

## THE CONVERGENCE BALL OF INEXACT NEWTON-LIKE METHOD IN BANACH SPACE UNDER WEAK LIPSCHITZ CONDITION

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ABSTRACT. We present a local convergence analysis for inexact Newton-like method in a Banach space under weaker Lipschitz condition. The convergence ball is enlarged and the estimates on the error distances are more precise under the same computational cost as in earlier studies such as [6, 7, 11, 18]. Some special cases are considered and applications for solving nonlinear systems using the Newton-arithmetic mean method are improved with the new convergence technique.

### 1. Introduction

In this study we are concerned with the problem of approximating a solution  $x^*$  of the nonlinear equation

$$(1.1) \quad F(x) = 0,$$

where  $F$  is a Fréchet differentiable operator defined on an open ball  $U(x^*, r)$  with center  $x^*$  and radius  $r > 0$  of a Banach space  $X$  with values in a Banach space  $Y$ . Many problems in Applied Sciences and other disciplines can be brought in a form like (1.1) using Mathematical Modelling [9, 12, 16, 18, 19]. The solutions of these equations can be found in closed form. That is why most solution methods for these equations are iterative. In Applied Sciences, the practice of numerical analysis is essentially connected to variants of Newton's method [1]-[19]. The study about convergence matter of Newton's method is usually centered

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on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton's method, while the local one is, based on the information around a solution, to find estimates of the radii of the convergence balls. There is a plethora of studies under different Lipschitz-type hypotheses on the operators involved (cf. [1]-[19]) and the references there in). In the present paper we study the local convergence of the Inexact Newton-Like Method(INLM) defined by

For  $n = 0$  step 1 until convergence do.

Find the step  $r_n$  which satisfies

$$(1.2) \quad A_n r_n = -F(x_n).$$

Set

$$x_{n+1} = x_n + r_n,$$

where  $x_0$  is an initial point. If  $A_n = F'(x_n)$ , (INLM) reduces to Newton's method and if  $A_n = F'(x_0)$ , (INLM) reduces to the simplified Newton's method. Other choices for operator  $A_n$  are possible [2]-[5]. We suppose: There exists  $x^* \in X$  such that  $F'(x^*) = 0$  and  $F'(x^*)^{-1} \in L(Y, X)$ . The radius Lipschitz condition with  $L$  average

$$(1.3) \quad \|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq \int_{\theta\rho(x)}^{\rho} L(u) du$$

for each  $x \in U(x^*, r)$ ,  $\rho(x) = \|x - x^*\|$ ,  $x_\theta = x^* + \theta(x - x^*)$ ,  $\theta \in [0, 1]$  for some nondecreasing function  $L$  holds and  $A(x)$  is invertible and

$$(1.4) \quad \|A(x)^{-1}F'(x)\| \leq a_1, \quad \|A(x)^{-1}F'(x) - I\| \leq a_2$$

for each  $x \in U(x^*, r)$ .

It follows from (1.3) that there exists a nondecreasing function  $L_0$  such that center Lipschitz condition with  $L_0$  average

$$(1.5) \quad \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \int_0^{\rho(x)} L_0(u) du, \quad \text{for each } x \in U(x^*, r).$$

Clearly,

$$(1.6) \quad L_0(u) \leq L(u), \quad \text{for each } u \in [0, r]$$

holds in general and  $\frac{L}{L_0}$  can be arbitrarily large [2], [5].

Hypothesis (1.3) has been used extensively in the study of Newton's method and together with (1.4) in the study of inexact Newton methods. In particular, convergence balls have been given under (1.3) in [7]. We

show under the same hypotheses that the convergence balls can be enlarged and the estimates on the distances  $\|x_n - x^*\|$  for each  $n = 1, 2, \dots$  can be more precise. These advantages are obtained since, using (1.5) and the Banach lemma on invertible operators [12], [13] we obtain the estimate

$$(1.7) \quad \|F'(x)^{-1}F'(x^*)\| \leq (1 - \int_0^{\rho(x)} L_0(u)du)^{-1} \text{ for each } x \in U(x^*, r),$$

whereas in earlier studies such as [6], [7], [11], [17], (1.4) and the Banach Lemma are used to obtain

$$(1.8) \quad \|F'(x)^{-1}F'(x^*)\| \leq (1 - \int_0^{\rho(x)} L(u)du)^{-1} \text{ for each } x \in U(x^*, r).$$

Estimate (1.7) is more accurate than (1.8) if  $L_0 < L$  and is obtained using (1.5) which is cheaper than (1.4). The exchange of (1.8) with (1.7) in the convergence analysis leads to the advantages already mentioned above. From now on we shall denote by (C) conditions (1.3)-(1.5) and by (C<sub>0</sub>) conditions (1.4) and (1.5). We shall also use the auxiliary Lemmas taken from [6], [18].

