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Range Kernel Orthogonality and Finite Operators

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Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H into itself. Let $A, B \in \mathcal{L}(H)$ we define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by $\delta_{A,B}(X) = AX - XB$, we note $\delta_{A,A} = \delta_A$. If the inequality $||T - (AX - XA)|| \geq ||T||$ holds for all $X \in \mathcal{L}(H)$ and for all $T \in ker\delta_A$, then we say that the range of δ_A is orthogonal to the kernel of δ_A in the sense of Birkhoff. The operator $A \in \mathcal{L}(H)$ is said to be finite [22] if $||I - (AX - XA)|| \geq 1(*)$ for all $X \in \mathcal{L}(H)$, where I is the identity operator. The well-known inequality (*), due to J. P. Williams [22] is the starting point of the topic of commutator approximation (a topic which has its roots in quantum theory [23]). In [16], the author showed that a paranormal operator is finite. In this paper we present some new classes of finite operators containing the class of paranormal operators and we prove that the range of a generalized derivation is orthogonal to its kernel for a large class of operators containing the class of normal operators.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H into itself. Given $A, B \in \mathcal{L}(H)$, we define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by $\delta_{A,B}(X) = AX - XB$, we note $\delta_{A,A} = \delta_A$.

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Recall that for elements a, b of a Banach algebra $\mathcal{V}$, we say that b is orthogonal to a , written $b \perp a$, provided

$$||a|| = \text{dist}(a, \mathbb{C}b),$$

so that the line $a + \mathbb{C}b$ is tangent to the ball of center 0 and radius $||a||$; when $V = H$ is a Hilbert space this agrees with the usual inner product $(a, b) = 0$. Thus when $b \perp a$ then the expression $||a + \lambda b||$ has global minimum when $\lambda = 0$.

We say that the operator $A \in \mathcal{L}(H)$ is finite if for all $X \in \mathcal{L}(H)$, we have $||I - (AX - XA)|| \geq 1$. The first important contribution to the study of commutators is due to A.Wintner who in 1947 proved that the identity element 1 in a unital, normed algebra $\mathcal{A}$ is not a commutator, that is, there are no elements A and B such that $1 = AB - BA$. Like much good mathematics, Wintner's theorem has its roots in physics. Indeed, it was prompted by the fact that the linear maps P and Q representing the quantum-mechanical momentum and position, respectively, satisfy the commutation relation $PQ - QP = (-ih/2\pi)I$, where h is the Planck's constant and I the identity operator on the underlying Hilbert space. The related topic of approximation by commutators $AX - XA$ or by generalized commutator $AX - XB$, which has attracted much interest, has its roots in quantum theory. The most striking property of Heisenberg's infinite matrices for the position and momentum is that they do not commute.

$$[X, P] = XP - PX = ih$$

and this result did not have a clear physical interpretation in the beginning. In March 1926, working in Bohr's institute, Heisenberg realized that the non-commutativity implies the uncertainty principle. This was a clear physical interpretation for the non-commutativity, and it laid the foundation for what became known as the Copenhagen interpretation of quantum mechanics. Heisenberg showed that the commutation relations implies an uncertainty. Any two variables that do not commute cannot be measured simultaneously-the more precisely one is known, the less precisely the other can be known. The Heisenberg Uncertainty principle may be mathematically formulated as saying that there exists a pair A, X of linear transformation and a non-zero scalar α such that

$$AX - XA = \alpha I. \tag{1.1}$$

Clearly, (1) cannot hold for square matrices A and X. To see this, just take the trace of both sides (1). Nor (1) hold for bounded linear operators A and X. This prompts the question: How close $AX - XA$ to be the identity? In 1973 J. H. Anderson proved the remarkable result that there exists a bounded linear operator A such that I belongs to the closure in the norm topology of the set of the commutators $AB - BA$, that is, $I \in \overline{R(\delta_A)}$. In other words the distance between the identity operator and the commutator $AX - XA$ is minimal and equal to zero. Hence Anderson's result minimizes the distance between the identity operator and the commutator $AX - XA$. Now by the inequality $||AX - XB - I|| \geq 1$ we also minimize the distance between the identity and $AX - XB$. Here the distance is maximal and equal to 1.

