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On Skew Centralizing Traces of Permuting n-Additive Mappings

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Abstract. Let R be a ring and $D : R^n \longrightarrow R$ be n-additive mapping. A map $d : R \longrightarrow R$ is said to be the trace of D if $d(x) = D(x, x, ...x)$ for all $x \in R$. Suppose that α, β are endomorphisms of R. For any $a, b \in R$, let $< a, b >_{(\alpha,\beta)}= a\alpha(b) + \beta(b)a$. In the present paper under certain suitable torsion restrictions it is shown that $D = 0$ if R satisfies either $< d(x), x^m >_{(\alpha,\beta)}= 0$, for all $x \in R$ or $<< d(x), x >_{(\alpha,\beta)}, x^m >_{(\alpha,\beta)}= 0$, for all $x \in R$. Further, if $< d(x), x >\in Z(R)$, the center of R, for all $x \in R$ or $< d(x)x - xd(x), x >= 0$, for all $x \in R$, then it is proved that d is commuting on R. Some more related results are also obtained for additive mapping on R.

1. Introduction

Throughout this paper R will denote an associative ring with the center $Z(R)$. A ring R is said to be n-torsion free if $nx = 0$ implies that $x = 0$ for all $x \in R$. If R is $n!$-torsion free, then it is d-torsion free for any divisor d of $n!$. Recall that the ring R is said to be prime if the product of any two nonzero ideals of R is nonzero. Equivalently, $aRb = \{0\}$ with $a, b \in R$ implies that $a = 0$ or $b = 0$. A ring R is said to be semiprime if it has no nonzero nilpotent ideals. Equivalently, $aRa = \{0\}$ with $a \in R$ implies that $a = 0$. As usual, we denote the commutator $xy - yx$ by $[x, y]$ and the skew commutator $xy + yx$ by $< x, y >$. Let α, β be endomorphisms of R. For the convenience the sum $x\alpha(y) + \beta(y)x$ and $x\alpha(y) - \beta(y)x$ will be denoted by $< x, y >_{(\alpha,\beta)}$ and $[x, y]_{(\alpha,\beta)}$ respectively. A mapping $f : R \to R$ is said to be (α, β)-centralizing on R, if $[f(x), x]_{(\alpha,\beta)} \in Z(R)$ for all $x \in R$. In the special case when $[f(x), x]_{(\alpha,\beta)} = 0$ for all $x \in R$, the mapping f is called (α, β)-commuting on R. A mapping $f : R \to R$ is said to be (α, β)-skew centralizing on R, if

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$< f(x), x >_{(\alpha,\beta)} \in Z(R)$ for all $x \in R$. In particular, if $< f(x), x >_{(\alpha,\beta)} = 0$ for all $x \in R$, then f is called (α, β)-skew commuting on R. If $\alpha = \beta = 1$ (the identity map on R), then f is called simply centralizing, commuting, skew centralizing and skew commuting respectively. The following example due to Jung and Chang [9] assures that there exists map $f : R \longrightarrow R$ which is (α, β)-skew commuting on R but not skew commuting on R. Let $R = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} \middle| \ w, x, y, z \in \mathbb{Z} \right\}$ be the ring of all 2×2 matrices over $\mathbb{Z}$, the ring of integers. Let $\alpha, \beta : R \to R$ be mappings defined by

$$\alpha \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} -w & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \beta \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & -x \\ 0 & 0 \end{pmatrix}.$$

Define the mapping $f : R \to R$ by

$$f \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}.$$

Then f is (α, β)-skew commuting on R but not skew commuting on R.

The study of centralizing and commuting mappings was initiated by a well known theorem due to Posner [18] which states that the existence of a nonzero centralizing derivation on a prime ring R must be commutative. This theorem has been extended by many authors in different ways (see eg., Bresar [7], Vukman [20] and references therein). Also Bell and Lucier [6] obtained some results concerning skew commuting and skew centralizing additive maps in which the condition of primeness is replaced by the existence of a left identity. Further Jung and Chang [9] obtained the similar results for biadditive maps in rings with left identity. Deng and Bell [5] extended the notion of commuting to n-commuting, where n is an arbitrary positive integer, by defining a mapping $f : R \longrightarrow R$ to be n-commuting on R if $[x^n, f(x)] = 0$ for all $x \in R$. By the analogy with the definition of n-commuting introduced by them, for $n \geq 2$, Park and Jung [15] introduced the concept of n-skew commuting (resp. n-skew centralizing) mapping on R. A mapping $f : R \to R$ is said to be n-skew commuting (resp. n-skew centralizing) on R if $< f(x), x^n >= 0$ (resp. $< f(x), x^n > \in Z(R)$) for all $x \in R$. A map $f : R \to R$ is said to be (α, β)-n-skew commuting (resp. (α, β)-n-skew centralizing) on R if $< f(x), x^n >_{(\alpha,\beta)} = 0$ (resp. $< f(x), x^n >_{(\alpha,\beta)} \in Z(R)$) holds for all $x \in R$. One interesting topic of all related works is to study the skew commuting and skew centralizing mappings involving the traces of symmetric biadditive maps on rings which was done by Jung and Chang [9].