LEMMA 1.1. *Let  $f(t) = \frac{1}{t^\delta} \int_0^t L(u)u^{\delta-1}du$ ,  $\delta \geq 1, 0 \leq t \leq r$ , where  $L$  is a positive integrable nondecreasing function in  $[0, r]$ . Then, function  $g$  is nondecreasing monotonically with respect to  $t$ .*

LEMMA 1.2. *Let  $g(t) = \frac{1}{t^2} \int_0^t L(u)(\delta t - u)du$ ,  $\delta \geq 1, 0 \leq t \leq r$ , where  $L$  is a positive integrable and nondecreasing monotonically function in  $[0, r]$ . Then, function  $g$  is nondecreasing monotonically with respect to  $t$ .*

## 2. Local convergence of (INLM)

In this section first we present the local convergence of (INLM) under the (C) conditions.

THEOREM 2.1. *Suppose the (C) conditions and let  $r$  satisfy*

$$(2.1) \quad \frac{a_1 \int_0^r L(u)udu}{r(1 - \int_0^r L_0(u)du)} + a_2 \leq 1.$$

*Then (INLM) is convergent for all  $x_0 \in U(x^*, r)$  and*

$$(2.2) \quad \|x_{n+1} - x^*\| \leq \frac{a_1 \int_0^{\rho(x_0)} L(u) u du}{\rho(x_0)^2 (1 - \int_0^{\rho(x_0)} L_0(u) du)} \|x_n - x^*\|^2 + a_2 \|x_n - x^*\|, \quad n = 0, 1, \dots$$

where

$$(2.3) \quad q := \frac{a_1 \int_0^{\rho(x_0)} L(u) u du}{\rho(x_0) (1 - \int_0^{\rho(x_0)} L_0(u) du)} + a_2 < 1.$$

**Proof.** Choosing  $x_0 \in U(x^*, r)$  where  $r$  satisfies (2.1), then  $q$  determined by (2.3) is less than 1. In fact, from the monotonicity of  $L$  and Lemma 1.1, we have

$$\begin{aligned} q &= \frac{a_1 \int_0^{\rho(x_0)} L(u) u du}{\rho(x_0)^2 (1 - \int_0^{\rho(x_0)} L_0(u) du)} \rho(x_0) + a_2 \\ &< \frac{a_1 \int_0^r L(u) u du}{r^2 (1 - \int_0^r L_0(u) du)} r + a_2 \leq 1. \end{aligned}$$

Then, if  $x_n \in U(x^*, r)$ , we have by (INLM)

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - A_n^{-1} (F(x_n) - F(x^*)) \\ &= x_n - x^* - \int_0^1 A_n^{-1} F'(x_\theta) d\theta (x_n - x^*) \\ &= A_n^{-1} F'(x_n) \int_0^1 F'(x_n)^{-1} F'(x^*) (F'(x^*)^{-1} \\ &\quad \times (F'(x_n) - F'(x_\theta))) (x_n - x^*) d\theta \\ &\quad + A_n^{-1} (A_n - F'(x_n)) (x_n - x^*), \end{aligned}$$

where  $x_\theta = x^* + \theta(x_n - x^*)$ . Using (1.3), (1.4), (1.5) and (1.7) in turn, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|A_n^{-1} F'(x_n)\| \int_0^1 \|F'(x_n)^{-1} F'(x^*)\| \\ &\quad \times \|(F'(x^*)^{-1} (F'(x_n) - F'(x_\theta)))\| \|x_n - x^*\| d\theta \\ &\quad + \|A_n^{-1} (A_n - F'(x_n))\| \|x_n - x^*\| \\ &\leq \frac{a_1}{1 - \int_0^{\rho(x_n)} L_0(u) du} \int_0^1 \int_{\theta\rho(x_n)}^{\rho(x_n)} L(u) du \rho(x_n) d\theta + a_2 \rho(x_n) \end{aligned}$$

$$= \frac{a_1 \int_0^{\rho(x_n)} L(u) u du}{1 - \int_0^{\rho(x_n)} L_0(u) du} + a_2 \rho(x_n).$$