Let $A \in \mathcal{L}(H)$, the approximate reduced spectrum of A, $\sigma_{ra}(A)$, is the set of scalars λ for which there exists a normed sequence $\{x_n\}$ in H satisfying

$$(A - \lambda)x_n \to 0, \text{ and } (A - \lambda)^* x_n \to 0.$$

J. P. Williams in [22], proved that the class of finite operators, $\mathcal{F}$, contains every normal, hyponormal operators. In [13], J. P. Williams results are generalized to a more general classes of operators containing the classes of normal and hyponormal operators.

For any operator A in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = \mid A \mid^2 - \mid A^* \mid^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$ and p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where $p > 0$. This definition is due to Aluthge [4] and many authors studied interesting properties of p-hyponormal operators by using Aluthge transform (see [4], [17]). An operator $A \in \mathcal{L}(H)$ is said to be normaloid if $||A|| = r(A)$, where $r(A)$ is the spectral radius of A, paranormal if

$$||Ax||^2 \leq ||A^2x||||x||, \text{ for all } x \in H.$$

An operator $A \in B(H)$ is said to be spectraloid if $\omega(A) = r(A)$, where $r(A)$ (resp. $\omega(A)$) spectral radius (resp. numerical radius) of A. We have

$$\text{hyponormal} \subset p-\text{hyponormal} \subset \text{ paranormal} \subset \text{normaloid} \subset \text{spectraloid}.$$

A is said to be a class $\mathcal{Y}_\alpha$ operator for $\alpha \geq 1$ (or $A \in \mathcal{Y}_\alpha$) if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda I)^*(A - \lambda I) \text{ for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \cup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator A is M-hyponormal, i.e., there exists a positive number M such that

$$(A - \lambda I)(A - \lambda I)^* \leq M^2 (A - \lambda I)^*(A - \lambda I) \text{ for all } \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ operators (see [21]). A is said to dominant if for any $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(A - \lambda I)(A - \lambda I)^* \leq M_\lambda^2 (A - \lambda I)^*(A - \lambda I).$$

It is obvious that dominant operators are M-hyponormal. But it is known that there exists a dominant operator which is not a class $\mathcal{Y}_1$ operator, and also there exists a class $\mathcal{Y}_2$ operator which is not dominant. We have

$$\text{normal} \Rightarrow \text{hyponormal} \Rightarrow M\text{-hyponormal} \Rightarrow \text{class } \mathcal{Y}_2$$

J. H. Anderson and Foias [2] proved that if A and B are normal, T is an operator such that $AT = TB$, then

$$||T - \delta_{A,B}(X)|| \geq ||T||, \text{ for all } X \in \mathcal{L}(H). \tag{2}$$

Hence the range of $\delta_{A,B}$ is orthogonal to the kenel of $\delta_{A,B}$. The inequality $||T - (AX - XA)|| \geq ||T||$, for all $X \in \mathcal{L}(H)$ and for all $T \in ker\delta_A$ means that the range of δ_A is orthogonal to the kernel of δ_A in the sense of Birkhoff. In [16], the author showed that a paranormal operator is finite. In this paper we will extend the last inequality for a large class of operators containing the class of normal operators and we prove that a class $\mathcal{Y}$ operator and a spectraloid operator are finite.

2. Main Results

We begin by the following well known proposition.

