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Fuzzy Prime Ideals of Pseudo-ŁBCK-algebras

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Abstract. Pseudo-ŁBCK-algebras are commutative pseudo-BCK-algebras with relative cancellation property. In the paper, we introduce fuzzy prime ideals in pseudo-ŁBCK-algebras and investigate some of their properties. We also give various characterizations of prime ideals and fuzzy prime ideals. Moreover, we present conditions for a pseudo-ŁBCK-algebra to be a pseudo-ŁBCK-chain.

1. Introduction

In 1958, C.C. Chang [1] introduced MV (Many Valued) algebras. In 1966, Y. Imai and K. Iséki [12] introduced the notion of BCK-algebra, an algebraic formulation of the BCK system in combinatory logic. In 1996, P. Hájek ([8], [9]) invented Basic Logic (BL for short) and BL-algebras, structures that correspond to this logical system. The class of BL-algebras contains the MV-algebras. G. Georgescu and A. Iorgulescu [4] (1999), and independently J. Rachůnek [19] introduced pseudo-MV-algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV-algebras, the pseudo-BL-algebras [5] (2000), and the pseudo-BCK-algebras [6] (2001) were introduced and studied. The paper [6] contains basic properties of pseudo-BCK-algebras and their connections with pseudo-MV-algebras and with pseudo-BL-algebras. Y. B. Jun [15] obtained some char-

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acterizations of pseudo-BCK-algebras. A. Iorgulescu ([13], [14]) studied particular classes of pseudo-BCK-algebras. A. Walendziak [21] considered maximal ideals in pseudo-BCK-algebras. J. Kühr ([17], [18]) investigated commutative pseudo-BCK-algebras with the relative cancellation property (pseudo-ŁBCK-algebras).

The concept of a fuzzy set was introduced by L. A. Zadeh [23]. Since then this idea has been applied to other algebraic structures such as semigroups, groups, rings, modules, vector spaces and topologies. Fuzzy ideals of BCK-algebras were introduced by O. G. Xi in [22]. Recently, we applied the concept of a fuzzy ideal to pseudo-BCK-algebras ([2]). In this paper, we introduce and investigate the notion of a fuzzy prime ideal in pseudo-ŁBCK-algebras. We give various characterizations of fuzzy prime ideals and establish the so called prime extension property for fuzzy ideals. Moreover, using the concept of a fuzzy prime ideal we present conditions for an ideal to be prime and also for a pseudo-ŁBCK-algebra to be a pseudo-ŁBCK-chain.

2. Preliminaries

The notion of pseudo-BCK-algebras is defined by Georgescu and Iorgulescu [6] as follows:

Definition 2.1. A pseudo-BCK-algebra is a structure $(A; \leq, *, \circ, 0)$, where "$\leq$" is a binary relation on a set A, "$*$" and "$\circ$" are binary operations on A and "0" is an element of A, verifying the axioms: for all $x, y, z \in A$,
(pBCK-1) $(x * y) \circ (x * z) \leq z * y, \quad (x \circ y) * (x \circ z) \leq z \circ y$,
(pBCK-2) $x * (x \circ y) \leq y, \quad x \circ (x * y) \leq y$,
(pBCK-3) $x \leq x$,
(pBCK-4) $0 \leq x$,
(pBCK-5) $(x \leq y \text{ and } y \leq x) \;\Rightarrow\; x = y$,
(pBCK-6) $x \leq y \Leftrightarrow x * y = 0 \Leftrightarrow x \circ y = 0$.

Note that every pseudo-BCK-algebra satisfying $x * y = x \circ y$ for all $x, y \in A$ is a BCK-algebra.

Example 2.2.([10], Example 2.4) Let $A = \{0, a, b, c\}$ and define binary operations "$*$" and "$\circ$" on A by the following tables:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	b	b	0

$\circ$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	a	0

Then $(A; \leq, *, \circ, 0)$ is a pseudo-BCK-algebra, where $0 < a < b < c$.

Let $(A; \leq, *, \circ, 0)$ be a pseudo-BCK-algebra. Then the algebra $(A; *, \circ, 0)$ satisfies the following identities and quasi-identity:
(A1) $[(x * y) \circ (x * z)] \circ (z * y) = 0$,

(A2) $[(x \circ y) * (x \circ z)] * (z \circ y) = 0$,
(A3) $x * 0 = x$,
(A4) $x \circ 0 = x$,
(A5) $0 * x = 0$,
(A6) $x * y = 0 = y * x \Rightarrow x = y$.