Taking  $n = 0$  above, we obtain  $\|x_1 - x^*\| \leq q \|x_0 - x^*\| < \|x_0 - x^*\|$ , i.e.,  $x_1 \in U(x^*, r)$ . By mathematical induction, all  $x_n$  belong to  $U(x^*, r)$  and  $\rho(x_n) = \|x_n - x^*\|$  decreases monotonically. Therefore, for all  $n = 0, 1, \dots$ , we get

$$\begin{aligned} \|x_{n+1} - x^*\| &= \frac{a_1 \int_0^{\rho(x_n)} L(u) u du}{\rho(x_n)^2 (1 - \int_0^{\rho(x_n)} L_0(u) du)} \rho(x_n)^2 + a_2 \rho(x_n) \\ &\leq \frac{a_1 \int_0^{\rho(x_0)} L(u) u du}{\rho(x_0)^2 (1 - \int_0^{\rho(x_0)} L_0(u) du)} \rho(x_n)^2 + a_2 \rho(x_n), \end{aligned}$$

which implies (2.2). The proof of the theorem is complete.

Next, we present the local convergence of (INLM) under the  $(C_0)$  conditions.

**THEOREM 2.2.** *Suppose the  $(C_0)$  conditions and let  $r > 0$  satisfy*

$$(2.4) \quad \frac{a_1 \int_0^r L_0(u) (2r - u) du}{r(1 - \int_0^r L_0(u) du)} + a_2 \leq 1.$$

*Then (INLM) is convergent for all  $x_0 \in U(x^*, r)$  and*

$$(2.5) \quad \begin{aligned} \|x_{n+1} - x^*\| &\leq \frac{a_1 \int_0^{\rho(x_n)} L_0(u) (2\rho(x_0) - u) du}{\rho(x_0) (1 - \int_0^{\rho(x_0)} L_0(u) du)} \|x_n - x^*\|^2 \\ &+ a_2 \|x_n - x^*\|, \quad n = 0, 1, \dots \end{aligned}$$

where

$$(2.6) \quad q_0 := \frac{a_1 \int_0^{\rho(x_0)} L_0(u) (2\rho(x_0) - u) du}{\rho(x_0) (1 - \int_0^{\rho(x_0)} L_0(u) du)} + a_2 < 1.$$

*Proof.* Arbitrarily choosing  $x_0 \in U(x^*, r)$  where  $r$  satisfies (2.4), then  $q_0$  determined by (2.6) is less than 1. in fact, from the monotonicity of  $L$  and Lemma 1.1, we have

$$\begin{aligned} q_0 &= \frac{a_1 \int_0^{\rho(x_0)} L_0(u) (2\rho(x_0) - u) du}{\rho(x_0)^2 (1 - \int_0^{\rho(x_0)} L_0(u) du)} \rho(x_0) + a_2 \\ &< \frac{a_1 \int_0^r L_0(u) (2r - u) du}{r^2 (1 - \int_0^r L_0(u) du)} r + a_2 \leq 1. \end{aligned}$$

Now if  $x_n \in U(x^*, r)$ , we have by (INLM)

$$\begin{aligned}
x_{n+1} - x^* &= x_n - x^* - A_n^{-1}(F(x_n) - F(x^*)) \\
&= x_n - x^* - \int_0^1 A_n^{-1}F'(x_\theta)d\theta(x_n - x^*) \\
&= A_n^{-1}F'(x_n) \int_0^1 F'(x_n)^{-1}F'(x^*)(F'(x^*)^{-1} \\
&\quad \times (F'(x_n) - F'(x_\theta)))(x_n - x^*)d\theta \\
&\quad + A_n^{-1}(A_n - F'(x_n))(x_n - x^*),
\end{aligned}$$