For a fixed positive integer n, a map $D : R^n \to R$ is said to be permuting if $D(x_1, x_2, ..., x_n) = D(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ for all $\pi \in S_n$ and $x_i \in R$ where $i = 1, 2, ..., n$. The notion of permuting n-derivation was defined by Park [14] as follows: a permuting map $D : R^n \to R$ is said to be permuting n-derivation if it is n-additive, i.e., $D(x_1, x_2, ..., x_i + x'_i, ..., x_n) = D(x_1, x_2, ..., x_i, ..., x_n) + D(x_1, x_2, ..., x'_i, ..., x_n)$ and $D(x_1, x_2, ..., x_i x'_i, ..., x_n) = x_i D(x_1, x_2, ..., x'_i, ..., x_n) + D(x_1, x_2, ..., x_i, ..., x_n) x'_i$

holds for all $x_i, x_i' \in R,\ 1 \leq i \leq n$. Let $n \geq 2$ be a fixed integer. A map $d : R \longrightarrow R$ defined by $d(x) = D(x, x, ..., x)$ for all $x \in R$, where $D : R^n \longrightarrow R$ is a permuting map, is called the trace of D. Moreover, it can be easily seen that $D(x_1, x_2, ..., -x_i, ..., x_n) = -D(x_1, x_2, ..., x_i, ..., x_n)$ for all $x_i \in R,\ i = 1, 2, ..., n$. Various results with respect to the traces of permuting n-derivation are obtained, see for reference [14]. The main objective of this paper is to consider some special skew commuting (skew centralizing) mappings, (α, β)-skew commuting mappings, which involves the traces of permuting n-additive maps. The results obtained in this paper generalize, extend and compliment several results obtained earlier. For example Theorem 3 of [9], Theorem 4 of [9], Theorem 5 of [17], etc.− to mention a few only.

2. Main Results

For any additive mapping $\alpha, \beta : R \longrightarrow R$ and $x, y, z \in R$, we will use the following basic identities without any specific mention; $< x, y + z >_{(\alpha,\beta)} = < x, y >_{(\alpha,\beta)} + < x, z >_{(\alpha,\beta)}$ and $< x + y, z >_{(\alpha,\beta)} = < x, z >_{(\alpha,\beta)} + < y, z >_{(\alpha,\beta)}$. If $D : R^n \longrightarrow R$ is a permuting n-additive mapping with the trace d, then it can be easily seen that

$$d(x + y) = d(x) + d(y) + \sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{x, x, ..., x}_{n-i \text{ times}}, \underbrace{y, y, ..., y}_{i \text{ times}}) \text{ for all } x, y \in R.$$

Using similar arguments as used in the proof of Theorem 2.3 of [14], one can easily obtain the following lemma.

Lemma 2.1. *Let $n \geq 2$ be a fixed integer and R be a $n!$-torsion free ring. Suppose that $D : R^n \longrightarrow R$ is a permuting $n-additive$ map with the trace $d : R \longrightarrow R$. If $d(x) = 0$, then $D(x_1, x_2, ..., x_n) = 0$.*

The following lemma will be used frequently throughout the text.

Lemma 2.2.([14], Lemma 2.4) *Let n be a fixed positive integer and R be a $n!$-torsion free ring. Suppose that $y_1, y_2, ..., y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + ... + \lambda^n y_n = 0$ (or $\in Z(R)$) for $\lambda = 1, 2, ..., n$. Then $y_i = 0$ (or $y_i \in Z(R)$) for all i.*

Recently, Jung and Chang [9] proved that if R is a $(n + 1)!$ -torsion free ring with left identity e and $G : R \times R \to R$ is a symmetric bi-additive mapping with the trace g of G, such that g is n-(α, β)-skew commuting on R, then $G = 0$. We begin with n-additive mapping $D : R^n \to R$ with the trace d of D, such that d is m-(α, β)-skew commuting on R, then $D = 0$.

