KYUNGPOOK Math. J. 55(2015), 29-39
http://dx.doi.org/10.5666/KMJ.2015.55.1.29
pISSN 1225-6951 eISSN 0454-8124
© Kyungpook Mathematical Journal

Functional Equations associated with Generalized Bernoulli Numbers and Polynomials

Cheon Seoung Ryoo
Department of Mathematics, Hannam University, Daejeon 306-791, Korea
e-mail : ryoocs@hnu.ac.kr

Dmitry Victorovich Dolgy
Hanrimwon, Kwangwoon University, Seoul 139-701, Korea
School of Natural Sciences, Far Eeastern Federal University, Vladivostok, Russia
e-mail : d_dol@mail.ru

Hyuck In Kwon
Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea
e-mail : sura@kw.ac.kr

Yu Seon Jang*
Department of Applied Mathematics, Kangnam University, Yongin 446-702, Korea
e-mail : ysjang@kangnam.ac.kr

Abstract. In this paper, we investigate the functional equations of the multiple Dirichlet and Hurwitz L-functions associated with Bernoulli numbers and polynomials attached to Dirichlet character.

1. Introduction

Euler's zeta function is defined for any real number s greater than 1 by the infinite sum:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

It connects by a continuous parameter all series from (1.1). In 1734 Leonhard

* Corresponding Author.
Received April 22, 2014; accepted December 11, 2014.
2010 Mathematics Subject Classification: 11B68, 11S80.
Key words and phrases: Euler zeta function, Dirichlet L-series, Hurwitz L-function, Generalized Bernoulli numbers and polynomials attached to χ.

Euler (1707-1783) found something amazing; namely he determined all values $\zeta(2n)$, $n \in \mathbb{N}$, a truly remarkable discovery.Ref. [5], [9] He also found a beautiful relationship between prime numbers and $\zeta(s)$ whose significance for current mathematics cannot be overestimated.Ref. [1], [5] This can be written more concisely as an infinite product over all primes p:

$$\zeta(s) = \prod_{p\,:\,\text{prime}} \frac{1}{1-p^{-s}}, \quad s > 1. \tag{1.2}$$

Many authors have studied various functions in connection with the Euler's zeta function under various hypotheses.Ref. [7], [11]-[14], [16], [17] Bernhard Riemann (1826-1866) extended the $\zeta(s)$ as a function of a complex variable $s = x + iy$ rather than a real variable s by meromorphic function.Ref. [5] In the half plane Re$(s)> 1$ the zeta function is given explicitly by series (1.1), and it is therefore subject to the estimate $|\zeta(s)| \le \zeta(\mathrm{Re}(s))$. Riemann recognized that there is a rather simple relationship between $\zeta(s)$ and $\zeta(1-s)$ as following;

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \tag{1.3}$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ is the gamma function of s. it has good control of the behavior of the zeta function also in the half plane Re$(s)< 0$.Ref. [2], [20]

In number theory, Dirichlet characters are certain arithmetic functions which arise from completely multiplicative characters on the units of $\mathbb{Z}/k\mathbb{Z}$. A Dirichlet character is any function χ from the integers $\mathbb{Z}$ to the complex numbers $\mathbb{C}$ such that χ has the following properties:
(i) There exists a positive integer τ such that $\chi(n) = \chi(n+\tau)$ for all n.
(ii) If $\gcd(n, \tau) > 1$ then $\chi(n) = 0$; if $\gcd(n, \tau) = 1$ then $\chi(n) \neq 0$.
(iii) $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n.

The unique character of period 1 is called the trivial character and the smallest positive integer τ in (i) and (ii) is called the conductor of χ. Dirichlet characters are used to define Dirichlet L-functions, which are meromorphic functions with a variety of interesting analytic properties. If χ is a Dirichlet character, The L-series attached to χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \mathrm{Re}(s) > 1. \tag{1.4}$$