where  $x_\theta = x^* + \theta(x_n - x^*)$ . Using (1.4), (1.5) and (1.7) in turn, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \|A_n^{-1}F'(x_n)\| \int_0^1 \|F'(x_n)^{-1}F'(x^*)\| \\
&\quad \times \|F'(x^*)^{-1}(F'(x_n) - F'(x_\theta))\| \|x_n - x^*\| d\theta \\
&\quad + \|A_n^{-1}(A_n - F'(x_n))\| \|x_n - x^*\| \\
&\leq \frac{a_1}{1 - \int_0^{\rho(x_n)} L_0(u)du} \int_0^1 \left( \int_0^{\rho(x_n)} \right. \\
&\quad \left. + \int_0^{\theta\rho(x_n)} \right) L_0(u)du \rho(x_n) d\theta + a_2\rho(x_n) \\
&= \frac{a_1 \int_0^{\rho(x_n)} L_0(u)(2\rho(x_n) - u)du}{1 - \int_0^{\rho(x_n)} L_0(u)du} + a_2\rho(x_n).
\end{aligned}$$

Taking  $n = 0$  above, we obtain  $\|x_1 - x^*\| \leq q_0\|x_0 - x^*\| < \|x_0 - x^*\|$ , i.e.,  $x_1 \in U(x^*, r)$  this shows that (INLM) can be continued an infinite number of times. By mathematical induction, all  $x_n$  belong to  $U(x^*, r)$  and  $\rho(x_n) = \|x_n - x^*\|$  decreases monotonically. Therefore, for all  $n = 0, 1, \dots$ , we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \frac{a_1 \int_0^{\rho(x_n)} L_0(u)(2\rho(x_n) - u)du}{\rho(x_n)^2(1 - \int_0^{\rho(x_n)} L_0(u)du)} \rho(x_n)^2 + a_2\rho(x_n) \\
&\leq \frac{a_1 \int_0^{\rho(x_0)} L_0(u)(2\rho(x_0) - u)du}{\rho(x_0)^2(1 - \int_0^{\rho(x_0)} L_0(u)du)} \rho(x_n)^2 + a_2\rho(x_n),
\end{aligned}$$

which implies (2.5). The proof of the theorem is complete.  $\square$

REMARK 2.3. If  $L_0 = L$ , then the results obtained here are reduced to the corresponding ones in [7]. Otherwise, i.e., if  $L_0 < L$ , then our

results constitute an improvement. Indeed, let us denote by  $\bar{q}$  and  $\bar{q}_0$  the quantities used in [7] and obtained from (2.3) and (2.6), respectively, if we replace  $L_0$  by  $L$ . Then, we have that

$$q < \bar{q}$$

and

$$q_0 < \bar{q}_0.$$

Moreover, the radii of convergence obtained from (2.3) and (2.6) are larger (see also next Section). It is worth noticing that in our Theorem 2.2 we use (1.5), whereas in the corresponding Theorem 3.2 in [7] less accurate, more expensive and more difficult to verify condition (1.3) is used.

### 3. Special cases and applications

We specialize functions  $L_0$  and  $L$  in this Section. First, we consider the classical case where

$$(3.1) \quad L_0(u) = L_0 > 0, \text{ and } L(u) = L > 0$$

are constant functions. Then, Theorem 2.1 and Theorem 2.2, reduce, respectively to

**COROLLARY 3.1.** *Suppose the (C) conditions and (3.1) hold and*

$$(3.2) \quad r = \frac{2(1 - a_2)}{La_1 + 2L_0(1 - a_2)}.$$

*Then (INLM) is convergent for all  $x_0 \in U(x^*, r)$ ,*

$$(3.3) \quad q = \frac{a_1 L \|x_0 - x^*\|}{2(1 - L_0 \|x_0 - x^*\|)} + a_2 < 1$$

*and the estimate (2.2) holds.*

**COROLLARY 3.2.** *Suppose the (C<sub>0</sub>) conditions and (3.1) hold and*

$$(3.4) \quad r = \frac{2\delta}{(3 + 2\delta)L_0},$$

*where  $\delta = \frac{1-a_2}{a_1}$ . Then (INLM) is convergent for all  $x_0 \in U(x^*, r)$ ,*

$$(3.5) \quad q = \frac{3a_1 L_0 \|x_0 - x^*\|}{2(1 - L_0 \|x_0 - x^*\|)} + a_2 < 1$$