Theorem 2.3. *Let $n \geq 2$ and $m \geq 1$ be fixed integers and let R be a $(m + n - 1)!$-torsion free ring with left identity e. Suppose that $D : R^n \longrightarrow R$ is a permuting n-additive mapping with trace $d : R \longrightarrow R$. If d is (α, β)-m-skew commuting on R,* *where α, β are endomorphism and epimorphism of R respectively, then $D = 0$.*

Proof. It is given that, for all $x \in R$

$$< d(x), x^m >_{(\alpha,\beta)} = d(x)\alpha(x^m) + \beta(x^m)d(x) = 0. \tag{2.1}$$

Since β is an epimorphism, $\beta(e)$ is also a left identity of R. Hence using (2.1), we have

$$< d(e), e^m >_{(\alpha,\beta)} = < d(e), e >_{(\alpha,\beta)} = d(e)\alpha(e) + d(e) = 0. \tag{2.2}$$

Since R is also 2-torsion free, multiplying by $\alpha(e)$ from right side gives $d(e)\alpha(e) = 0$. Hence by (2.2), we find that $d(e) = 0$.
Substituting $e + kx$ for x, where $1 \leq k \leq m + n - 1$, in the hypothesis we obtain,

$$< d(e + kx), (e + kx)^m >_{(\alpha,\beta)} = 0, \text{ for all } x \in R.$$

This implies that,

$$< d(e) + d(kx) + \sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{kx, kx, ..., kx}_{i \text{ times}}), (e + kx)^m >_{(\alpha,\beta)} = 0, \tag{2.3}$$

or $kP_1(x, e) + k^2P_2(x, e) + ... + k^{(m+n-1)}P_{(m+n-1)}(x, e) = 0$ for all $x \in R$, where $P_t(x, e)$ is the sum of terms involving x and e such that $P_t(x, ke) = k^tP_t(x, e), t = 1, 2, ..., m + n - 1$. Using hypothesis and Lemma 2.2, we have,

$$P_t(x, e) = 0 \text{ for all } x \in R, \text{ and for all } t = 1, 2, ..., m + n - 1. \tag{2.4}$$

In particular, we have for all $x \in R, P_1(x, e) = 0$. This yields that $n < D(x, e, e, ..., e), e >_{(\alpha,\beta)} = 0$. Since R is $(m + n - 1)!$-torsion free, we find that $< D(x, e, e, ..., e), e >_{(\alpha,\beta)} = 0$ for all $x \in R$, or $D(x, e, e, ..., e)\alpha(e) + \beta(e)D(x, e, ..., e) = 0$. Since $\beta(e)$ is left identity we get

$$D(x, e, e, ..., e)\alpha(e) + D(x, e, ..., e) = 0.$$

Multiply by $\alpha(e)$ from right and use the torsion restriction to get $D(x, e, e, ..., e)\alpha(e) = 0$. Hence above equation reduces to

$$D(x, e, ..., e) = 0. \tag{2.5}$$

Also from (2.4), we have $P_2(x, e) = 0$ for all $x \in R$, that is

$$\binom{n}{2} < D(x, x, e, ..., e), e >_{(\alpha,\beta)} + n < D(x, e, e, ..., e), x + (n - 1)xe >_{(\alpha,\beta)} = 0.$$

Since R is $(m + n - 1)!$-torsion free, in view of (2.5), the above equation reduces to $< D(x, x, e, ..., e), e >_{(\alpha,\beta)} = 0$. Now applying the same technique as used to obtain (2.5), we get

$$D(x, x, e, ..., e) = 0. \tag{2.6}$$

Proceeding in the similar manner we get,

$$D(\underbrace{x,x,...,x}_{n-i \text{ times}},\underbrace{e,e,...,e}_{i \text{ times}}) = 0, \text{ for all } 1 \le i \le n-1. \tag{2.7}$$

