and all integers n. Hence, we get $f(0) = 0$ for $p \notin \{0,1,2\}$ and $\|f(0)\| \le 3\theta$ for $p = 0$.

From the definitions of $J_n f(x)$ and $Df(x,y,z,w)$, applying a long calculation, we get

$$
\begin{aligned}
J_n f(x) - J_{n+1} f(x) = &-\frac{1}{2}\Big(16^{\tau_{-s,n}}\big(Df(-4^{\tau_{s,n}}x, 4^{\tau_{s,n}}x, 4^{\tau_{s,n}}x, 4^{\tau_{s,n}}x) \\
&\qquad + Df(4^{\tau_{s,n}}x, -4^{\tau_{s,n}}x, -4^{\tau_{s,n}}x, -4^{\tau_{s,n}}x)\big)s \\
&\qquad + 4^{\tau_{-t,n}}\big(Df(-4^{\tau_{t,n}}x, 4^{\tau_{t,n}}x, 4^{\tau_{t,n}}x, 4^{\tau_{t,n}}x) \\
&\qquad - Df(4^{\tau_{t,n}}x, -4^{\tau_{t,n}}x, -4^{\tau_{t,n}}x, -4^{\tau_{t,n}}x)\big)t\Big) \\
&+ 3 \cdot 16^{\tau_{-s,n}} f(0)
\end{aligned}
\tag{2.3}
$$

for all $x \in X\backslash\{0\}$ and all nonnegative integers n, where $s = \operatorname{sign}(2-p)$, $t = \operatorname{sign}(1-p)$, and $\tau_{k,n}$ are the integers defined by $\tau_{k,n} = k(n+1/2) - 1/2$ for $k \in \{-1, 1\}$.

It follows from (2.1) and (2.3) that

$$
\begin{aligned}
&\|J_n f(x) - J_{n+m} f(x)\| \\
&= \sum_{j=n}^{n+m-1} \|J_j f(x) - J_{j+1} f(x)\| \\
&\le \frac{1}{2} \sum_{j=n}^{n+m-1} \Big(\big\|16^{\tau_{-s,j}} Df(-4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x)s \\
&\qquad + 4^{\tau_{-t,j}} Df(-4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x)t\big\| \\
&\qquad + \big\|16^{\tau_{-s,j}} Df(4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x)s \\
&\qquad - 4^{\tau_{-t,j}} Df(4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x)t\big\| \\
&\qquad + \|6 \cdot 16^{\tau_{-s,j}} f(0)\|\Big) \qquad (2.4) \\
&\le \begin{cases}
\sum_{j=n}^{n+m-1} \left(\frac{\theta}{4^j} + \frac{9\theta}{16^{j+1}}\right) & (\text{for } \ p = 0), \\
\sum_{j=n}^{n+m-1} 4^{-j}\theta\|4^j x\|^p & (\text{for } \ p < 0 \ \text{ or } \ 0 < p < 1), \\
\sum_{j=n}^{n+m-1} \left(\frac{\theta\|4^j x\|^p}{4^{2j+1}} + \frac{4^{j+1}\theta\|x\|^p}{4^{(j+1)p}}\right) & (\text{for } \ 1 < p < 2), \\
\sum_{j=n}^{n+m-1} 4^{2j+1}\theta\|4^{-j-1}x\|^p & (\text{for } \ p > 2)
\end{cases} \\
&\le \begin{cases}
\frac{4\theta}{3 \cdot 4^n} + \frac{3\theta}{5 \cdot 16^n} & (\text{for } \ p = 0), \\
\frac{4^{np}\theta\|x\|^p}{4^{n-1}(4-4^p)} & (\text{for } \ p < 0 \ \text{ or } \ 0 < p < 1), \\
\frac{4^{np}\theta\|x\|^p}{4^{2n-1}(16-4^p)} + \frac{4^{n+1}\theta\|x\|^p}{4^{np}(4^p-4)} & (\text{for } \ 1 < p < 2), \\
\frac{4^{2n+1}\theta\|x\|^p}{4^{np}(4^p-16)} & (\text{for } \ p > 2)
\end{cases}
\end{aligned}
$$

for all $x \in X\backslash\{0\}$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $\lim_{n\to\infty} f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in X$.

Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Moreover, putting $n = 0$ and letting $m \to \infty$ in (2.4), we obtain the inequality (2.2).

From the definition of F, we get

$$\begin{aligned}
&DF(x,y,z,w)\\
&= \lim_{n\to\infty}\bigg(\frac{16^{-sn}}{2}\big(Df(4^{sn}x,4^{sn}y,4^{sn}z,4^{sn}w) + Df(-4^{sn}x,-4^{sn}y,-4^{sn}z,-4^{sn}w)\big)\\
&\qquad + \frac{4^{-tn}}{2}\big(Df(4^{tn}x,4^{tn}y,4^{tn}z,4^{tn}w) - Df(-4^{tn}x,-4^{tn}y,-4^{tn}z,-4^{tn}w)\big)\bigg)\\
&\le \lim_{n\to\infty}\big(4^{-sn(2-p)} + 4^{-tn(1-p)}\big)\theta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\big)\\
&= 0
\end{aligned}$$

for all $x, y, z, w \in X\backslash\{0\}$. By Lemma 2.2, F is a quadratic-additive mapping.