*and estimate (2.6) holds.*

REMARK 3.3. If  $a_1 = 1$  and  $a_2 = 0$ , then (INLM) specializes to the Newton's method and the radius  $r$  in (3.2) specializes to the convergence radius given by Traub [16], [17] and Rheinboldt [15] if  $L_0 = L$ ,

$$r_{TR} = \frac{2}{2L}.$$

However, our radius  $r^*$  is larger if  $L_0 < L$ , since

$$r_{TR} < r^* = \frac{2}{L + 2L_0}.$$

Notice also that

$$\frac{r_{TR}}{r^*} \rightarrow \frac{1}{3} \text{ as } \frac{L_0}{L} \rightarrow 0.$$

Another popular choice for function  $L$  is given by [6], [7], [11], [17]

$$(3.6) \quad L(u) = \frac{2\gamma}{(1 - \gamma u)^3} \text{ for some } \gamma > 0 \text{ and each } u \in (0, \frac{1}{\gamma}).$$

In this case condition (1.3) is satisfied so that

$$\|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq \frac{1}{(1 - \gamma\|x - x^*\|)^2} - \frac{1}{(1 - \theta\gamma\|x - x^*\|)^2}$$

Moreover, condition (1.5) is also satisfied for  $L_0$  defined by

$$(3.7) \quad L(u) = \frac{2\gamma_0}{(1 - \gamma_0 u)^3} \text{ for some } \gamma_0 \in (0, \gamma] \text{ and each } u \in (0, \frac{1}{\gamma_0})$$

so that

$$\|F'(x^*)^{-1}(F'(x) - F'(x_\theta))\| \leq \frac{1}{(1 - \gamma_0\|x - x^*\|)^2} - 1.$$

As in Corollary 3.1 and Corollary 3.2 we can obtain the specialized results for Theorem 2.1 and Theorem 2.2 where  $L$  and  $L_0$  are given by (3.6) and (3.7), respectively. However, we leave the details to the motivated reader. In the rest of this section we suppose that  $X$  and  $Y$  are finite dimensional spaces of  $\mathbb{R}^k$  ( $k$  a natural number). Next, we improve the results in [8], [11] for the Newton-arithmetic mean (NAM) for solving the system of equations

$$(3.8) \quad F(x) = 0, \quad F : U(x^*, r) \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k.$$

E. Galligani considered the following two splittings of the matrix  $F'(x)$

$$(3.9) \quad F'(x) = M_1(x) - N_1(x) = M_2(x) - N_2(x),$$

where the spectral radius  $\rho(M_1(x)^{-1}N_1(x)) < 1$ ,  $\rho(M_1(x)^{-2}N_2(x)) < 1$ .

Combining the splittings (3.9), (NAM) can be described as follows:

$$\begin{aligned}
& \text{Choose the initial guess } x_0 \\
& \text{For } n = 0 \text{ step 1 until convergence } d_0 \\
& \quad w_n^{(0)} = 0 \\
& \quad \text{For } j = 1, \dots, j_n \text{ do} \\
& \quad \quad M_1(x_n)z_1 = N_1(x_n)w_n^{(j-1)} - F(x_n) \\
& \quad \quad M_2(x_n)z_2 = N_2(x_n)w_n^{(j-1)} - F(x_n) \\
& \quad \quad w_n^{(j)} = \frac{1}{2}(z_1 + z_2) \\
& \quad \text{Set } x_{n+1} = x_n + w_n^{(j_n)}.
\end{aligned}$$

Here  $\{j_n\}$  denotes a sequence of positive integers.

In fact, at each outer iteration  $n$ , the (NAM) method generates the vectors

$$\begin{aligned}
w_n^{(1)} &= -M(x_n)^{-1}F(x_n), \\
w_n^{(2)} &= -(H(x_n) + I)M(x_n)^{-1}F(x_n), \\
&\vdots \\
w_n^{(j_n)} &= -\left(\sum_{j=0}^{j_n-1} (H(x_n))^j M(x_n)^{-1}F(x_n)\right),
\end{aligned}$$

where

$$(3.10) \quad M(x)^{-1} = \frac{1}{2}(M_1(x)^{-1} + M_2(x)^{-1}),$$

$$(3.11) \quad H(x) = \frac{1}{2}(M_1(x)^{-1}N_1(x) + M_2(x)^{-1}N_2(x)) = I - M(x)^{-1}F'(x).$$

If we set

$$(3.12) \quad A(x_n)^{-1} = \sum_{j=0}^{j_n-1} (H(x_n))^j M(x_n)^{-1},$$

then we have

$$(3.13) \quad x_{n+1} = x_n - A(x_n)^{-1}F(x_n).$$

Thus, the (NAM) method can be regarded as a class Newton-like method in which  $F'(x_n)$  has been replaced by the matrix  $A(x_n)$  given by (3.12).