Again expanding $P_t(x,e)$ in (2.4) and using (2.7) we find that $< d(x), e >_{(\alpha,\beta)} = 0$, implies $d(x)\alpha(e) + d(x) = 0$. On right multiplying by $\alpha(e)$ the above equation reduces to $2d(x)\alpha(e) = 0$ and hence $d(x)\alpha(e) = 0$. Therefore, we have $d(x) = 0$ for all $x \in R$. Hence in view of Lemma 2.1 we conclude that $D = 0$. □

Corollary 2.4.([9], Theorem 1)*Let R be a 2-torsion free ring with left identity e and α, β be endomorphism and epimorphism of R respectively. Let $G : R \times R \to R$ be a symmetric bi-additive mapping and g the trace of G. If g is (α,β)-skew commuting on R, then $G = 0$.*

Using similar techniques as used in the proof of Corollary 2 of [9] we have;

Corollary 2.5. *Let $n \ge 2$ be a fixed integer, R be a $n!$-torsion free ring with left identity e and α, β be endomorphism and epimorphism of R respectively. If f is an additive map on R such that the mapping $x \mapsto < f(x), x >_{(\alpha,\beta)}$ is (α,β)-skew commuting on R, then $f = 0$.*

Proof. Define a map $D : R^n \longrightarrow R$ by

$$\begin{aligned} D(x_1, x_2, ..., x_n) &= < f(x_1), x_2 >_{(\alpha,\beta)} + < f(x_2), x_3 >_{(\alpha,\beta)} + ... \\ &\quad + < f(x_{n-1}), x_n >_{(\alpha,\beta)} + < f(x_n), x_1 >_{(\alpha,\beta)} \end{aligned}$$

for all $x_1, x_2, ..., x_n \in R$ and a mapping $d : R \longrightarrow R$ by $d(x) = D(x, x, ..., x)$ for all $x \in R$. It can easily be shown that D is permuting n-additive map and d is the trace of D. In view of the hypothesis, using torsion restriction on R, we have $d(x) = n < f(x), x >_{(\alpha,\beta)}$ which is (α,β) skew commuting on R, and so by Theorem 2.3 we obtain $d = 0$, that is, f is (α,β)-skew-commuting on R and hence it follows that

$$f(e)\alpha(e) + \beta(e)f(e) = f(e)\alpha(e) + f(e) = 0 \tag{2.8}$$

implies $2f(e)\alpha(e) = 0 = f(e)\alpha(e)$. This in view of (2.8) yields that $f(e) = 0$. Therefore, $f(x+e) = f(x)$ for all $x \in R$. Since $< f(x+e), x+e >_{(\alpha,\beta)} = 0$, from the above relation we find that $f(x)\alpha(e) + f(x) = 0$ for all $x \in R$. On right multiplying by $\alpha(e)$ and using torsion restriction on R we have, $f(x)\alpha(e) = 0$, which results in $f(x) = 0$ for all $x \in R$. □

Theorem 2.6. *Let $n \ge 2$ be a fixed integer and R be a $n!$-torsion free ring with left identity e which admits a permuting n-additive map $D : R^n \longrightarrow R$ with trace $d : R \to R$. If d is skew centralizing on R, then d is commuting on R.*

Proof. Since e is left identity, we first remark that the relation $[x, e]y = 0$ for all $x, y \in R$. It is given that

$$< d(x), x > = d(x)x + xd(x) \in Z(R) \text{ for all } x \in R. \tag{2.9}$$

Hence (2.9) becomes,

$$< d(e), e >= d(e)e + d(e) \in Z(R). \tag{2.10}$$

On commuting the above equation with e we get $[d(e), e]e + [d(e), e] = 0$. On right multiplying by e we have, $2[d(e), e]e = 0$ or $[d(e), e]e = 0$. Using this relation, above equation reduces to $[d(e), e] = 0$. Also we have, $d(e)e = d(e)$ and hence from (2.10) we get $2d(e) \in Z(R)$, that is $d(e) \in Z(R)$. Substituting $e+kx$ for x, where $1 \leq k \leq n$ in the hypothesis we obtain that, for all $x \in R$, $< d(e + kx), e + kx >\in Z(R)$. This implies that,

$$< d(e) + d(kx) + \sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{kx, kx, ..., kx}_{i \text{ times}}), e + kx >\in Z(R),$$

or $kP_1(x, e) + k^2P_2(x, e) + ... + k^nP_n(x, e) \in Z(R)$ for all $x \in R$, where $P_t(x, e)$ is the sum of terms involving x and e such that $P_t(x, ke) = k^tP_t(x, e), t = 1, 2, ..., n$. Using hypothesis and Lemma 2.2, we have,

$$P_t(x, e) \in Z(R), \text{ for all } x \in R, \text{ for all } t = 1, 2, ..., n. \tag{2.11}$$

