KYUNGPOOK Math. J. 55(2015), 73-81
http://dx.doi.org/10.5666/KMJ.2015.55.1.73
pISSN 1225-6951 eISSN 0454-8124

Refined Stability Results of Functional Equation in Four Variables

Hark-Mahn Kim and Soon Lee*
Department of Mathematics, Chungnam National University, 79 Daehangno, Yuseong-gu, Daejeon 305-764, Korea
e-mail : hmkim@cnu.ac.kr *and* lsmath6256@hanmail.net

Abstract. In this paper, we present the general solution of the functional equation

$$rf(\frac{x+y+z+w}{s})+rf(\frac{x+y-z-w}{s})+rf(\frac{x-y+z-w}{s})+rf(\frac{x-y-z+w}{s})$$
$$=tf(x)+tf(y)+tf(z)+tf(w)$$

and improve the Hyers–Ulam stability of the equation.

1. Introduction

In 1940, S.M. Ulam [12] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms: We are given a group G_1 and a metric group G_2 with metric $\varphi(\cdot,\ \cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:G_1\to G_2$ satisfies $\varphi(f(x\ y),f(x)\ f(y))<\delta$ for all $x,\ y\in G_1$, then a homomorphism $h:G_1\to G_2$ exists with $\varphi(f(x),h(x))<\varepsilon$ for all $x\in G_1$?

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. D.H. Hyers [6] showed that if $\varepsilon>0$ and $f:X\to Y$ such that

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x,y\in X$, then there exists a unique additive mapping $T:X\to Y$ such that

$$\|f(x)-T(x)\|\le\varepsilon$$

for all $x\in X$.

* Corresponding Author.
Received March 30, 2013; accepted July 18, 2013.
2010 Mathematics Subject Classification: 39B52, 39B82.
Key words and phrases: quadratic functional equation, Hyers–Ulam stability.

In 1950 T. Aoki [1] and in 1951 D.G. Bourgin [2] provided a generalized the Hyers theorem for additive mapping and in 1978 Th.M. Rassias [10] generalized the Hyers theorem for liner mapping by allowing the Cauchy difference to be unbounded. Let $f : X \to Y$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then Th.M. Rassias proved that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in X$. And then, the result of Th.M. Rassias theorem has been generalized by P. Găvruta [5] by allowing the Cauchy difference to be a generalized control function.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

which may be originated from this parallelogram equality, is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A Hyers–Ulam stability problem for the quadratic functional equation was proved by F. Skof [11] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. P.W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In S. Czerwik [4] proved the Hyers–Ulam stability of the quadratic functional equation. In the last decade, S. Lee and K. Jun [7] and S. Lee and C. Park [8] have proved the Hyers–Ulam stability of quadratic type functional equation with three variables.

Recently, C. Park [9] has investigated the Hyers–Ulam stability of the following functional equation.

$$\begin{aligned} &rf(\frac{x + y + z + w}{s}) + rf(\frac{x + y - z - w}{s}) + rf(\frac{x - y + z - w}{s}) \\ &+rf(\frac{x - y - z + w}{s}) = tf(x) + tf(y) + tf(z) + tf(w) \end{aligned} \tag{1.1}$$

for all $x, y, z, w \in X$ under the assumption of an even mapping $f : X \to Y$ with $f(0) = 0$. Throughout this paper, we now assume that r, s, t are nonzero real numbers, and that X and Y are a normed linear space with norm $\| \cdot \|$ and a Banach space with norm $\| \cdot \|$, respectively. In this paper, we establish the general solution of the functional equation (1.1) and then improve the Hyers–Ulam stability of the functional equation without the even condition and $f(0) = 0$ for a mapping $f : X \to Y$.

2. Stability of the Functional Equation in Four Variables

First of all, we solve the general solution of the equation (1.1).

Lemma 2.1. If a mapping $f : X \to Y$ satisfies the equation (1.1) for all $x, y, z, w \in X$, then $f - f(0)$ is quadratic, and $f(0) = 0$ if $r \neq t$.

Proof. First, we prove the case $r \neq t$.