Proposition 2.1.([5, Berberian technique]) *Let H be a complex Hilbert space. Then there exists a Hilbert space $\tilde{H} \supset H$ and $\varphi : \mathcal{L}(H) \to \mathcal{L}(\tilde{H})$ $(A \mapsto \tilde{A})$ satisfying:*

*φ is an *-isometric isomorphism preserving the order such that*

(i) $\varphi(A^*) = \varphi(A)^*$, $\varphi(I) = \tilde{I}$, $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$,

$\varphi(AB) = \varphi(A)\varphi(B)$, $||\varphi(A)|| = ||A||$, $\varphi(A) \leq \varphi(B)$, if $A \leq B$, for all $A, B \in \mathcal{L}(H)$, $\alpha, \beta \in \mathbb{C}$.

(ii) $\sigma(A) = \sigma(\tilde{A})$ and $\sigma_a(A) = \sigma_a(\tilde{A}) = \sigma_p(\tilde{A})$, where $\sigma_a(A)$ is the approximate spectrum of A and $\sigma_p(A)$ is the point spectrum of A.

In the following theorems we will present some classes of finite operators.

Lemma 2.2. *If S is class $\mathcal{Y}$, then $\sigma_{ar}(S) \neq \phi$.*

Proof. It is known that $\sigma_{ar}(S) \subset \sigma_a(S)$. Since $\sigma_a(S) \neq \phi$, it suffices to prove that $\sigma_a(S) \subset \sigma_{ar}(S)$. If $S \in \mathcal{Y}$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$\| \mid SS^* - S^*S \mid^{\frac{\alpha}{2}} x \| \leq k_\alpha \| (S - \lambda I)x \| \text{ for all } x \in H \text{ and for all } \lambda \in \mathbb{C}.$$

Since

$$(S - \mu I)(S - \mu I)^* = SS^* - S^*S + (S - \mu I)^*(S - \mu I) \text{ for all } \mu \in \mathbb{C},$$

then

$$\mid \langle (SS^* - S^*S)x, x \rangle \mid \leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2, \text{ for all } x \in H.$$

Indeed, consider the polar decomposition of the operator $SS^* - S^*S = VD$, where $D = |SS^* - S^*S|$. Then V is a Hermitian partial isometry which commutes with D because $SS^* - S^*S$ is Hermitian. Hence, for any $x \in H$ such that $\|x\| = 1$

$$\begin{aligned} |\langle (SS^* - S^*S)\, x, x\rangle| &\leq \left| \left\langle |SS^* - S^*S|^{\frac{1}{2}}\, x, |SS^* - S^*S|^{\frac{1}{2}}\, V^* x \right\rangle \right| \\ &\leq \left\| |SS^* - S^*S|^{\frac{1}{2}}\, x \right\| \left\| |SS^* - S^*S|^{\frac{1}{2}}\, V^* x \right\| \end{aligned}$$

$$= \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\| \left\| V^* |SS^* - S^*S|^{\frac{1}{2}} x \right\| \leq \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2 .$$

Consequently

$$\|(S - \mu I)^* x\|^2 \leq \|(S - \mu I)x\|^2 + \left\| |SS^* - S^*S|^{\frac{1}{2}} x \right\|^2 , \tag{3}$$

for all $\mu \in \mathbb{C}$ and for all $x \in H$. Let $\lambda \in \sigma_a(S)$, then there exists a normed sequence $\{x_n\}_n \subset H$ such that $\|(S - \lambda I)x_n\| \to 0$. Therefore for $\lambda = \mu$, $x_n = x$, and for all n we get

$$\begin{aligned} |||SS^* - S^*S|^{\frac{1}{2}} x_n||^{2\alpha} &\leq |||SS^* - S^*S|^{\frac{\alpha}{2}} x_n||^2 ||x_n||^{2(\alpha-1)} \\ &\leq k_\alpha^2 ||(S - \mu I)x_n||^2 ||x_n||^{2(\alpha-1)}. \end{aligned} \tag{4}$$

The first inequality holds by Holder-McCarthy inequality, (i.e., $\langle Tx, x\rangle^\alpha \leq \langle T^\alpha x, x\rangle \langle x, x\rangle^{\alpha-1}$ for all $\alpha \geq 1$ and $T \geq 0$).