In particular, for all $x \in R$, we have

$$P_1(x, e) =< d(e), x > +n < D(x, e, ..., e), e >\in Z(R).$$

Since $d(e) \in Z(R)$, we have

$$2xd(e) + n(D(x, e, ..., e)e + D(x, e, ..., e)) \in Z(R). \tag{2.12}$$

On commuting the above equation with e and using the fact that $[x, e]y = 0$ for all $x, y \in R$ we obtain, $n([D(x, e, ..., e), e]e + [D(x, e, ..., e), e]) = 0$ and hence $[D(x, e, ..., e), e]e + [D(x, e, ..., e), e] = 0$. Since R is $n!$-torsion free ring, on right multiplying by e we have, $2[D(x, e, ..., e), e]e = 0$ and hence, $[D(x, e, ..., e), e]e = 0$. Therefore, we get

$$[D(x, e, ..., e), e] = 0, \text{ for all } x \in R, \tag{2.13}$$

that is, $D(x, e, ..., e)e = D(x, e, ..., e)$. Now it follows from (2.12) that

$$2xd(e) + 2nD(x, e, ..., e) \in Z(R). \tag{2.14}$$

Since R is $n!$- torsion free, (2.14) yields that

$$2n[D(x, e, ..., e), x] = [D(x, e, ..., e), x] = 0, \text{ for all } x \in R. \tag{2.15}$$

Also from (2.11) we have $P_2(x, e) \in Z(R)$ for all $x \in R$, that is

$$\binom{n}{2} < D(x, x, e, ..., e), e > +n < D(x, e, e, ..., e), x >\in Z(R).$$

On using (2.15) above equation reduces to,

$$\binom{n}{2}(D(x,x,e,...,e)e + D(x,x,e,...,e)) + 2nxD(x,e,...,e) \in Z(R). \tag{2.16}$$

On commuting (2.16) with e and using (2.13) we get,

$$\binom{n}{2}([D(x,x,e,...,e),e]e + [D(x,x,e,...,e),e]) = 0.$$

Since R is $n!$-torsion free, we have $[D(x,x,e,...,e),e]e + [D(x,x,e,...,e),e] = 0$. On right multiplying by e and using torsion restriction on R we find that $[D(x,x,e,...,e),e]e = 0$, which further reduces to, $[D(x,x,e,...,e),e] = 0$, or, $D(x,x,e,...,e)e = D(x,x,e,...,e)$, for all $x \in R$. Therefore one can rewrite (2.16) as

$$\binom{n}{2}2D(x,x,e,...,e) + 2nxD(x,e,e,...,e) \in Z(R).$$

Commuting the above equation with x and using (2.15) yields that $\binom{n}{2}2[D(x,x,e,...,e),x] = 0$. Now since R is $n!$-torsion free we obtain that $[D(x,x,e,...,e),x] = 0$. On proceeding in the same manner, we obtain for $1 \leq i \leq n-1$

$$[D(\underbrace{x,x,...,x}_{n-i \text{ times}},\underbrace{e,e,...,e}_{i \text{ times}}),e] = 0. \tag{2.17}$$

Also,

$$[D(\underbrace{x,x,...,x}_{n-i \text{ times}},\underbrace{e,e,...,e}_{i \text{ times}}),x] = 0 \text{ for all } x \in R. \tag{2.18}$$

Again from (2.11) and using (2.18) we find that,
$< d(x), e > +n < D(x,x,...,x,e), x > \in Z(R)$. On simplification we obtain that, $d(x)e + d(x) + n(D(x,x,...,x,e)x + xD(x,x,...,x,e)) \in Z(R)$. This further yields that

$$d(x)e + d(x) + 2nxD(x,x,...,x,e) \in Z(R), \text{ for all } x \in R. \tag{2.19}$$

Now on commuting the above expression with e and using (2.17) we get, $[d(x),e]e$ $+ [d(x),e] = 0 = [d(x),e]e$, for all $x \in R$. Therefore, $[d(x),e] = 0$ for all $x \in R$, or we have $d(x)e = d(x)$. Thus (2.19) can be rewritten as $2d(x) + 2nxD(x,x,...,x,e)$. On commuting with x and using (2.18) we find that, $[d(x),x] = 0$ for all $x \in R$. □

Theorem 2.7. *Let $n \geq 2$ and $m \geq 1$ be fixed positive integers and R be a $(m+n)!$-torsion free ring with left identity e. If R admits a permuting n-additive map $D : R^n \longrightarrow R$ such that the trace $d : R \longrightarrow R$ satisfies $<< d(x), x >_{(\alpha,\beta)}, x^m >_{(\alpha,\beta)} = 0$*

for all $x \in R$, where α, β are endomorphism and epimorphism of R respectively, then $D = 0$.