Letting $x = y = z = w := 0$ in (1.1), one has $f(0) = 0$. Putting $y = z = w := 0$ in (1.1), we have

$$4rf(\frac{x}{s}) = tf(x) \tag{2.1}$$

for all $x \in X$. Using (2.1) and (1.1), we get

$$\begin{aligned} &f(x+y+z+w) + (x+y-z-w) + f(x-y+z-w) + f(x-y-z+w) \\ &= 4f(x) + 4f(y) + 4f(z) + 4f(w) \end{aligned} \tag{2.2}$$

for all $x, y, z, w \in X$. We note that $f(y) = f(-y)$ by putting $x = z = w := 0$ in the last equation. Putting $z = w := 0$ in (2.2), we deduce

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. So f is quadratic.

Next, we prove the case $r = t$. In this case, we see the functional equation

$$\begin{aligned} &f(\frac{x+y+z+w}{s}) + f(\frac{x+y-z-w}{s}) + f(\frac{x-y+z-w}{s}) + f(\frac{x-y-z+w}{s}) \\ &= f(x) + f(y) + f(z) + f(w) \end{aligned}$$

for all $x, y, z, w \in X$. Hence, putting $f(x) - f(0) := Q(x), x \in X$, one can easily observe the relation

$$\begin{aligned} &Q(\frac{x+y+z+w}{s}) + Q(\frac{x+y-z-w}{s}) + Q(\frac{x-y+z-w}{s}) \\ &+Q(\frac{x-y-z+w}{s}) = Q(x) + Q(y) + Q(z) + Q(w), \quad Q(0) = 0. \end{aligned} \tag{2.3}$$

Putting $y = z = w := 0$ in (2.3), we have $4Q(\frac{x}{s}) = Q(x)$, and so

$$\begin{aligned} &Q(x+y+z+w) + Q(x+y-z-w) + Q(x-y+z-w) \\ &+Q(x-y-z+w) = 4(Q(x) + Q(y) + Q(z) + Q(w)) \end{aligned} \tag{2.4}$$

for all $x, y, z, w \in X$. We observe that $Q(y) = Q(-y)$ by setting $x = z = w := 0$ in (2.4). Thus, by letting $z = w := 0$ in (2.4), one arrives at

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

for all $x, y \in X$. So Q is quadratic. □

We now prove the Hyers–Ulam stability of the functional equation without the even condition and $f(0) = 0$ for a mapping $f : X \to Y$. Given a mapping $f : X \to Y$, we set for notational convenience

$$\begin{aligned} &Df(x,y,z,w) \\ &= rf(\frac{x+y+z+w}{s}) + rf(\frac{x+y-z-w}{s}) + rf(\frac{x-y+z-w}{s}) \\ &\quad + rf(\frac{x-y-z+w}{s}) - [tf(x) + tf(y) + tf(z) + tf(w)] \end{aligned}$$

for all $x, y, z, w \in X$.

Theorem 2.2. *Let $f : X \to Y$ be a mapping for which there is a function $\varphi : X^4 \to [0, \infty)$ such that*

$$\Phi(x,y,z,w) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty, \tag{2.5}$$

$$\|Df(x,y,z,w)\| \le \varphi(x,y,z,w) \tag{2.6}$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + \frac{1}{3}(1 - \frac{4r}{t})f(0) - Q(x)\| \le \frac{1}{4|t|}[\Phi(2x,0,0,0) + 2\Phi(x,x,0,0)] \tag{2.7}$$

for all $x \in X$, where $4|r-t|\|f(0)\| \le \varphi(0,0,0,0)$.

Proof. Putting $y = z = w := 0$ in (2.6), we get

$$\|4rf(\frac{x}{s}) - tf(x) - 3tf(0)\| \le \varphi(x,0,0,0)$$

and then replacing x by $2x$, we have

$$\|4rf(\frac{2x}{s}) - tf(2x) - 3tf(0)\| \le \varphi(2x,0,0,0) \tag{2.8}$$

for all $x \in X$. Putting $y := x$ and $z = w := 0$ in (2.6), we have

$$\|2rf(\frac{2x}{s}) + 2rf(0) - 2tf(x) - 2tf(0)\| \le \varphi(x,x,0,0) \tag{2.9}$$