This function can be extended to a meromorphic function on the whole complex plane and are generalizations of the Riemann zeta-function. This can be expressed by the partial zeta functions as follows;Ref. [18], [19]

$$L(s,\chi) = \sum_{a=1}^{\tau} \chi(a)\tau^{-s}\zeta\left(s, \frac{a}{\tau}\right), \tag{1.5}$$

where $\zeta(s,x)$ are Hurwitz zeta function defined by

$$\zeta(s,x)=\sum_{n=1}^{\infty}\frac{1}{(n+x)^s},\quad \mathrm{Re}(s)>1,\ x>0. \tag{1.6}$$

The generalized Bernoulli numbers attached to χ, $B_{n,\chi}, n=0,1,\cdots$, are defined by the exponential generating function to be

$$\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}=\sum_{a=1}^{\tau}\frac{\chi(a)te^{at}}{e^{\tau t}-1},\quad |t|<\frac{2\pi}{\tau}. \tag{1.7}$$

There is a relation between $B_{n,\chi}$ and $B_n(x)$ as following;

$$B_{n,\chi}=\tau^{n-1}\sum_{a=1}^{\tau}\chi(a)B_n\left(\frac{a}{\tau}\right), \tag{1.8}$$

where $B_n(x), n=0,1,\cdots$ are Bernoulli polynomials defined by

$$\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt},\quad |t|<2\pi. \tag{1.9}$$ Ref. [3], [6], [8], [15]

For the special values at non-positive integers $s=1-n$ $(n=1,2,\cdots)$, $L(s,\chi)$ can be expressed by the generalized Bernoulli numbers $B_{n,\chi}$ as following;

$$L(1-n,\chi)=-\frac{B_{n,\chi}}{n}. \tag{1.10}$$

This was found by many authors including Washington.Ref. [4], [10], [20], [21] In this paper, we investigate the functional equations of the multiple Dirichlet and Hurwitz L-functions associated with Bernoulli numbers and polynomials attached to Dirichlet character.

2. Functional Equations

For any natural number $k\in\mathbb{N}$, the high-order Bernoulli polynomials with order k, $B_n^{(k)}(x), n=0,1,\cdots$, are defined by the exponential generating functions to be

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}\right)^k e^{xt}. \tag{2.1}$$

When $x=0$, the numbers $B_n^{(k)}=B_n^{(k)}(0)$, $n=1,2,\cdots$ are called the higher-order Bernoulli numbers with order k. In complex plane $\mathbb{C}$, the gamma function is defined as an improper integral for Re$(s)>0$

$$\Gamma(s)=\int_0^{\infty}t^{s-1}e^{-t}dt. \tag{2.2}$$

Replacing s by $1-s$, we know that for Re$(s)<1$

$$\Gamma(1-s)=\int_0^\infty \frac{e^{-t}}{t^s}dt. \tag{2.3}$$

For $s=n\in\mathbb{N}$ in (2.3), since the complex function e^{-z}/z^n has a pole of order n at 0, from the Cauchy Residue Theorem, we know that the contour integral is

$$\oint_{C_0}\frac{e^{-z}}{z^n}dz=2\pi i\lim_{z\to 0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[z^n f(z)\right]=\frac{(-1)^{n-1}}{(n-1)!}2\pi i, \tag{2.4}$$

where C_0 is any circle centered at 0 and $i=\sqrt{-1}$. Observe that for any complex s with Re$(s)>1$

$$\int_0^\infty \frac{t}{e^t-1}t^{s-1}dt=\sum_{m=0}^\infty \frac{1}{(m+1)^s}\int_0^\infty e^{-y}y^{s-1}dy. \tag{2.5}$$

So we have

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{1}{e^t-1}t^{s-2}dt. \tag{2.6}$$

Consider the Generalized Hurwitz zeta functions, the multiple Hurwitz zeta functions with order k are defined by

$$\zeta_k(s,x)=\sum_{n_1,\cdots,n_k=0}^\infty \frac{1}{(n_1+\cdots+n_k+x)^s} \tag{2.7}$$

for Re$(s)>1$ and $x>0$. Observe that

$$\begin{aligned}\zeta_k(s,x)=&\frac{1}{\Gamma(s)}\sum_{n=0}^\infty \binom{n+k-1}{n}\int_0^\infty e^{-(n+x)t}t^{s-1}dt\\ =&\frac{1}{\Gamma(s)}\sum_{n=0}^\infty \binom{n+k-1}{n}\frac{1}{(n+x)^s}\int_0^\infty e^{-y}y^{s-1}dy.\end{aligned} \tag{2.8}$$