By applying (3) and (4) we deduce that

$$\|(S - \mu I)^* x\|^2 \leq \|(S - \mu I)x\|^2 + k_\alpha^{\frac{2}{\alpha}} \|(S - \mu I)x\|^{\frac{2}{\alpha}} , \text{ for all } n.$$

Therefore $\|(S - \mu I)^* x\| \to 0$ and $\lambda \in \sigma_{ar}(S)$, that is, $\sigma_{ar}(S) \neq \phi$. □

Theorem 2.3. *Let $A \in \mathcal{L}(H)$ be class $\mathcal{Y}$. Then A is finite.*

Proof. It is well known [13] that if $\sigma_{ar}(A) \neq \phi$, then A is finite. Hence it suffices to apply the previous lemma. □

Now we are ready to prove that a specraloid operator is finite. For this we need the following lemma.

Lemma 2.4.([13]) *Let $A \in \mathcal{L}(H)$. Then $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$.*

Theorem 2.5. *Let $A \in \mathcal{L}(H)$ be spectraloid. Then A is finite.*

Proof. Since A is spectraloid, we have $\omega(A) = r(A)$. Then there exists $\lambda \in \sigma(A) \subset \overline{W(A)}$ such that $|\lambda| = \omega(A)$. Thus $\lambda \in \partial W(A)$. This implies that $\partial W(A) \cap \sigma(A) \neq \emptyset$. Now by applying Lemma 2.2, we get the result. □

In [7] authors, Furuta, Ito and Yamazaki introduced the class A operators, respectively class $A(k)$ of operators defined as follows: for each $k > 0$, an operator T is class $A(k)$ operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2, \tag{2.1}$$

An operator $T \in \mathcal{L}(H)$ is said to be absolute -k-paranormal if $|||T|^k Tx|| \geq ||Tx||^{k+1}$ for every unit vector $x \in H$. On other hand Fujii, Izumino and Nakamoto [9] introduced p-paranormal operators for $p > 0$ as another generalization of paranormal operators. An operator T is said to be p-paranormal if $|||T|^p U|T^p x|| \geq |||T|^p x||^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. In [7] the authors showed inclusion relations among these classes. Fujii, Jung, S.H. Lee, M.Y.Lee and Nakamoto [8] introduced class $A(p,r)$ as further generalization of class $A(k)$. An operator $T \in A(p,r)$ for $p > 0$ and $r > 0$ if $(|T^*|^r|T^*|^{2p})^{\frac{r}{p+r}} \geq |T^*|^{2r}$ and class $AI(p,r)$ is class of invertible operators which belong to class $A(p,r)$. Yamazaki and Yanagida [24] introduced the notion of absolute-(p,r)-paranormal operators. It is a further generalization of classes of both absolute-k-paranormal operators and p-paranormal operators as a parallel concept of class $A(p,r)$. An operator T is said to be absolute-(p,r)-paranormal if $|||T|^p|T^*|^r x||^r \geq |||T^*|^r x||^{p+r}$ for every unit vector $x \in H$ and for positive real numbers $p > 0$ and $r > 0$ or equivalently $|||T|^p|T^*|^r x||^r||x|| \geq |||T^*|^r x||^{p+r}$ for every $x \in H$ and for positive real numbers $p > 0$ and $r > 0$. Concerning the connections between all these classes we have

$$\text{normal} \Rightarrow \text{hyponormal} \Rightarrow p-\text{hyponormal} \Rightarrow \text{Class}A \Rightarrow \text{paranormal} \Rightarrow$$

$$\Rightarrow \text{absolute-}k\text{-paranormal} \Rightarrow \text{absolute-}(p,r)\text{-paranormal}$$

$$\Rightarrow \text{normaloid} \Rightarrow \text{spectraloid} \Rightarrow \text{finite.}$$

For more details concerning the connections between these operators the reader is referred to [24].