By the proof of Theorem 1.1.10 from [17], if $(A; *, \circ, 0)$ is an algebra of type $\langle 2, 2, 0\rangle$ satisfying (A1)-(A6), then the relation $\leq$ defined by

$$x \leq y \Leftrightarrow x * y = 0$$

is a partial order making the structure $(A; \leq, *, \circ, 0)$ into a pseudo-BCK-algebra.

Therefore pseudo-BCK-algebras can be treated as pure algebras with binary operations $*$ and $\circ$, and a constant 0 (see [20]).

Definition 2.3. A pseudo-BCK-algebra is *commutative* (see [17], [18]) if it satisfies the identities:

$$x * (x \circ y) = y * (y \circ x),$$
$$x \circ (x * y) = y \circ (y * x).$$

Example 2.4.([18], Example 3.1) Let $(G; +, -, 0, \wedge, \vee)$ be a lattice-ordered group and let $G^+ = \{x \in G : x \geq 0\}$ be its positive cone. Then upon defining

$$x * y = x - (x \wedge y) \quad \text{and} \quad x \circ y = -(x \wedge y) + x,$$

$(G^+; *, \circ, 0)$ is a commutative pseudo-BCK-algebra.

Example 2.5. The pseudo-BCK-algebra A from Example 2.2 is not commutative, since e.g. $a * (a \circ b) = a$ while $b * (b \circ a) = 0$.

By Theorem 4.2 of [18] commutative pseudo-BCK-algebras can be defined by the following identities:
(C1) $x \circ (x * y) = y \circ (y * x) = x * (x \circ y) = y * (y \circ x)$,
(C2) $(x \circ y) * z = (x * z) \circ y$,
(C3) $x * x = 0 = x \circ x$,
(C4) $x * 0 = x = x \circ 0$.

Definition 2.6. We say that a commutative pseudo-BCK-algebra $(A; *, \circ, 0)$ satisfies the *relative cancellation property*, (RCP) for short, if for every $a, x, y \in A$,

$$a \leq x, y \quad \text{and} \quad x * a = y * a \quad \text{imply} \quad x = y.$$

The relative cancellation property can be equivalently defined by:

$$a \leq x, y \quad \text{and} \quad x \circ a = y \circ a \quad \text{imply} \quad x = y$$

for all $a, x, y \in A$ (see [18], p. 477).

Example 2.7. The commutative pseudo-BCK-algebra $(G^+; *, \circ, 0)$ from Example 2.4 satisfies (RCP). Indeed, let $a, x, y \in G^+$. Suppose that $a \leq x, y$ and let $x * a = y * a$. Then $x - (x \wedge a) = y - (y \wedge a)$ and hence $x - a = y - a$. Consequently, $x = y$.

Example 2.8. Consider the set $A = \{0, 1, 2, 3\}$ with the binary operation "$*$" given as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	1	1	0

It is easy to see that $(A; *, 0)$ is a commutative BCK-algebra. Observe that $1 < 2, 3$ and $2 * 1 = 3 * 1$ while $2 \neq 3$. Therefore A does not satisfy the (RCP).

We shall refer to commutative pseudo-BCK-algebras with (RCP) briefly as *pseudo-ŁBCK-algebras*. We borrow the name "pseudo-ŁBCK-algebra" from [3] (see also [17] and [18]).

Let $(A; *, \circ, 0)$ be a commutative pseudo-BCK-algebra. From Proposition 1.15 of [6] it follows that A is a meet-semilattice with respect to its natural order (that is, $\inf\{x, y\}$ exists for any two elements x and y), where

$$x \wedge y := \inf\{x, y\} = x \circ (x * y). \tag{1.1}$$

Theorem 6.8 of [18] shows that $(A; *, \circ, 0)$ is a pseudo-ŁBCK-algebra if and only if A satisfies (C1)-(C4) and the following identities:

$$(x * y) \wedge (y * x) = 0 = (x \circ y) \wedge (y \circ x). \tag{1.2}$$

A *pseudo-ŁBCK-chain* is a pseudo-ŁBCK-algebra such that its partial order is linear.