Then the generalized Hurwitz zeta function can be expressed as follows;

$$\zeta_k(s,x)=\sum_{n=0}^\infty \binom{n+k-1}{n}\frac{1}{(n+x)^s}. \tag{2.9}$$

Lemma 2.1. *For $k\in\mathbb{N}$, we have*

$$\zeta_k(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty \left(\frac{1}{1-e^{-t}}\right)^k e^{-xt}t^{s-1}dt,$$

where Re$(s)>1$ *and* $x>0$.

Proof. Since

$$\frac{e^t}{e^t-1}=1+\frac{1}{e^t}+\frac{1}{e^{2t}}+\cdots=\sum_{n=0}^{\infty}\left(\frac{1}{e^t}\right)^n,$$

thus we have

(2.10)

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{t}{1-e^{-t}}\right)^k e^{-xt}t^{s-k-1}dt&=\int_0^\infty\left(\sum_{n_1=0}^{\infty}e^{-n_1t}\cdots\sum_{n_k=0}^{\infty}e^{-n_kt}\right)e^{-xt}t^{s-1}dt\\&=\sum_{n_1,\cdots,n_k=0}^{\infty}\int_0^\infty e^{-(n_1+\cdots+n_k+x)t}t^{s-1}dt\\&=\sum_{n_1,\cdots,n_k=0}^{\infty}\frac{1}{(n_1+\cdots+n_k+x)^s}\Gamma(s).\end{aligned}$$

Therefore, dividing $\Gamma(s)$ on the both side of (2.10), we have the desired result. The proof is complete. □

Suppose F is analytic in the annulus $R_1<|z|<R_2$ for some $R_1,R_2(>0)\in\mathbb{R}$. Let

$$H(s)=\oint_C F(z)z^{s-1}dz, \tag{2.11}$$

where the integral is over the following path C consisting of
i) the horizontal line segment I_1 from M to $\sqrt{\varepsilon_2^2-\varepsilon_1^2}+i\varepsilon_1$;
ii) the circular arc $C_{\varepsilon_1,\varepsilon_2}$ of radius ε_2 traced counterclockwise from $\sqrt{\varepsilon_2^2-\varepsilon_1^2}+i\varepsilon_1$ to $\sqrt{\varepsilon_2^2-\varepsilon_1^2}-i\varepsilon_1$;
iii) the horizontal line segment I_2 from $\sqrt{\varepsilon_2^2-\varepsilon_1^2}-i\varepsilon_1$ to M,
where ε_1 and ε_2 ($\varepsilon_1<\varepsilon_2$) are any small numbers and M is arbitrary large. Then $\oint_C=\oint_{I_1}+\oint_{C_{\varepsilon_1,\varepsilon_2}}+\oint_{I_2}$. Using the contour integral on C, for the special values at non-positive integers $s=1-n$ ($n=1,2,\cdots$) we have the relations between $\zeta_k(1-n,x)$ and $B_n^{(k)}(x)$ in the following theorem.

Theorem 2.2. *For* $k,n\in\mathbb{N}$ *and* $x>0$, *we have*

$$\zeta_k(1-n,x)=(-1)^k\frac{(n-1)!}{(n+k-1)!}B_{n+k-1}^{(k)}(x),$$

where $B_j^{(k)}(x), j=0,1,\cdots$ *are the Bernoulli polynomials with order* k.