Proof. We have, $<< d(x), x >_{(\alpha,\beta)}, x^m >_{(\alpha,\beta)} = 0$ for all $x \in R$. This yields that $<< d(e), e >_{(\alpha,\beta)}, e^m >_{(\alpha,\beta)} = < d(e)\alpha(e) + d(e), e >_{(\alpha,\beta)} = 0$ or

$$d(e)\alpha(e) + d(e)\alpha(e) + d(e)\alpha(e) + d(e) = 0.$$

On right multiplying by $\alpha(e)$ we get, $4d(e)\alpha(e) = 0$. This implies that $d(e)\alpha(e) = 0$ and hence $d(e) = 0$. Now on replacing x by $e + kx$ for $1 \leq k \leq m + n$ in our hypothesis we get, $<< d(e + kx), e + kx >_{(\alpha,\beta)}, (e + kx)^m >_{(\alpha,\beta)} = 0$ for all $x \in R$ or,

$$<< d(e)+d(kx)+\sum_{i=1}^{n-1}\binom{n}{i}D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{kx, kx, ..., kx}_{i \text{ times}}), e+kx >_{(\alpha,\beta)}, (e+kx)^m >_{(\alpha,\beta)}$$

$= 0$. Using hypothesis and $d(e) = 0$ we have,

(2.20)

$$\begin{aligned} 0 \;=\; & << d(kx), e >_{(\alpha,\beta)}, (e + kx)^m >_{(\alpha,\beta)} \\ & + << d(kx), kx >_{(\alpha,\beta)}, (e + kx)^m >_{(\alpha,\beta)} \\ & + << \textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{kx, kx, ..., kx}_{i \text{ times}}), e >_{(\alpha,\beta)}, (e + kx)^m >_{(\alpha,\beta)} \\ & + << \textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{kx, kx, ..., kx}_{i \text{ times}}), kx >_{(\alpha,\beta)}, (e + kx)^m >_{(\alpha,\beta)} \end{aligned}$$

or $kP_1(x, e) + k^2P_2(x, e) + ... + k^{(m+n)}P_{(m+n)}(x, e) = 0$ for all $x \in R$, where $P_t(x, e)$ is the sum of terms involving x and e such that $P_t(x, ke) = k^tP_t(x, e), t = 1, 2, ..., m + n$. Using hypothesis and Lemma 2.2, we have

$$P_t(x, e) = 0, \text{ for all } x \in R, \text{ for all } t = 1, 2, ..., m + n. \tag{2.21}$$

In particular, for all $x \in R$, $P_1(x, e) = 0$ that is,

$$n << D(x, e, ..., e), e >_{(\alpha,\beta)}, e >_{(\alpha,\beta)} = 0.$$

Torsion restriction implies $<< D(x, e, ..., e), e >_{(\alpha,\beta)}, e >_{(\alpha,\beta)} = 0$. Simplifying the latter relation we find that, $3D(x, e, ..., e)\alpha(e) + D(x, e, ..., e) = 0$. On right multiplying by $\alpha(e)$ we obtain $4D(x, e, ..., e)\alpha(e) = 0$. Since R is $(m + n)!$-torsion free we have, $D(x, e, ..., e)\alpha(e) = 0$. Hence the above equation reduces to

$$D(x, e, ..., e) = 0 \text{ for all } x \in R. \tag{2.22}$$

Also from (2.21) we have $P_2(x, e) = 0$ for all $x \in R$. Therefore from (2.20) we find that,

$$\begin{aligned} 0 \;=\; & \tbinom{n}{2} << D(x, x, e, ..., e), e >_{(\alpha,\beta)}, e >_{(\alpha,\beta)} \\ & +n << D(x, e, ..., e), e >_{(\alpha,\beta)}, x + (n - 1)x >_{(\alpha,\beta)} \\ & +n << D(x, e, ..., e), x >_{(\alpha,\beta)}, e >_{(\alpha,\beta)} . \end{aligned}$$