Example 2.9. Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV-algebra and we put $x \odot y = (y^- \oplus x^-)^\sim$. Define

$$x * y = x \odot y^- \quad \text{and} \quad x \circ y = y^\sim \odot x.$$

By 4.1.3 of [17], $(M; *, \circ, 0)$ is a commutative pseudo-BCK-algebra. If we put

$$x \leq y \Leftrightarrow x^- \oplus y = 1,$$

then $(M; \leq)$ is a lattice and applying Proposition 1.24 of [7] we have (for all $x, y \in M$)

$$(x * y) \wedge (y * x) = (x \odot y^-) \wedge (y \odot x^-) = 0$$

and similarly, $(x \circ y) \wedge (y \circ x) = 0$. Thus M is a pseudo-ŁBCK-algebra.

Definition 2.10. Let $(A; *, \circ, 0)$ be a pseudo-BCK-algebra. A subset $I \subseteq A$ is called an *ideal* of A if it satisfies for all $x, y \in A$:

(I1) $0 \in I$,
(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

An ideal I of A is *proper* if $I \neq A$.

Example 2.11. Let A be the pseudo-BCK-algebra from Example 2.2. Then it is easy to see that $\{0\}, \{0, a\}$ and A are the only ideals of A.

Proposition 2.12. Let I be an ideal of a pseudo-BCK-algebra A. For any $x, y \in A$, if $y \in I$ and $x \leq y$, then $x \in I$.

Proof. Straightforward. □

Proposition 2.13. Let I be a subset of a pseudo-BCK-algebra A. Then I is an ideal of A if and only if it satisfies conditions (I1) and

(I2') for all $x, y \in A$, if $x \circ y \in I$ and $y \in I$, then $x \in I$.

Proof. It suffices to prove that if (I2) is satisfied, then (I2') is also satisfied. The proof of the converse of this implication is analogous. Suppose that $x \circ y \in I$ and $y \in I$. From (pBCK-2) we know that $x * (x \circ y) \leq y$. Then, by Proposition 2.12, $x * (x \circ y) \in I$. Hence, since $x \circ y \in I$, by (I2), $x \in I$. □

Let A be a pseudo-BCK-algebra. Denote by $\mathrm{Id}(A)$ the set of all ideals of A. Note that $\mathrm{Id}(A)$ is a distributive lattice.

Definition 2.14. Let A be a pseudo-BCK-algebra. We say that a proper ideal P of A is *prime* if for every $I, J \in \mathrm{Id}(A)$, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Since the lattice $\mathrm{Id}(A)$ is distributive, it is easy to see that $P \in \mathrm{Id}(A)$ is prime if and only if, for all $I, J \in \mathrm{Id}(A)$, $P = I$ or $P = J$ whenever $P = I \cap J$.

Example 2.15.([11]) Let $A = \{0, a, b, c\}$ and define a binary operation "$*$" on A by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $(A; *, 0)$ is a BCK-algebra, so also a pseudo-BCK-algebra. It is easy to see that $\{0, a, b\}$ and $\{0, a, c\}$ are prime ideals of A and $\{0, a\}$ is not a prime ideal of A.

Proposition 2.16.([16], Theorem 3.9) Let $(A; *, \circ, 0)$ be a pseudo-ŁBCK-algebra. Then for any proper ideal P of A, the following are equivalent:
(i) P is prime;

(ii) for all $x, y \in A$, if $x \wedge y \in P$, then $x \in P$ or $y \in P$;
(iii) for all $x, y \in A$, if $x \wedge y = 0$, then $x \in P$ or $y \in P$;
(iv) for all $x, y \in A$, if $x * y \in P$ or $y * x \in P$;
(v) for all $x, y \in A$, if $x \circ y \in P$ or $y \circ x \in P$.

From Proposition 3.2.5 of [17] we obtain

Proposition 2.17. Let J be a proper ideal of a pseudo-LBCK-algebra A and $a \in A - J$. Then there is a prime ideal P of A such that $J \subseteq P$ and $a \notin P$.