Proof. Let

$$F(z)=\left(\frac{1}{1-e^{-z}}\right)^k e^{-xz}.$$

Then

$$H(s) = \left(e^{2\pi is} - 1\right) \int_{I_2} F(t)t^{s-1}dt + \oint_{C_{\varepsilon_1,\varepsilon_2}} F(z)z^{s-1}dz.$$

Letting $\varepsilon_2 \to 0$ and $M \to \infty$, we have

$$\oint_{C_{\varepsilon_1,\varepsilon_2}} F(z)z^{s-1}dz = 0$$

and, from Lemma 2.1, we get

$$H(s) = \left(e^{2\pi is} - 1\right) \Gamma(s)\zeta_k(s,x). \tag{2.12}$$

Now, for $s = 1-n$ ($n \in \mathbb{N}$), letting $\varepsilon_1 \to 0$ and taking $\varepsilon_2 > 1$, we have

$$H(1-n) = \oint_{C_{\varepsilon_2}} \frac{F(z)}{z^n}dz = \sum_{m=0}^{\infty}(-1)^m \frac{B_m^{(k)}(x)}{m!} \oint_{C_{\varepsilon_2}} \frac{1}{z^{m-(n+k)}}dz,$$

where C_{ε_2} is circle with radius ε_2 centered at zero. From the Residue theorem, this implies that

$$H(1-n) = (-1)^{n+k-1} 2\pi i \frac{B_{n+k-1}^{(k)}(x)}{(n+k-1)!}. \tag{2.13}$$

And also, from (2.4), we can see easily that

$$\lim_{s\to 1-n} \left(e^{2\pi is} - 1\right) \Gamma(s) = \frac{(-1)^{n-1}}{(n-1)!} 2\pi i. \tag{2.14}$$

Therefore, from (2.12), (2.13) and (2.14), we have the desired result. This is completion of the proof. □

The generalized Bernoulli polynomials attached to χ with the conductor τ are defined by

$$\sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!} = \sum_{a=1}^{\tau} \frac{\chi(a)te^{(a+x)t}}{e^{\tau t}-1}. \tag{2.15}$$

In particular, when $x = 0$, $B_{n,\chi} = B_{n,\chi}(0), n = 0, 1, \cdots$ are the generalized Bernoulli numbers attached to χ. From the definition of $B_{n,\chi}(x)$, we know that

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!} &= \frac{1}{\tau}\sum_{a=1}^{\tau}\chi(a)\left\{\frac{(\tau t)e^{[(a+x)/\tau]\tau t}}{e^{\tau t}-1}\right\} \\ &= \frac{1}{\tau}\sum_{a=1}^{\tau}\chi(a)\sum_{n=0}^{\infty} B_n\left(\frac{a+x}{\tau}\right)\frac{\tau^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left\{\tau^{n-1}\sum_{a=1}^{\tau}\chi(n)B_n\left(\frac{a+x}{\tau}\right)\right\}\frac{t^n}{n!}. \end{aligned} \tag{2.16}$$

So, comparing the both side of (2.16), we have

$$B_{n,\chi}(x) = \tau^{n-1}\sum_{a=1}^{\tau}\chi(n)B_n\left(\frac{a+x}{\tau}\right). \tag{2.17}$$

And also, the complex Hurwitz L-function $L(s,x,\chi)=\sum_{n=1}^{\infty}\chi(n)(n+x)^{-s}$ can be expressed in the following lemma.

Lemma 2.3. *For* $\mathrm{Re}(s)>1$ *and* $x>0$*, we have*

$$L(s,x,\chi)=\frac{1}{\Gamma(s)}\int_0^{\infty}\sum_{a=1}^{\tau}\chi(a)\frac{e^{-(a+x)t}}{1-e^{-\tau t}}t^{s-1}dt,$$

where τ *is the conductor of the Dirichlet character* χ*.*

Proof. Observe that

$$\begin{aligned}\int_0^{\infty}\sum_{a=1}^{\tau}\chi(a)\frac{e^{-(a+x)t}}{1-e^{-\tau t}}t^{s-1}dt &=\sum_{a=1}^{\tau}\chi(a)\int_0^{\infty}\sum_{m=0}^{\infty}e^{-(a+m\tau+x)t}t^{s-1}dt\\ &=\sum_{a=1}^{\tau}\chi(a)\sum_{m=0}^{\infty}\frac{1}{(a+m\tau+x)^s}\int_0^{\infty}e^{-y}y^{s-1}dt\\ &=\sum_{a=1}^{\tau}\sum_{m=0}^{\infty}\frac{\chi(a+m\tau)}{(a+m\tau+x)^s}\Gamma(s).\end{aligned}$$