for all $x \in X$. By (2.8) and (2.9), we have

$$\|tf(2x) - 4tf(x) + (4r-t)f(0)\| \leq \varphi(2x,0,0,0) + 2\varphi(x,x,0,0)$$

and so

$$\|f(2x) - 4f(x) + \frac{(4r-t)}{t}f(0)\| \leq \frac{1}{|t|}[\varphi(2x,0,0,0) + 2\varphi(x,x,0,0)]$$

for all $x \in X$. Let $F(x) = f(x) + \frac{1}{3}(1 - \frac{4r}{t})f(0)$, $x \in X$. Then we lead to the crucial inequality due to the last functional inequality

$$\begin{aligned} \|F(2x) - 4F(x)\| &\leq \frac{1}{|t|}[\varphi(2x,0,0,0) + 2\varphi(x,x,0,0)], \\ \|F(x) - \frac{F(2x)}{4}\| &\leq \frac{1}{4|t|}[\varphi(2x,0,0,0) + 2\varphi(x,x,0,0)] \end{aligned} \tag{2.10}$$

for all $x \in X$. Hence

$$\begin{aligned} \|\frac{F(2^n x)}{4^n} - \frac{F(2^{n+1}x)}{4^{n+1}}\| &= \frac{1}{4^n}\|F(2^n x) - \frac{F(2 \cdot 2^n x)}{4}\| \\ &\leq \frac{1}{4^{n+1}|t|}[\varphi(2^{n+1}x,0,0,0) + 2\varphi(2^n x, 2^n x,0,0)] \end{aligned} \tag{2.11}$$

for all $x \in X$ and all nonnegative integers n. Thus, it follows from (2.11) that

$$\left\|\frac{F(2^m x)}{4^m} - \frac{F(2^n x)}{4^n}\right\| \leq \sum_{k=m}^{n-1} \frac{1}{4^{k+1}|t|}[\varphi(2^{k+1}x,0,0,0) + 2\varphi(2^k x, 2^k x,0,0)] \tag{2.12}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that the sequence $\{\frac{F(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$ by the convergence of Φ. Since Y is complete, the sequence $\{\frac{F(2^n x)}{4^n}\}$ converges in Y for all $x \in X$. So one can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{F(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. Obviously, $Q(0) = 0$. It follows from (2.6) and the definition of Q that

$$\begin{aligned} \|DQ(x,y,z,w)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y, 2^n z, 2^n w)\| \\ &\leq \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) \end{aligned}$$

for all $x, y, z, w \in X$. By Lemma 2.1, the mapping Q is quadratic. Putting $m = 0$ and letting $n \to \infty$ in (2.12), we get the desired approximation (2.7).

Now, let $Q^{'} : X \to Y$ be another quadratic mapping satisfying (2.7). Then we have

$$\begin{aligned} \| Q(x) - Q^{'}(x) \| &= \frac{1}{4^n} \|Q(2^n x) - Q^{'}(2^n x)\| \\ &\leq \frac{1}{4^n} \Big(\|Q(2^n x) - f(2^n x) - \frac{1}{3}(1 - \frac{4r}{t}) f(0)\| \\ &\qquad + \|f(2^n x) + \frac{1}{3}(1 - \frac{4r}{t}) f(0) - Q^{'}(2^n x)\| \Big) \\ &\leq \frac{2}{4^{n+1}|t|} [\Phi(2^{n+1}x, 0, 0, 0) + 2\Phi(2^n x, 2^n x, 0, 0)], \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q^{'}(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Corollary 2.3. Let θ and p $(0 < p < 2)$ be positive real numbers. Let $f : X \to Y$ be a mapping such that

$$\|Df(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + \frac{1}{3}(1 - \frac{4r}{t}) f(0) - Q(x)\| \leq \frac{\theta}{|t|} (\frac{4 + 2^p}{4 - 2^p}) \|x\|^p$$

for all $x \in X$.