3. Fuzzy Prime Ideals

We now review some fuzzy logic concepts. First, for $\Gamma \subseteq [0;1]$ we define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a *fuzzy set* in A is a function $\mu : A \to [0;1]$.

For any fuzzy sets μ and ν in a pseudo-BCK-algebra A, we define

$$\mu \leq \nu \text{ iff } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

It is easy to check that this relation is an order relation in the set of fuzzy sets in A.

Definition 3.1.([2]) A fuzzy set μ in a pseudo-BCK-algebra A is called a *fuzzy ideal* of A if it satisfies for all $x, y \in A$:
(d1) $\mu(0) \geq \mu(x)$,
(d2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$.

Proposition 3.2. Let μ be a fuzzy ideal of a pseudo-BCK-algebra A. For any $x, y \in A$, if $x \leq y$, then $\mu(x) \geq \mu(y)$.

Proof. If $x \leq y$, then $x * y = 0$. Hence, by (d2), we have $\mu(x) \geq \mu(x * y) \wedge \mu(y) = \mu(0) \wedge \mu(y) = \mu(y)$. □

Denote by $\mathcal{FI}(A)$ the set of all fuzzy ideals of a pseudo-BCK-algebra A.

Example 3.3. Let A be the pseudo-BCK-algebra from Example 2.2. Let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = 0, \\ \alpha_2 & \text{if } x = a, \\ \alpha_3 & \text{if } x \in \{b, c\}. \end{cases}$$

It is easily checked that μ satisfies (d1) and (d2). Thus $\mu \in \mathcal{FI}(A)$.

Example 3.4. Let I be an ideal of a pseudo-BCK-algebra A and let $\alpha, \beta \in [0,1]$, with $\alpha > \beta$. Define $\mu_I^{\alpha,\beta}$ as follows:

$$\mu_I^{\alpha,\beta}(x) := \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise.} \end{cases}$$

We denote $\mu_I^{\alpha,\beta} = \mu$. Since $0 \in I$, $\mu(0) = \alpha \geq \mu(x)$ for all $x \in A$. To prove (d2), let $x, y \in A$. If $x \in I$, then $\mu(x) = \alpha \geq \mu(x * y) \wedge \mu(y)$. Suppose now that $x \notin I$. By the definition of an ideal, $x * y \notin I$ or $y \notin I$. Therefore, $\mu(x * y) \wedge \mu(y) = \beta = \mu(x)$. Thus μ is a fuzzy ideal of A.
In particular the characteristic function χ_I of I:

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

is the fuzzy ideal of A.

Proposition 3.5.([2], Proposition 3.5) A fuzzy set μ in a pseudo-BCK-algebra A is a fuzzy ideal of A if and only if it satisfies (d1) and

(d2') $\mu(x) \geq \mu(x \circ y) \wedge \mu(y)$ for all $x, y \in A$.

Proposition 3.6.([2], Theorem 3.8) Let μ be a fuzzy set in a pseudo-BCK-algebra A. Then $\mu \in \mathfrak{FI}(A)$ if and only if its nonempty level subset

$$U(\mu; \alpha) := \{x \in A : \mu(x) \geq \alpha\}$$

is an ideal of A for all $\alpha \in [0; 1]$.

Corollary 3.7. If μ is a fuzzy ideal of a pseudo-BCK-algebra A, then the set

$$A_b := \{x \in A : \mu(x) \geq \mu(b)\}$$

is an ideal of A for every $b \in A$.

By Corollary 3.7, we have the following.

Corollary 3.8. If μ is a fuzzy ideal of a pseudo-BCK-algebra A, then the set

$$A_\mu := \{x \in A : \mu(x) = \mu(0)\}$$

is an ideal of A.

Example 3.9. Let μ be as in Example 3.3. One can easily check that for all $\alpha \in [0, 1]$ we have

$$U(\mu; \alpha) = \begin{cases} \emptyset & \text{if } \alpha > \alpha_1, \\ \{0\} & \text{if } \alpha_2 < \alpha \leq \alpha_1, \\ \{0, a\} & \text{if } \alpha_3 < \alpha \leq \alpha_2, \\ A & \text{if } \alpha \leq \alpha_3. \end{cases}$$

Since $\{0\}, \{0, a\}$ and A are all ideals of A, this is an another proof (by Theorem 3.6) that μ is a fuzzy ideal of A.