Now, we show that F is unique. Let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.2). It is easy to show that $F'(0) = 0$ for all quadratic-additive mapping F'. It follows from (2.3) that

$$\begin{aligned}
F'(x) - J_nF'(x) &= \sum_{j=0}^{n-1}(J_jF'(x) - J_{j+1}F'(x))\\
&= -\frac{1}{2}\sum_{j=0}^{n-1}\Big(16^{\tau_{-s,j}}\big(DF'(4^{-\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x, 4^{\tau_{s,j}}x)\\
&\qquad + DF'(4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x, -4^{\tau_{s,j}}x)\big)s\\
&\qquad + 4^{\tau_{-t,j}}\big(DF'(-4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x, 4^{\tau_{t,j}}x)\\
&\qquad - DF'(4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x, -4^{\tau_{t,j}}x)\big)t\Big)\\
&= 0
\end{aligned}$$

for all $n \in \mathbb{N}$ and for all $x \in X$.

Since F and F' are quadratic-additive, replacing x with 4^nx in (2.2), we have

$$\begin{aligned}
\|F(x) - F'(x)\| &= \|J_nF(x) - J_nF'(x)\|\\
&\le \frac{16^{-sn}}{2}\Big(\|(F-f)(4^{sn}x)\| + \|(f-F')(4^{sn}x)\|\\
&\qquad + \|(F-f)(-4^{sn}x)\| + \|(f-F')(-4^{sn}x)\|\Big)\\
&\quad + \frac{4^{-tn}}{2}\Big(\|(F-f)(4^{tn}x)\| + \|(F'-f)(4^{tn}x)\|\\
&\qquad + \|(F-f)(-4^{tn}x)\| + \|(F'-f)(-4^{tn}x)\|\Big)\\
&\le \left(\frac{8\theta}{|16-4^p|} + \frac{8\theta}{|4^p-4|}\right)\big(4^{-sn(2-p)} + 4^{-tn(1-p)}\big)\|x\|^p
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and all positive integers n provided $p \neq 0$. We also have

$$\|F(x) - F'(x)\| \leq \frac{49}{15}\big(16^{-n} + 4^{-n}\big)\theta$$

for all $x \in X\backslash\{0\}$ and all positive integers n provided $p = 0$.

Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F(x) = F'(x)$ for all $x \in X$. This proves the uniqueness of F. Since

$$\begin{aligned}
&\|f(x) - F(x)\| \\
&\quad \leq \|Df((3k-1)x, kx, kx, kx) - DF((3k-1)x, kx, kx, kx)\| \\
&\qquad + 3\|(F-f)((4k-1)x)\| + 3\|(f-F)((3k-1)x)\| \\
&\qquad + \|(f-F)((1-3k)x)\| + 9\|(f-F)(kx)\| + 3\|(f-F)(-kx)\| \\
&\quad \leq \left((3k-1)^p + 3k^p + \frac{4\big(3(4k-1)^p + 4(3k-1)^p + 12k^p\big)}{4-4^p}\right)\theta\|x\|^p
\end{aligned}$$

holds for all $x \in X\backslash\{0\}$ and all $k \in \mathbb{R}$, we conclude that $f(x) = F(x)$ for all $x \in X\backslash\{0\}$ provided $p < 0$ by letting $k \to \infty$ in the above inequality. Since $f(0) = 0$, f is itself a quadratic-additive mapping. □

In the case of $p \geq 0$, x, y, z, w can take 0 in the inequality (2.1). For this case, we obtain a little bit different result from Theorem 2.3.

Theorem 2.4. *Let θ be a nonnegative constant. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z,w)\| \leq \theta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p\big) \tag{2.5}$$

for all $x, y, z, w \in X$ with a nonnegative real constant $p \notin \{1, 2\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta\|x\|^p}{2^p-4} & (\textit{for} \;\; p > 2), \\ \left(\frac{\theta}{4-2^p} + \frac{\theta}{2^p-2}\right)\|x\|^p & (\textit{for} \;\; 1 < p < 2), \\ \frac{\theta\|x\|^p}{2-2^p} & (\textit{for} \;\; 0 < p < 1), \\ \frac{19}{9}\theta & (\textit{for} \;\; p = 0) \end{cases} \tag{2.6}$$

for all $x \in X$.