Using (2.22), and torsion restriction of R, the above equation reduces to $<< D(x,x,e,...,e), e>_{(\alpha,\beta)}, e>_{(\alpha,\beta)} = 0$, which on simplification becomes

$$D(x,x,e,...,e) \text{ for all } x \in R. \tag{2.23}$$

On proceeding in the same way for $1 \leq i \leq n-1$ we find that,

$$D(\underbrace{x,x,...,x}_{n-i \text{ times}}, \underbrace{e,e,...,e}_{i \text{ times}}) = 0. \tag{2.24}$$

Also, since $P_n(x,e) = 0$, $d(x) = 0$. Hence in view of Lemma 2.1 we conclude that $D = 0$. $\square$

Corollary 2.8. *Let $n \geq 2$ and $m \geq 1$ be fixed positive integers and R be a $n!$-torsion free ring with left identity e. If R admits a permuting n-additive map $D : R^n \longrightarrow R$ such that the trace $d : R \longrightarrow R$ satisfies $<< d(x), x>, x^m >= 0$ for all $x \in R$, then $D = 0$.*

Using similar techniques as used in Theorem 5 of [17], we obtain that;

Theorem 2.9. *Let $n \geq 2$ and $m \geq 1$ be fixed positive integers and R be a $n!$-torsion free ring with left identity e. If f is an additive map on R such that the mapping $x \mapsto < f(x), x >$ is m-skew centralizing on R, then f is commuting on R.*

Proof. We define a mapping $D : R^n \longrightarrow R$ by

$$D(x_1, x_2, ..., x_n) = [f(x_1), x_2] + [f(x_2), x_3] + ... + [f(x_{n-1}), x_n] + [f(x_n), x_1]$$

for all $x_1, x_2, ..., x_n \in R$. Then it can be easily seen that D is permuting n-additive mapping on R, also $d(x) = D(x, x..., x) = n[f(x), x]$ for all $x \in R$ is the trace of D. Since it follows from the hypothesis that $<< f(x), x>, x^m > \in Z(R)$ for all $x \in R$, on commuting it with x we obtain

$$[< f(x), x > x^m + x^m < f(x), x >, x] = 0 \text{ for all } x \in R.$$

This implies that $[< f(x), x >, x]x^m + x^m[< f(x), x >, x] = 0$ for all $x \in R$. Since $[< y, x >, x] =< [y, x], x >$ for all $x, y \in R$, the latter verification yields that $< [f(x), x], x > x^m + x^m < [f(x), x], x >= 0$ for all $x \in R$. Since R is $n!$-torsion free, we obtain $< d(x), x > x^m + x^m < d(x), x >= 0$ for all $x \in R$. This implies that $<< d(x), x >, x^m >= 0$ for all $x \in R$. Hence it follows from Corollary 2.8 that $d = 0$ on R and so f is commuting on R. $\square$

Theorem 2.10. *Let $n \geq 2$ be a fixed integer and R be a $(n+1)!$-torsion free ring with left identity e which admits a permuting n-additive mapping $D : R^n \longrightarrow R$ with trace $d : R \longrightarrow R$ satisfying $< [d(x), x], x >= 0$ for all $x \in R$. Then d is commuting on R.*

Proof. By our assumption $< [d(x), x], x >= 0$, for all $x \in R$ and hence we have

$$< [d(e), e], e >= [d(e), e]e + [d(e), e] = 0. \tag{2.25}$$

On right multiplying by e and using torsion restriction, (2.25) becomes $[d(e), e]e = 0$ which further reduces to $[d(e), e] = 0$. Now considering $[d(x + e), x + e]$ and using (2.25) we get,

(2.26)
$$\begin{aligned}
[d(x+e), x+e] &= [d(x) + d(e) + \textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{x, x, ..., x}_{i \text{ times}}), x + e] \\
&= [d(x), x] + [d(x), e] + [d(e), x] \\
&\quad +[\textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{x, x, ..., x}_{i \text{ times}}, x] \\
&\quad +[\textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{x, x, ..., x}_{i \text{ times}}, e].
\end{aligned}$$