Let A be a pseudo-BCK-algebra. Let $\mu_t \in \mathfrak{FI}(A)$ for $t \in T$. The meet $\bigwedge_{t \in T} \mu_t$ of fuzzy ideals μ_t of A is defined as follows:

$$\left(\bigwedge_{t \in T} \mu_t\right)(x) = \bigwedge\{\mu_t(x) : t \in T\}.$$

Proposition 3.10.([2], Theorem 3.14) Let $\mu_t \in \mathfrak{FI}(A)$ for $t \in T$. Then $\bigwedge_{t\in T} \mu_t \in \mathfrak{FI}(A)$.

Now, we introduce and study fuzzy prime ideals in a pseudo-ŁBCK-algebra.

Definition 3.11. A fuzzy ideal μ of a pseudo-ŁBCK-algebra A is said to be *fuzzy prime* if it is non-constant and satisfies:

$$\mu\,(x \wedge y) = \mu\,(x) \vee \mu\,(y)$$

for all $x, y \in A$.

Theorem 3.12. *Let μ be a non-constant fuzzy ideal of a pseudo-ŁBCK-algebra A. Then the following are equivalent:*
(i) *μ is a fuzzy prime ideal of A,*
(ii) *for all $x, y \in A$, if $\mu\,(x \wedge y) = \mu\,(0)$, then $\mu\,(x) = \mu\,(0)$ or $\mu\,(y) = \mu\,(0)$,*
(iii) *for all $x, y \in A$, $\mu(x * y) = \mu(0)$ or $\mu(y * x) = \mu(0)$,*
(iv) *for all $x, y \in A$, $\mu(x \circ y) = \mu(0)$ or $\mu(y \circ x) = \mu(0)$.*

Proof. (i) $\Rightarrow$ (ii): Assume that μ is a fuzzy prime ideal of A. Let $x, y \in A$ be such that $\mu\,(x \wedge y) = \mu\,(0)$. Then $\mu\,(x) \vee \mu\,(y) = \mu\,(x \wedge y) = \mu\,(0)$ and hence $\mu\,(x) = \mu\,(0)$ or $\mu\,(y) = \mu\,(0)$.

(ii) $\Rightarrow$ (iii): Since A satisfies (1.2), $(x * y) \wedge (y * x) = 0$. Then $\mu((x * y) \wedge (y * x)) = \mu(0)$ and by (ii), we obtain $\mu(x * y) = \mu(0)$ or $\mu(y * x) = \mu(0)$.

(iii) $\Rightarrow$ (i): Let $x, y \in A$. Suppose that, for instance, $\mu(x * y) = \mu(0)$. By (1.1), $x \wedge y = x \circ (x * y)$ and hence applying Proposition 3.5 we obtain $\mu(x) \geq \mu(x \wedge y) \wedge \mu(x * y)$. Then $\mu(x) \geq \mu(x \wedge y)$. Since $x \wedge y \leq x$, we conclude that $\mu(x) \leq \mu(x \wedge y)$. Consequently, $\mu(x) = \mu(x \wedge y)$. From (d2) it follows that $\mu(x) \geq \mu(x * y) \wedge \mu(y)$. Then $\mu(x) \geq \mu(0) \wedge \mu(y) = \mu(y)$. Finally, we have $\mu(x \wedge y) = \mu(x) = \mu(x) \vee \mu(y)$. So μ is a fuzzy prime ideal of A.

Analogously, the implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) can be proved. □

Theorem 3.13. *Let A be a pseudo-ŁBCK-algebra and let $\mu \in \mathfrak{FI}(A)$. Then μ is a fuzzy prime ideal of A if and only if A_μ is a prime ideal of A.*

Proof. Suppose that μ is a fuzzy prime ideal of a pseudo-ŁBCK-algebra A. Since μ is non-constant, A_μ is proper. Let $x, y \in A$ and $x \wedge y \in A_\mu$. Then $\mu\,(0) = \mu\,(x \wedge y) = \mu\,(x) \vee \mu\,(y)$. Hence $\mu\,(x) = \mu\,(0)$ or $\mu\,(y) = \mu\,(0)$. This means that $x \in A_\mu$ or $y \in A_\mu$. Therefore, by Proposition 2.16, A_μ is a prime ideal of A.