In the following theorem we will show that an absolute-(p,r)-paranormal operator is finite.

Theorem 2.6. *Let $A \in \mathcal{L}(H)$. If A is absolute-(p,r)-paranormal, then A is finite.*
Proof. Since an absolute-(p,r)-paranormal operator A is normaloid [18], hence A is spectraloid. But a spectraloid operator A is finite by Theorem 2.2, then A is finite. □

Lemma 2.7. *If A is class $\mathcal{Y}$ (resp. absolute-(p,r)-paranormal) and T is a normal operator such that $AT = TA$, then for every $\lambda \in \sigma_p(T)$ (point spectrum of T),*

$$|\lambda| \leq ||T - (AX - XA)||, \textit{ for all} X \in \mathcal{L}(H).$$

Proof. Let $\lambda \in \sigma_p(T)$ and M_λ be the eigenspace associated with λ. Since $AT = TA$, we have $T^*A = AT^*$ by Fuglede-Putnam's theorem. Hence M_λ reduces both A and T. According to the decomposition of H, we can write A, T and X as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix} \text{and} X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

where A_1 is class $\mathcal{Y}$ (resp. absolute-(p,r-paranormal)[21, 18]. By applying Theorem 2.1 and Theorem 2.3, we get

$$||T - (AX - XA)|| = \left\| \begin{bmatrix} \lambda - (A_1X_1 - X_1A_1) & * \\ * & * \end{bmatrix} \right\| \geq ||\lambda - (A_1X_1 - X_1A_1)||$$

$$\geq |\lambda| ||1 - (A_1(\frac{X_1}{\lambda}) - (\frac{X_1}{\lambda})A_1)|| \geq |\lambda|$$

□

Theorem 2.8. *If A is class $\mathcal{Y}$ (resp. spectraloid), then for every normal operator T such that $AT = TA$, we have*

$$||T - (AX - XA)|| \geq ||T||, \textit{ for all} X \in \mathcal{L}(H).$$

Hence the range of δ_A is orthogonal to the kernel of δ_A.

Proof. Let $\lambda \in \sigma(T) = \sigma_a(T)$[10], then it follows from Proposition 2.1 that $\tilde{T}$ is normal, $\tilde{A}$ is finite, $\tilde{T}A = \tilde{T}\tilde{A} = \tilde{A}\tilde{T}$ and $\lambda \in \sigma_p(\tilde{A})$. By applying Lemma 2.3, we get for all $X \in \mathcal{L}(H)$

$$|\lambda| \leq ||\tilde{T} - (\tilde{A}\tilde{X} - \tilde{X}\tilde{A})|| = ||T - (AX - XA)||.$$

Hence

$$\sup_{\lambda \in \sigma(\tilde{T})} |\lambda| = ||\tilde{T}|| = ||T|| = r(T) \leq ||T - (AX - XA)||,$$

for all $X \in \mathcal{L}(H)$. □

Theorem 2.9. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$ be a class $\mathcal{Y}$ (resp. absolute-(p,r)-paranormal) operator. Then a is finite.*

Proof. It is known [[11], p.97] that there exists a *-isometric homomorphism φ and a Hilbert space $H(\varphi : \mathcal{A} \longmapsto \mathcal{L}(H)$). Since $a \in \mathcal{A}$ is a class $\mathcal{Y}$ operator (resp. absolute-(p,r)-paranormal operator), $\varphi(a)$ is a class $\mathcal{Y}$ operator (resp. absolute-(p,r)-paranormal operator). Since φ is isometric, it results from the previous theorem that a is finite. □

Corollary 2.10. *Let $A \in \mathcal{L}(H)$ be class $\mathcal{Y}$ (resp. absolute-(p,r)-paranormal). Then $T = A + K$ is finite, where K is a compact operator.*