Since χ is the Dirichlet character with the conductor τ, this implies that

$$\int_0^{\infty}\sum_{a=1}^{\tau}\chi(a)\frac{e^{-(a+x)t}}{1-e^{-\tau t}}t^{s-1}dt=\sum_{n=0}^{\infty}\frac{\chi(n)}{(n+x)^s}\Gamma(s).$$

So the desired result is obtained. The proof is complete. □

Theorem 2.4. *For* $n\in\mathbb{N}$*, we have*

$$L(1-n,x,\chi)=-\frac{B_{n,\chi}(x)}{n},$$

where $B_{n,\chi}(x)$ *are the generalized Bernoulli polynomials attached to* χ*.*

Proof. Let

$$F(z)=\sum_{a=1}^{\tau}\chi(a)\frac{e^{-(a+x)z}}{1-e^{-\tau z}}.$$

Then

$$H(s)=\left(e^{2\pi is}-1\right)\int_{I_2}F(t)t^{s-1}dt+\oint_{C_{\varepsilon_1,\varepsilon_2}}F(z)z^{s-1}dz,$$

where I_1, I_2 and $C_{\varepsilon_1,\varepsilon_2}$ are defined in (2.11). Letting $\varepsilon_2 \to 0$ and $M \to \infty$, we have

$$\oint_{C_{\varepsilon_1,\varepsilon_2}} F(z) z^{s-1} dz = 0$$

and, from Lemma 2.3, we get

$$H(s) = \left(e^{2\pi i s} - 1\right) \Gamma(s) L(s, x, \chi). \tag{2.18}$$

Now, for $s = 1 - n$ ($n \in \mathbb{N}$), letting $\varepsilon_1 \to 0$ and taking $\varepsilon_2 > 1$, we have

$$H(1-n) = \oint_{C_{\varepsilon_2}} \frac{F(z)}{z^n} dz = \sum_{m=0}^{\infty} (-1)^m \frac{B_{n,\chi}(x)}{m!} \oint_{C_{\varepsilon_2}} \frac{1}{z^{n-m+1}} dz,$$

where C_{ε_2} is circle with radius ε_2 centered at zero. From the Residue theorem, this implies that

$$H(1-n) = (-1)^n 2\pi i \frac{B_{n,\chi}(x)}{n!}. \tag{2.19}$$

Since

$$\lim_{s \to 1-n} \left(e^{2\pi i s} - 1\right) \Gamma(s) = \frac{(-1)^{n-1}}{(n-1)!} 2\pi i,$$

so, from (2.18) and (2.19), we have the desired result. This is completion of the proof. □

The multiple Dirichlet's L-functions are defined by

$$L_k(s, x) = \sum_{n_1,\cdots,n_k=0}^{\infty} \frac{\prod_{j=1}^{k} \chi(n_j)}{(n_1 + \cdots + n_k + x)^s}, \tag{2.20}$$

where τ is the conductor of the Dirichlet character χ.

Lemma 2.5. *For $k \in \mathbb{N}$ and* Re$(s)> 1$, *we have*

$$L_k(s, \chi) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{a=1}^{\tau} \chi(a) \frac{e^{-at}}{1 - e^{-\tau t}} \right)^k t^{s-1} dt,$$

where τ is the conductor of the Dirichlet character χ.

Proof. Observe that

$$\begin{aligned} &\int_0^{\infty} \left(\sum_{a=1}^{\tau} \chi(a) \frac{e^{-at}}{1 - e^{-\tau t}} \right)^k t^{s-1} dt \\ &= \sum_{a_1,\cdots,a_k=1}^{\tau} \prod_{j=1}^{k} \chi(a_j) \sum_{m_1,\cdots,m_k=0}^{\infty} \int_0^{\infty} e^{-(m_1\tau + \cdots + m_k\tau)t} t^{s-1} dt \\ &= \sum_{a_1,\cdots,a_k=1}^{\infty} \sum_{m_1,\cdots,m_k=1}^{\infty} \frac{\chi(a_1 + m_1\tau) \cdots \chi(a_1 + m_1\tau)}{(m_1\tau + a_1 + \cdots + m_k\tau + a_k)^s} \Gamma(s). \end{aligned}$$