Conversely, assume that A_μ is a prime ideal of A. Since A_μ is proper, μ is non-constant. Let $x, y \in A$. From (1.2) we have $(x * y) \wedge (y * x) = 0 \in A_\mu$. Hence, by Proposition 2.16, $x * y \in A_\mu$ or $y * x \in A_\mu$, i.e., $\mu(x * y) = \mu(0)$ or $\mu(y * x) = \mu(0)$. Thus, by Theorem 3.12, μ is a fuzzy prime ideal of A. □

Corollary 3.14. If μ is a fuzzy prime ideal of a pseudo-ŁBCK-algebra A, then the set

$$\mathrm{Ker}\mu = \{x \in A : \mu\,(x) = 1\}$$

is either empty or a prime ideal of A.

Theorem 3.15. *Let A be a pseudo-ŁBCK-algebra, $P \in \mathrm{Id}(A)$ and $\alpha, \beta \in [0,1]$ with $\alpha > \beta$. Then P is a prime ideal of A if and only if $\mu_P^{\alpha,\beta}$ is a fuzzy prime ideal of A.*

Proof. Assume that P is a prime ideal of a pseudo-ŁBCK-algebra A. Since P is proper, $\mu_P^{\alpha,\beta}$ is non-constant. Let $x, y \in A$. Then, by (1.2), $(x*y) \wedge (y*x) = 0 \in P$. Hence, by Proposition 2.16, $x * y \in P$ or $y * x \in P$, i.e., $\mu_P^{\alpha,\beta}(x*y) = \alpha = \mu_P^{\alpha,\beta}(0)$ or $\mu_P^{\alpha,\beta}(y*x) = \alpha = \mu_P^{\alpha,\beta}(0)$. Thus, by Theorem 3.12, $\mu_P^{\alpha,\beta}$ is a fuzzy prime ideal of A.

Conversely, assume that $\mu_P^{\alpha,\beta}$ is a fuzzy prime ideal of A. Then, by Theorem 3.13, $P = A_{\mu_P^{\alpha,\beta}}$ is a prime ideal of A. □

Corollary 3.16. Let A be a pseudo-ŁBCK-algebra and let $P \in \mathrm{Id}(A)$. Then P is a prime ideal of A if and only if χ_P is a fuzzy prime ideal of A.

Theorem 3.17. *Let A $(\neq \{0\})$ be a pseudo-ŁBCK-algebra. Then the following are equivalent:*

(i) *A is a pseudo-ŁBCK-chain,*

(ii) *every non-constant fuzzy ideal of A is fuzzy prime,*

(iii) *every non-constant fuzzy ideal μ of A such that $\mu(0) = 1$ is fuzzy prime,*

(iv) *the fuzzy ideal $\chi_{\{0\}}$ of A is fuzzy prime.*

Proof. (i) ⇒ (ii): Assume that A is a pseudo-ŁBCK-chain and μ is a non-constant fuzzy ideal of A. Then for any $x, y \in A$, $x \leq y$ or $y \leq x$. Hence, $x * y = 0$ or $y * x = 0$. Thus $\mu(x*y) = \mu(0)$ or $\mu(y*x) = \mu(0)$. Therefore, by Theorem 3.12, μ is a fuzzy prime ideal of A.

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (iv): Obvious.

(iv) ⇒ (i): Let $x, y \in A$. Since $\chi_{\{0\}}$ is a fuzzy prime ideal of A, we have, by Theorem 3.12, $\chi_{\{0\}}(x*y) = \chi_{\{0\}}(0) = 1$ or $\chi_{\{0\}}(y*x) = \chi_{\{0\}}(0) = 1$. Hence $x * y \in \{0\}$ or $y * x \in \{0\}$, i.e., $x * y = 0$ or $y * x = 0$. Thus, $x \leq y$ or $y \leq x$. It follows that A is a pseudo-ŁBCK-chain. □

Theorem 3.18. *Let μ be a non-constant fuzzy set in a pseudo-ŁBCK-algebra A. Then μ is a fuzzy prime ideal of A if and only if for each $\alpha \in [0,1]$, $U(\mu;\alpha)$ is empty or is a prime ideal of A if it is proper.*