We now assume that the matrices  $M_1(x_n), M_2(x_n), M(x_n)$  are all nonsingular and  $H(x)$  is convergent at  $x_n \in D$  (i.e., the spectral radius  $\rho(H(x_n)) < 1$ ). Thus from (3.12),

$$\begin{aligned} A(x_n)^{-1} &= (I - (H(x_n))^{j_n})(I - H(x_n))^{-1}M(x_n)^{-1} \\ (3.14) \quad &= (I - (H(x_n))^{j_n})(M(x_n)^{-1}F'(x_n))^{-1}M(x_n)^{-1} \\ &= (I - (H(x_n))^{j_n})F'(x_n)^{-1}. \end{aligned}$$

Using the above notation and Theorem 2.1 we present the local result for (NAM).

**THEOREM 3.4.** *Suppose that the (C) conditions holds,  $M_1(x), M_2(x), M(x), H(x)$  are invertible for all  $x \in U(x^*, r)$  and*

$$(3.15) \quad \|I - H(x)\| \leq a_1, \quad \|H(x)\| \leq a_2 \leq 1.$$

Let  $r > 0$  satisfy

$$(3.16) \quad \frac{a_1 \int_0^r L(u)u du}{r(1 - \int_0^r L_0(u)du)} + a_2 \leq 1.$$

Then (NAM) is convergent for all  $x_0 \in U(x^*, r)$  and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \frac{a_1 \int_0^{\rho(x_0)} L(u)u du}{\rho(x_0)^2(1 - a_2)(1 - \int_0^{\rho(x_0)} L_0(u)du)} \|x_n - x^*\|^2 \\ (3.17) \quad &+ a_2 \|x_n - x^*\|, \quad n = 0, 1, \dots \end{aligned}$$

where

$$(3.18) \quad q := \frac{a_1 \int_0^{\rho(x_0)} L(u)u du}{\rho(x_0)(1 - a_2)(1 - \int_0^{\rho(x_0)} L_0(u)du)} + a_2 < 1.$$

*Proof.* In fact, From (3.14) and (3.15), we have

$$\begin{aligned} \|A(x_k)^{-1}F'(x_k) - I\| &= \|(H(x_k))^{j_k}\| \leq \|H(x_k)\|^{j_k} \leq a_2, \\ \|A(x_k)^{-1}F'(x_k)\| &= \|I - (H(x_k))^{j_k}\| \\ &\leq \|I - H(x_k)\| \sum_{j=0}^{j_k-1} \|H(x_k)\|^j \\ &\leq \frac{\|I - H(x_k)\|}{1 - \|H(x_k)\|} \leq \frac{a_1}{1 - a_2}. \end{aligned}$$

The results now follows from Theorem 2.1.  $\square$

COROLLARY 3.5. Suppose that the (C) conditions hold  $L_0$  and  $L$  are constant functions,  $M_1(x), M_2(x), M(x), H(x)$  are invertible for all  $x \in U(x^*, r)$  and (3.15) holds. Let

$$(3.19) \quad r = \frac{2(1 - a_2)^2}{La_1 + 2L_0(1 - a_2)^2} > 0.$$

Then (NAM) is convergent for all  $x_0 \in U(x^*, r)$  and for

$$(3.20) \quad q = \frac{a_1 L \|x_0 - x^*\|}{2(1 - a_2)(1 - L_0 \|x_0 - x^*\|)} + a_2 < 1.$$

the inequality (3.17) holds.

REMARK 3.6. If  $L_0 = L$ , Theorem 3.4 and Corollary 3.5 reduce to the corresponding results in [7]. In particular if  $L_0$  and  $L$  are constant functions our results present the corresponding ones [11] in affine invariant form. The advantages of this approach have been explained in [9]. If  $L_0 < L$  our results extend the convergence ball and improve the error estimates on the distances  $\|x_n - x^*\|$  for each  $n = 1, 2, \dots$ .

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