Proof. Since

$$\|f(0)\| = \left\|\frac{1}{12}Df(0,0,0,0)\right\| \leq \frac{\|0\|^p\theta}{3},$$

we get $f(0) = 0$ for $p \notin \{0,1,2\}$ and $\|f(0)\| \le \theta/3$ for $p = 0$. From the definitions of $J_n f(x)$ and $Df(x,y,z,w)$, we get

$$
\begin{aligned}
&J'_n f(x) - J'_{n+1} f(x)\\
&\quad = -\frac{1}{4}\Big(4^{\tau_{-s,n}}\big(Df(2^{\tau_{s,n}}x, 2^{\tau_{s,n}}x, 0, 0) + Df(-2^{\tau_{s,n}}x, -2^{\tau_{s,n}}x, 0, 0)\big)s\\
&\qquad\qquad + 2^{\tau_{-t,n}}\big(Df(2^{\tau_{t,n}}x, 2^{\tau_{t,n}}x, 0, 0) - Df(-2^{\tau_{t,n}}x, -2^{\tau_{t,n}}x, 0, 0)\big)t\Big)\\
&\qquad + 4^{\tau_{-s,n}+1} f(0)
\end{aligned} \tag{2.7}
$$

for all $x \in X$ and all nonnegative integers n, where $s = \operatorname{sign}(2-p)$, $t = \operatorname{sign}(1-p)$, and $\tau_{k,n}$ are the integers defined by $\tau_{k,n} = k(n+1/2) - 1/2$ for $k \in \{-1,1\}$.

It follows from (2.5) and (2.7) that

$$
\begin{aligned}
&\|J'_n f(x) - J'_{n+m} f(x)\|\\
&\quad = \sum_{j=n}^{n+m-1} \|J'_j f(x) - J'_{j+1} f(x)\|\\
&\quad \le \frac{1}{4}\sum_{j=n}^{n+m-1}\Big(\big\|4^{\tau_{-s,j}} Df\big(2^{\tau_{s,j}}x, 2^{\tau_{s,j}}x, 0, 0\big)s + 2^{\tau_{-t,j}} Df\big(2^{\tau_{t,j}}x, 2^{\tau_{t,j}}x, 0, 0\big)t\big\|\\
&\qquad\qquad + \big\|4^{\tau_{-s,j}} Df\big(-2^{\tau_{s,j}}x, -2^{\tau_{s,j}}x, 0, 0\big)s\\
&\qquad\qquad\quad - 2^{\tau_{-t,j}} Df\big(-2^{\tau_{t,j}}x, -2^{\tau_{t,j}}x, 0, 0\big)t\big\|\\
&\qquad\qquad + \big\|16\cdot 4^{\tau_{-s,j}} f(0)\big\|\Big)
\end{aligned} \tag{2.8}
$$

$$
\le \begin{cases}
\sum_{j=n}^{n+m-1}\left(\frac{\theta}{2^j} + \frac{\theta}{3\cdot 4^j}\right) & (\text{for } \ p = 0),\\
\sum_{j=n}^{n+m-1} 2^{-j-1}\theta\|2^j x\|^p & (\text{for } \ 0 < p < 1),\\
\sum_{j=n}^{n+m-1} \left(2^{-2j-2}\theta\|2^j x\|^p + 2^j\theta\|2^{-j-1}x\|^p\right) & (\text{for } \ 1 < p < 2),\\
\sum_{j=n}^{n+m-1} 2^{2j}\theta\|2^{-j-1}x\|^p & (\text{for } \ p > 2)
\end{cases}
$$

$$
\le \begin{cases}
\frac{2\theta}{2^n} + \frac{\theta}{9\cdot 4^n} & (\text{for } \ p = 0),\\
\frac{2^{np}\theta\|x\|^p}{2^n(2-2^p)} & (\text{for } \ 0 < p < 1),\\
\frac{2^{np}\theta\|x\|^p}{4^n(4-2^p)} + \frac{2^n\theta\|x\|^p}{2^{np}(2^p-2)} & (\text{for } \ 1 < p < 2),\\
\frac{4^n\theta\|x\|^p}{2^{np}(2^p-4)} & (\text{for } \ p > 2)
\end{cases}
$$

for all $x \in X$.

So, it is easy to show that the sequence $\{J'_n f(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J'_n f(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J'_n f(x)$$

for all $x \in X$. Moreover, putting $n = 0$ and letting $m \to \infty$ in (2.8), we get the inequality (2.6). The remaining part of the proof is similar to the proof of Theorem 2.3. □

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64–66.

[2] J. A. Baker, *The stability of certain functional equations*, Proc. Amer. Math. Soc., **112(3)**(1991), 729–732.

[3] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure and Appl. Math., **4(1)**(2003), Art. 4, http://jipam.vu.edu.au

[4] L. Cădariu and V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timisoara Ser. Mat.-Inform., **41**(2003), 25–48.

[5] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber., **346**(2004), 43–52.

[6] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations – a survey*, Ann. Funct. Anal., **3(1)**(2012), 151–164.

[7] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore (2002).

[8] P. Enflo and M. S. Moslehian, *An interview with Th. M. Rassias*, Banach J. Math. Anal., **1(2)**(2007), 252–260.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27**(1941), 222–224.

[10] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations of Several Variables, Birkhäuser, Basel (1998).

[11] S.-M. Jung, *On the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **204**(1996), 221–226.

[12] S.-M. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc., **126**(1998), 3137–3143.

[13] S.-M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl., **222**(1998), 126–137.

[14] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer Optimization and Its Applications Vol. 48, Springer, New York (2011).

[15] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. (N. S.), **37(3)**(2006), 361–376.

[16] M. S. Moslehian and Th. M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math., **1(2)**(2007), 325–334.

[17] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**(2003), 91–96.

[18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.

[19] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York (1964).