Proof. Assume μ is a fuzzy prime ideal of A. For each $\alpha \in [0,1]$, if $U(\mu;\alpha) \neq \emptyset$, then it is an ideal of A, by Proposition 3.6. Let $x, y \in A$. If $U(\mu;\alpha)$ is proper and $x \wedge y \in U(\mu;\alpha)$, then $\mu(x) \vee \mu(y) = \mu(x \wedge y) \geq \alpha$. Hence $\mu(x) \geq \alpha$ or $\mu(y) \geq \alpha$, i.e., $x \in U(\mu;\alpha)$ or $y \in U(\mu;\alpha)$. Therefore, by Proposition 2.16, $U(\mu;\alpha)$ is a prime ideal of A.

Conversely, let $x, y \in A$. Assume that $\mu(x \wedge y) > \mu(x) \vee \mu(y)$. Take

$$\beta = \frac{1}{2}\left[\mu(x \wedge y) + (\mu(x) \vee \mu(y))\right].$$

Then $\mu(x \wedge y) > \beta > \mu(x) \vee \mu(y)$ and hence $x \wedge y \in U(\mu;\beta)$, $x \notin U(\mu;\beta)$ and $y \notin U(\mu;\beta)$. This is a contradiction. Therefore μ is a fuzzy prime ideal of A. □

Theorem 3.19. (Prime extension property for fuzzy ideals) *Let μ be a fuzzy prime ideal of a pseudo-ŁBCK-algebra A and ν any non-constant fuzzy ideal of A such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. Then ν is a fuzzy prime ideal of A.*

Proof. Let $x, y \in A$. Since μ is a fuzzy prime ideal of A, we conclude from Theorem 3.12 that $\mu(x * y) = \mu(0)$ or $\mu(y * x) = \mu(0)$. Let $\mu(x * y) = \mu(0)$. Then $\nu(x * y) = \nu(0)$, because $\mu \leq \nu$ and $\mu(0) = \nu(0)$. Similarly, if $\mu(y * x) = \mu(0)$, then $\nu(y * x) = \nu(0)$. Therefore, from Theorem 3.12 it follows that ν is a fuzzy prime ideal of A. □

Let $0 \leq t < 1$ be a real number. If $\alpha \in [0, 1]$, α^t shall mean the positive root. Let $\mu : A \to [0, 1]$ be a fuzzy set in a pseudo-BCK-algebra A. We define $\mu^t : A \to [0, 1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in A$. It is easily verified that if μ is a fuzzy ideal of A, then so is μ^t, and if $\mu(0) = 1$, then $A_{\mu^t} = A_\mu$.

Theorem 3.20. *Let μ be a fuzzy prime ideal of a pseudo-ŁBCK-algebra A such that $\mu(0) = 1$. Then for every $0 \leq t < 1$, μ^t is a fuzzy prime ideal of A.*

Proof. Since μ is non-constant, μ^t is also non-constant. Next, we have $\mu^t(0) = (\mu(0))^t = 1 = \mu(0)$ and $\mu \leq \mu^t$. This means, by Theorem 3.19, that μ^t is a fuzzy prime ideal of A. □

Theorem 3.21. *Let μ be a fuzzy prime ideal of a pseudo-ŁBCK-algebra A and $\alpha \in [0, \mu(0))$. Then $\mu \vee \alpha$ is a fuzzy prime ideal of A, where $(\mu \vee \alpha)(x) = \mu(x) \vee \alpha$.*

Proof. First, we prove that $\mu \vee \alpha$ is a fuzzy ideal of A. Let $x, y \in A$. We have $(\mu \vee \alpha)(0) = \mu(0) \vee \alpha \geq \mu(x) \vee \alpha = (\mu \vee \alpha)(x)$. Since $\mu(x) \geq \mu(x * y) \wedge \mu(y)$, we conclude that $\mu(x) \vee \alpha \geq (\mu(x * y) \vee \alpha) \wedge (\mu(y) \vee \alpha)$ and hence $(\mu \vee \alpha)(x) \geq (\mu \vee \alpha)(x * y) \wedge (\mu \vee \alpha)(y)$. Therefore $\mu \vee \alpha \in \mathfrak{FI}(A)$.