Proof. Defining $\varphi(x, y, z, w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and applying Theorem 2.2, we get the desired result. □

Remark 2.4. Let $f : X \to Y$ be a mapping for which there is a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.5) and (2.6). Then by putting $\tilde{f}(x) = f(x) - f(0)$, $x \in X$, we have from the functional inequality (2.6)

$$\|D\tilde{f}(x, y, z, w)\| \leq \varphi(x, y, z, w) + 4|r - t|\|f(0)\|,$$

which yields

$$\|4r\tilde{f}(\frac{2x}{s}) - t\tilde{f}(2x)\| \leq \varphi(2x, 0, 0, 0) + 4|r - t|\|f(0)\|, \tag{2.13}$$

$$\|2r\tilde{f}(\frac{2x}{s}) - 2t\tilde{f}(x)\| \leq \varphi(x, x, 0, 0) + 4|r - t|\|f(0)\| \tag{2.14}$$

for all $x \in X$. By (2.13) and (2.14), one has

$$\|\tilde{f}(2x) - 4\tilde{f}(x)\| \leq \frac{1}{|t|}[\varphi(2x, 0, 0, 0) + 2\varphi(x, x, 0, 0) + 12|r - t|\|f(0)\|] \tag{2.15}$$

for all $x \in X$. Thus, by applying the same argument as that of Theorem 2.2, one can prove that there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - f(0) - Q(x)\| \le \frac{1}{4|t|}[\Phi(2x, 0, 0, 0) + 2\Phi(x, x, 0, 0)] + \frac{4|r - t|\|f(0)\|}{|t|}$$

for all $x \in X$, where $4|r - t|\|f(0)\| \le \varphi(0, 0, 0, 0)$.

Next, we consider another stability problem which is an alternative stability result of Theorem 2.2.

Theorem 2.5. *Let $f : X \to Y$ be a mapping for which there is a function $\varphi : X^4 \to [0, \infty)$ such that*

$$\Psi(x, y, z, w) := \sum_{j=0}^{\infty} 4^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}, \frac{w}{2^j}) < \infty, \tag{2.16}$$

$$\|Df(x, y, z, w)\| \le \varphi(x, y, z, w) \tag{2.17}$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + \frac{1}{3}(1 - \frac{4r}{t})f(0) - Q(x)\| \le \frac{1}{|t|}[\Psi(x, 0, 0, 0) + 2\Psi(\frac{x}{2}, \frac{x}{2}, 0, 0)] \tag{2.18}$$

for all $x \in X$, where $4 \mid r - t \mid \|f(0)\| \le \varphi(0, 0, 0, 0)$.

Proof. First, we note that $f(0) = 0 = \varphi(0, 0, 0, 0)$ by the convergence of $\Psi(0, 0, 0, 0)$ if $r \ne t$. Replacing x by $\frac{x}{2}$ in (2.10), we get

$$\|F(x) - 4F(\frac{x}{2})\| \le \frac{1}{|t|}[\varphi(x, 0, 0, 0) + 2\varphi(\frac{x}{2}, \frac{x}{2}, 0, 0)] \tag{2.19}$$

for all $x \in X$. Using (2.19), we have

$$\begin{aligned} \|4^n F(\frac{x}{2^n}) - 4^{n+1} F(\frac{x}{2^{n+1}})\| &= 4^n \|F(\frac{x}{2^n}) - 4F(\frac{x}{2 \cdot 2^n})\| \\ &\le \frac{4^n}{|t|}\Big[\varphi(\frac{x}{2^n}, 0, 0, 0) + 2\varphi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0, 0)\Big] \end{aligned} \tag{2.20}$$

for all $x \in X$ and all positive integers n. By (2.20), we have

$$\|4^m F(\frac{x}{2^m}) - 4^n F(\frac{x}{2^n})\| \le \sum_{k=m}^{n-1} \frac{4^k}{|t|}[\varphi(\frac{x}{2^k}, 0, 0, 0) + 2\varphi(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, 0, 0)] \tag{2.21}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{4^n F(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n F(\frac{x}{2^n})\}$ converges in Y for all $x \in X$. So one can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 4^n F(\frac{x}{2^n}) = \lim_{n\to\infty} 4^n \Big(f(\frac{x}{2^n}) + \frac{1}{3}(1 - \frac{4r}{t})f(0)\Big)$$