Proof. Since the Calkin algebra $\mathcal{L}(H)/K(H)$ is a C^*- algebra, $[A] \in \mathcal{L}(H)/K(H)$ is class $\mathcal{Y}$ (resp. absolute-(p,r)-paranormal operator). Hence it follows from Theorem 2.5 that $[A]$ is finite and we have

$$\|I - TX - XT\| \geq \|[I] - [A][X] - [X][A]\| \geq \|I\| = 1.$$ □

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References

[1] J. H. Anderson, *On normal derivation*, Proc. Amer. Math. Soc., **38**(1973), 135-140. MR0312313

[2] J. H. Anderson and C. Foias, *Properties which normal operator share with normal derivation and related operators*, Pacific. Jour. Math., (1973), 313-325.

[3] T. Ando, *Operators with a norm condition*, Acta. Sci. Math.(Szeged), **33**(1972), 169-178.

[4] Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th., **13**(1990), 307-315.

[5] S. K. Berberian, *Approximate proper values*, Proc. Amer. Math. Soc., **13**(1962), 111-114.

[6] M. Fujii, C. Himeji and A. Matsumoto, *Theorems of Ando and Saito for p-hyponormal operators*, Math. Japonica, **39**(1994), 595-598.

[7] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and severel related classes*, Scientiae Mathematicae **1**(1998), 389-403.

[8] M. Fujii, D. Jung, S. H. Lee , M. Y. Lee and R. Nakamoto. *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japan, **51(3)**(2000), 395-402.

[9] M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Holder-Mc. Carthy inequality, Nihonkai Math. J., **1(5)**(1994), 61-67.

[10] P. R. Halmos, Hilbert space problem book, springer Verlag, New York (1962).

[11] D. A. Herrero, *Approximation of Hilbert space operator I*, Pitman Advanced publishing program, Boston, London-Melbourne (1982).

[12] I. H. Jeon, K. Tanahashi and A. Uchiyama,*On quasisimilarity for log-hyponormal operator*, Glasgow Math. J., **46**(2004), 169-176.

[13] S. Mecheri, *Finite operators.*, Demonstratio Math., **35**(2002), 355-366.

[14] S. Mecheri, *Non-normal derivation and orthogonality.*, Proc. Amer. Math. Soc., **133** (2005), 759-762.

[15] S. Mecheri, K. Tanahashi and A. Uchiyama, *Fuglede-Putnam's theorem for class* $\mathcal{Y}$ *or p-hyponormal operators*, Bull. Korean Math. Soc., **4**(2006), 747-753.

[16] S. Mecheri, *Finite Operators and Orthogonality*, Nihonkai Math. J., **19**(2008), 53-60.

[17] S. M. Patel, *O*n Intertwining p-hyponormal operators, Indian J. Math., **38**(1996), 287–290.

[18] D. Senthilkumar, P. Mahaswari, *Weyl's theorem for algebraically absolute-*(p, r)*-paranormal operators*, Banach J. Math. Anal., **5**(2011), 29-37.

[19] K. Tanahashi, *On log-hyponormal operators*, Integral equations Operator Theory., **34**(1999), 364-372.

[20] A. Uchiyama, *Inequalities of Purnam and BergerShaw for p-quasihyponormal operators*, Integr. Equat. Oper. Th., **34**(1999), 179-180.

[21] A. Uchiyama and T. Yoshino,*O*n the class $\mathcal{Y}$ operators, Nihonkai Math. J., **8**(1997), 179–194.

[22] J. P. Williams, *Finite operators.*, Proc. Amer. Math. Soc., **26**(1970), 129–135.

[23] H. Wielandt, *ber die Unbeschrnktheit der Operatoren der Quantenmechanik*. (German) Math. Ann., **121**(1949), 21.

[24] T. Yamazaki and M. Yanagida, A further generalization of paranormal operators, Sci. Math., 2000, 3: 23-31.