Since μ is non-constant, $\mu(x_0) < \mu(0)$ for some $x_0 \in A$. Then $(\mu \vee \alpha)(x_0) = \mu(x_0) \vee \alpha < \mu(0) \leq \mu(0) \vee \alpha = (\mu \vee \alpha)(0)$. This shows that $\mu \vee \alpha$ is a non-constant fuzzy ideal. Finally, since $(\mu \vee \alpha)(0) = \mu(0)$ and $\mu \leq \mu \vee \alpha$, we conclude from Theorem 3.19 that $\mu \vee \alpha$ is a fuzzy prime ideal of A. □

Theorem 3.22. *Let μ be a non-constant fuzzy ideal of a pseudo-ŁBCK-algebra A and $\mu(0) \neq 1$. Then there is a fuzzy prime ideal ν of A such that $\mu \leq \nu$.*

Proof. Since μ is a non-constant fuzzy ideal of A, we have A_μ is a proper ideal of A. Hence, by Proposition 2.17, there is a prime ideal P of A such that $A_\mu \subseteq P$. By Corollary 3.16, χ_P is a fuzzy prime ideal of A. Let $\nu = \chi_P \vee \alpha$, where $\alpha = \bigvee \{\mu(x) : x \in A - P\}$. Then $\alpha \leq \mu(0) < 1$. From Theorem 3.21 we see that ν is a fuzzy prime ideal of A. Moreover, $\mu(x) \leq \nu(x)$ for all $x \in A$. □

Theorem 3.23. *Let μ be a non-constant fuzzy ideal of a pseudo-ŁBCK-algebra A such that for any fuzzy ideals μ_1, μ_2 of A, $\mu_1 \wedge \mu_2 \leq \mu$ implies $\mu_1 \leq \mu$ or $\mu_2 \leq \mu$. Then μ is a fuzzy prime ideal of A.*

Proof. Assume that μ is not a fuzzy prime ideal of A. By Theorem 3.13, A_μ is not a prime ideal of A. Then there are $J_1, J_2 \in \mathrm{Id}(A)$ such that $A_\mu = J_1 \cap J_2$,

$A_\mu \subset J_1$, and $A_\mu \subset J_2$. Let $x_1, x_2 \in A$ with $x_1 \in J_1 - A_\mu$ and $x_2 \in J_2 - A_\mu$. Hence $\mu(x_1) < \mu(0)$ and $\mu(x_2) < \mu(0)$. Define fuzzy sets μ_1 and μ_2 in A as follows:

$$\mu_1(x) = \begin{cases} \mu(0) & \text{if } x \in J_1, \\ 0 & \text{if } x \notin J_1, \end{cases}$$
$$\mu_2(x) = \begin{cases} \mu(0) & \text{if } x \in J_2, \\ 0 & \text{if } x \notin J_2. \end{cases}$$

It is easy to prove that μ_1, μ_2 are fuzzy ideals and $\mu_1 \wedge \mu_2 \leq \mu$. Since $\mu_1(x_1) = \mu(0) > \mu(x_1)$ and $\mu_2(x_2) = \mu(0) > \mu(x_2)$, we see that $\mu_1 \nleq \mu$ and $\mu_2 \nleq \mu$. This is a contradiction. Thus μ is a fuzzy prime ideal of A. □

The converse of Theorem 3.23 does not hold, which is shown in the following example.

Example 3.24. Let A be a pseudo-BCK-algebra from Example 2.15. It is not difficult to verify that A is a pseudo-ŁBCK-algebra. Let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. Define a fuzzy ideal μ of A by

$$\mu(x) = \begin{cases} \alpha_2 & \text{if } x \in \{0, a, b\}, \\ \alpha_3 & \text{if } x = c. \end{cases}$$

Then μ is fuzzy prime. Define fuzzy ideals μ_1, μ_2 of A as follows:

$$\mu_1(x) = \alpha_2 \text{ for all } x \in A,$$
$$\mu_2(x) = \begin{cases} \alpha_1 & \text{if } x \in \{0, a\}, \\ \alpha_3 & \text{if } x \in \{b, c\}. \end{cases}$$

Then $\mu_1 \wedge \mu_2 \leq \mu$, but $\mu_1 \nleq \mu$ and $\mu_2 \nleq \mu$.

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