Since χ is the Dirichlet character with the conductor τ, this implies that

$$\int_0^\infty \left(\sum_{a=1}^{\tau} \chi(a)\frac{e^{-at}}{1-e^{-\tau t}}\right)^k t^{s-1}dt = \Gamma(s) \sum_{n_1,\cdots,n_k=1}^{\infty} \prod_{j=1}^{\tau} \frac{\chi(a_j)}{(n_1+\cdots+n_k+x)^s}.$$

So the desired result is obtained. The proof is complete. □

The generalized Bernoulli numbers attached to χ with order k, $B_{n,\chi}^{(k)}$, $n = 0, 1, \cdots$, are defined by

$$\sum_{n=0}^{\infty} B_{n,\chi}^{(k)} \frac{t^n}{n!} = \left(\sum_{a=1}^{\tau} \chi(a)\frac{te^{at}}{e^{\tau t}-1}\right)^k, \tag{2.21}$$

where τ is the conductor of the Dirichlet character χ. Observe that

(2.22)

$$\begin{aligned}\left(\sum_{a=1}^{\tau} \chi(a)\frac{te^{at}}{e^{\tau t}-1}\right)^k &= \sum_{a_1,\cdots,a_k=1}^{\tau} \prod_{j=1}^{k} \chi(a_j) \left(\frac{t}{e^{\tau t}-1}\right)^k e^{(a_1+\cdots+a_k)t} \\ &= \frac{1}{\tau^k} \sum_{a_1,\cdots,a_k=1}^{\tau} \prod_{j=1}^{k} \chi(a_j) \sum_{n=0}^{\infty} B_n^{(k)} \left(\frac{a_1+\cdots+a_k}{\tau}\right) \tau^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ \tau^{n-k} \sum_{a_1,\cdots,a_k=1}^{\tau} \prod_{j=1}^{k} \chi(a_j) B_n^{(k)} \left(\frac{a_1+\cdots+a_k}{\tau}\right) \right\} \frac{t^n}{n!}.\end{aligned}$$

Therefore, we get the functional equation

$$B_{n,\chi}^{(k)} = \tau^{n-k} \sum_{a_1,\cdots,a_k=1}^{\tau} \prod_{j=1}^{k} \chi(a_j) B_n^{(k)} \left(\frac{a_1+\cdots+a_k}{\tau}\right), \tag{2.23}$$

where $B_n^{(k)}(x), n = 0, 1, \cdots$ are the Bernoulli polynomials with order k. From Lemma 2.5, we have the following Theorem.

Theorem 2.6. *For $n \in \mathbb{N}$, we have*

$$L_k(1-n,\chi) = (-1)^k \frac{(n-1)!}{(n+k-1)!} B_{n+k-1,\chi}^{(k)},$$

where $B_{j,\chi}^{(k)}$, $j = 0, 1, \cdots$ are the generalized Bernoulli numbers with order k attached to χ.

Proof. Let

$$F(z) = \left(\sum_{a=1}^{\tau} \chi(a)\frac{te^{at}}{e^{\tau t}-1}\right)^k.$$

Then

$$H(s) = \left(e^{2\pi is} - 1\right) \int_{I_2} F(t)t^{s-1}dt + \oint_{C_{\varepsilon_1,\varepsilon_2}} F(z)z^{s-1}dz,$$

where I_1, I_2 and $C_{\varepsilon_1,\varepsilon_2}$ are defined in (2.11). Letting $\varepsilon_2 \to 0$ and $M \to \infty$, we have

$$\oint_{C_{\varepsilon_1,\varepsilon_2}} F(z)z^{s-1}dz = 0$$

and, from Lemma 2.5, we get

$$H(s) = \left(e^{2\pi is} - 1\right) \Gamma(s) L_k(1-n, \chi). \tag{2.24}$$

Now, for $s = 1 - n$ ($n \in \mathbb{N}$), letting $\varepsilon_1 \to 0$ and taking ε_2, we have