for all $x \in X$. We remark that $Q(0) = 0$ by the definition of Q if $r = t$, and also $Q(0) = 0$ because of $f(0) = 0 = \varphi(0,0,0,0)$ by the convergence of $\Psi(0,0,0,0)$ if $r \neq t$. Thus, $Q(0) = 0$ for any nonzero real numbers r, t. Also, we get from (2.17)

$$\begin{aligned}\|DQ(x,y,z,w)\| &= \lim_{n\to\infty} 4^n \|DF(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n})\| \\ &\leq \lim_{n\to\infty} 4^n \Big[\|Df(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n})\| + \frac{4|r-t|}{3}|1 - \frac{4r}{t}|\|f(0)\|\Big] \\ &\leq \lim_{n\to\infty} 4^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}) = 0\end{aligned}$$

for all $x, y, z, w \in X$. By Lemma 2.1, Q is quadratic. Putting $m = 0$ and letting $n \to \infty$ in (2.21), we get the estimation (2.18).

The proof of the uniqueness of Q is similar to that of Theorem 2.2. □

Corollary 2.6. Let θ and p ($p > 2$) be positive real numbers. Let $f : X \to Y$ be a mapping such that

$$\|Df(x,y,z,w)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + \frac{1}{3}(1 - \frac{4r}{t})f(0) - Q(x)\| \leq \frac{\theta}{|t|}(\frac{2^p + 4}{2^p - 4})\|x\|^p \tag{2.22}$$

for all $x \in X$.

Proof. Defining $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and applying Theorem 2.5 we get the desired result. □

Remark 2.7. Let $f : X \to Y$ be a mapping for which there is a function $\varphi : X^4 \to [0,\infty)$ satisfying (2.16) and (2.17). Then by putting $\tilde{f}(x) = f(x) - f(0)$, $x \in X$, we have from (2.17)

$$\|D\tilde{f}(x,y,z,w)\| \leq \varphi(x,y,z,w) + 4|r-t|\|f(0)\|.$$

First, we note that $f(0) = 0 = \varphi(0,0,0,0)$ by the convergence of $\Psi(0,0,0,0)$ if $r \neq t$, and so the term $4|r-t|\|f(0)\| = 0$ for any nonzero real numbers r, t. Thus, we get from (2.15)

$$\|\tilde{f}(x) - 4\tilde{f}(\frac{x}{2})\| \leq \frac{1}{|t|}[\varphi(x,0,0,0) + 2\varphi(\frac{x}{2}, \frac{x}{2}, 0, 0)]$$

for all $x \in X$. Hence, by applying the same argument as that of Theorem 2.5, one can prove that there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - f(0) - Q(x)\| \leq \frac{1}{4|t|}[\Psi(2x, 0, 0, 0) + 2\Psi(x, x, 0, 0)]$$

for all $x \in X$.

Acknowledgements. This research was supported by the Basic Research Program through the National Research Foundation (NRF) of Korea funded by the Ministry of Education (NRF-2012R1A1A2008139).

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, **2**(1950), 64–66.

[2] D. G. Bourgin, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc., **57**(1951), 223–237.

[3] P. W. Cholewa, Remarks on the shability of functional equations, Aequationes Math., **27**(1984), 76–86.

[4] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg, **62**(1992), 59–64.

[5] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl., **184**(1994), 431–436.

[6] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. U.S.A., **27**(1941), 222–224.

[7] S. Lee and K. Jun,Hyers–Ulam–Rassias stability of a quadratic type functional equation, Bull. Korean Math. Soc., **40**(2003), 183–193.

[8] S. Lee and C. Park, Hyers–Ulam–Rassias stability of a functional equation in three variables, J. Chungcheong Math. Soc., **16(2)**(2003), 11–21.

[9] C. Park, Hyers–Ulam–Rassias stability of an even functional equation in four variables, Kyungpook Math J., **44(2)**(2004), 299–304.

[10] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., **72**(1978), 297–300.

[11] F. Skof, Proprietà locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano, **53**(1983), 113–129.

[12] S. M. Ulam, A Collection of the Mathematical Problems, Interscience Publ. New York, 1960.