On replacing x by $e + kx$, where $1 \leq k \leq n + 1$ in the hypothesis and using (2.25), we obtain, for all $x \in R$,

$$< [d(e + kx), e + kx], e + kx >= 0.$$

This implies that,

$$\begin{aligned}
0 &= < [d(kx), (kx)] + [d(kx), e] + [d(e), kx] \\
&\quad +[\textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{x, x, ..., x}_{i \text{ times}}), e] \\
&\quad +[\textstyle\sum_{i=1}^{n-1} \binom{n}{i} D(\underbrace{e, e, ..., e}_{n-i \text{ times}}, \underbrace{x, x, ..., x}_{i \text{ times}}), kx], kx + e >
\end{aligned} \tag{2.27}$$

or $kP_1(x, e) + k^2P_2(x, e) + ... + k^nP_n(x, e) + k^{n+1}P_{n+1}(x, e) = 0$ for all $x \in R$, where $P_t(x, e)$ is the sum of terms involving x and e such that $P_t(x, ke) = k^tP_t(x, e)$, $t = 1, 2, ..., n, n + 1$. Using hypothesis and Lemma 2.2, we have,

$$P_t(x, e) = 0, \text{ for all } x \in R, \text{ for all } t = 1, 2, ...n + 1. \tag{2.28}$$

In view of (2.27), in particular, we find that

$$0 = P_1(x, e) =< [d(e), x], e > +n < [D(x, e, ..., e), e], e >, \text{ for all } x \in R$$

or

$$[d(e), x] + n[D(x, e, ..., e), e] = 0, \text{ for all } x \in R. \tag{2.29}$$

Also from (2.28), we obtain $P_2(x, e) = 0$, that is,

$$\begin{aligned}
0 &= < [d(e), x], x > + < n[D(x, e, ..., e), x], e > + < n[D(x, e, ..., e), e], x > \\
&\quad + < \binom{n}{2}[D(x, x, e, ..., e), e], e >
\end{aligned}$$

or

$$\begin{aligned}
0 &= < [d(e), x] + n[D(x, e, ..., e), e], x > + < n[D(x, e, ..., e), x] \\
&\quad +\binom{n}{2}[D(x, x, e, ..., e), e], e >= 0, \text{ for all } x \in R.
\end{aligned}$$

On using (2.29) we get, $< n[D(x,e,...,e),x] + \binom{n}{2}[D(x,x,e,...,e),e], e >= 0$ or

$$n[D(x,e,...,e),x] + \binom{n}{2}[D(x,x,e,...,e),e] = 0, \text{ for all } x \in R. \tag{2.30}$$

On proceeding in a similar manner, as the above we find that

$$\binom{n}{n-1}[D(e,x,...,x),e] + \binom{n}{n-2}[D(e,e,x,...,x),x] = 0. \tag{2.31}$$

Also, for $P_n(x,e) = 0$ we get,

$$\begin{aligned} 0 \ &= \ < [d(x),e], e > + < \tbinom{n}{n-2}[D(e,e,x,...,x),x], x > \\ &\quad + < \tbinom{n}{n-1}[D(e,x,..x),x], e > + < \tbinom{n}{n-1}[D(e,x,...,x),e], x >= 0. \end{aligned}$$

Using (2.31) we obtain,

$$[d(x),e] + \binom{n}{n-1}[D(e,x,...,x),x] = 0, \text{ for all } x \in R. \tag{2.32}$$

In view of (2.32); the relation (2.26) becomes $[d(x+e), x+e] = [d(x),x]$. Now $< [d(x+e), x+e], x+e >= 0$ implies that $< [d(x),x], x+e >= 0$. This on simplification reduces to $< [d(x),x], x > + < [d(x),x], e >= 0$ or $[d(x),x] = 0$ for all $x \in R$. □

Corollary 2.11. *Let $n \geq 1$ be a fixed integer and R be a $n!$-torsion free ring with left identity e. If R admits a permuting n-additive mapping $D : R^n \to R$ with trace $d : R \to R$ such that $< d(x), x >$ is commuting on R for all $x \in R$, then d is commuting on R.*

Proof. By our assumption $< d(x), x >$ is commuting on R, we have $[< d(x), x >, x] = 0$ for all $x \in R$. Using the fact $< [y,x], x >= [< y, x >, x]$ for all $x, y \in R$, we have $< [d(x),x], x >= 0$ for all $x \in R$. Hence in view of Theorem 2.10 we obtain the required result. □

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