$$H(1-n) = \oint_{C_{\varepsilon_2}} \frac{F(z)}{z^n} dz = \sum_{m=0}^{\infty} (-1)^m \frac{B_{m,\chi}^{(r)}}{m!} \oint_{C_{\varepsilon_2}} \frac{1}{z^{n+k-m}} dz,$$

where C_{ε_2} is circle with radius ε_2 centered at zero. From the Residue theorem, this implies that

$$H(1-n) = (-1)^{n+k-1} 2\pi i \frac{B_{n+k-1}^{(k)}}{(n+k-1)!}. \tag{2.25}$$

Since

$$\lim_{s \to 1-n} \left(e^{2\pi is} - 1\right) \Gamma(s) = \frac{(-1)^{n-1}}{(n-1)!} 2\pi i,$$

so, from (2.24) and (2.25), we have the desired result. This is completion of the proof. □

References

[1] L. V. Ahlfors, Complex Analysis, 3rd Edition, McGraw-Hill, 1979.

[2] A. Bayad, *Modular properties of elliptic Bernoulli and Euler functions*, Adv. Stud. Contemp. Math., **20(3)**(2010), 389–401.

[3] A. Bayad and T. Kim, *Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20(2)**(2010), 247–253.

[4] J. Choi, D. S. Kim, T. Kim, and Y. H. Kim, *Some arithmetic identities on Bernoulli and Euler numbers arising from the p-adic integrals on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math., **22(2)**(2012), 239–247.

[5] R. J. Dwilewicz and J. Minac, Values of the Riemann zeta function at integers, Materials Matematics 2009, no. 6, 1–26.

[6] K. W. Hwang, D. V. Dolgy, D. S. Kim, T. Kim, and S. H. Lee, *Some theorems on Bernoulli and Euler numbers*, Ars Combin., **109**(2013), 285–297.

[7] L. C. Jang and H. K. Pak, *Non-Archimedean integration associated with q-Bernoulli numbers*, Proc. Jangjeon math. Soc., **5(2)**(2002), 125–129.

[8] D. Kang, J. Jeong, S. J. Lee, and S. H. Rim, *A note on the Bernoulli polynomials arising from a non-linear differential equation*, Proc. Jangjeon math. Soc., **16(1)**(2013), 37–43.

[9] T. Kim, J. Choi and Y. H. Kim, *A note on the values of Euler zeta functions at positive integers*, Adv. Stud. Contemp. Math., **22(1)**(2012), 27–34.

[10] T. Kim, D. S. Kim, A. Bayad, and S. H. Rim, *Identities on the Bernoulli and the Euler numbers and polynomials*, Ars Combin., **107**(2012), 455–463.

[11] G. Kim, B. Kim and J. Choi, *The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers*, Adv. Stud. Contemp. Math., **17(2)**(2008), 137–145.

[12] A. Kudo, *A congruence of generalized Bernoulli number for the character of the first kind*, Adv. Stud. Contemp. Math., **2**(2000), 1–8.

[13] B. Kurt, *Some identities on the second kind Bernoulli polynomials of order α and second kind Euler polynomials of order α with the parameters a, b, c*, Proc. Jangjeon Math. Soc., **16(2)**(2013), 245–249.

[14] Q. M. Luo, *Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order*, Adv. Stud. Contemp. Math., **10(1)**(2005), 63–70.

[15] Q. M. Luo and F. Qi, *Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials*, Adv. Stud. Contemp. Math., **7(1)**(2003), 11–18.

[16] H. Ozden, I. N. Cangul and Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math., **18(1)**(2009), 41–48.

[17] J. W. Park, S. H. Rim, J. Seo, and J. Kwon, *A note on the modified q-Bernoulli polynomials*, Proc. Jangjeon math. Soc., **16(4)**(2013), 451–456.

[18] Y. Simsek, *Theorems on twisted L-function and twisted Bernoulli numbers*, Adv. Stud. Contemp. Math., **11(2)**(2005), 205–218.

[19] H. M. Srivastava, T. Kim and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys., **12(2)**(2005), 241–268.

[20] L. C. Washington, *Introduction to cyclotomic fields*, Springer-Verlag, New York, 1997.

[21] Z. Zhang and H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials*, Proc. Jangjeon Math. Soc., **11(2)**(2008), 191–198.