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Some Properties of Alexandrov Topologies

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Abstract

Alexandrov topologies are the topologies induced by relations. This paper addresses the properties of Alexandrov topologies as the extensions of strong topologies and strong cotopologies in complete residuated lattices. With the concepts of Zhang's completeness, the notions are discussed as extensions of interior and closure operators in a sense as Pawlak's the rough set theory. It is shown that interior operators are meet preserving maps and closure operators are join preserving maps in the perspective of Zhang's definition.

Keywords: Complete residuated lattices, Alexandrov topologies, Fuzzy partially ordered set, Meet and join

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1. Introduction

Pawlak [1, 2] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [3-7]. Zhang and Fan [8] and Zhang et al. [9] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy partially ordered sets. Alexandrov topologies [7, 10-12] were introduced the extensions of fuzzy topology and strong topology [13].

In this paper, we investigate the properties of Alexandrov topologies as the extensions of strong topologies and strong cotopologies in complete residuated lattices. Moreover, we study the notions as extensions of interior and closure operators. We give their examples.

Definition 1.1. [3, 4] An algebra $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(C2) $(L, \odot, \top)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, ^* \bot, \top)$ is a complete residuated lattice with a negation; i.e., $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$, $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$ and $\top_x(x) = \top, \top_x(x) = \bot,$ otherwise.

Lemma 1.2. [3, 4] For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) If $y \leq z$, then $x \odot y \leq x \odot z$.

(2) If $y \leq z$, then $x \to y \leq x \to z$ and $z \to x \leq y \to x$.

(3) $x \to y = \top$ iff $x \leq y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

(15) $x \to y \odot z \geq (x \to y) \odot z$ and $(x \to y) \to z \geq x \odot (y \to z)$.

Definition 1.3. [7, 10, 12, 13] A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies:

(T1) $\perp_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\perp_X(x) = \perp$ for $x \in X$.

(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(T4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

A subset $\tau \subset L^X$ satisfying (T1), (T3) and (T4) is called a *strong topology* if it satisfies:

(ST) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \wedge_{i\in\Lambda} A_i \in \tau$ for each finite index $\Lambda \subset \Gamma$.

A subset $\tau \subset L^X$ satisfying (T1), (T3) and (T4) is called a *strong cotopology* if it satisfies:

(SC) If $A_i \in \tau$ for $i \in \Gamma$, $\bigwedge_{i\in\Gamma} A_i, \vee_{i\in\Lambda} A_i \in \tau$ for each finite index $\Lambda \subset \Gamma$.

Remark 1.4. Each Alexandrov topology is both strong topology and strong cotopology.

Definition 1.5. [8, 9] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = \top$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = \top$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preordered set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially ordered set.

Example 1.6. (1) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy partially ordered set from Lemma 1.2 (8).

(2) Let τ be an Alexandrov topology. We define a function $e_\tau : \tau \times \tau \to L$ as $e_\tau(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (τ, e_τ) is a fuzzy partially ordered set.

Definition 1.7. [8, 9] Let (X, e_X) be a fuzzy partially ordered set and $A \in L^X$.

(1) A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x\in X}(A(x) \to e_X(x, y)) \leq e_X(x_0, y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x\in X}(A(x) \to e_X(y, x)) \leq e_X(y, x_1)$.

Remark 1.8. [8, 9] Let (X, e_X) be a fuzzy partially ordered set and $A \in L^X$.

(1) x_0 is a join of A iff

$$\bigwedge_{x\in X}(A(x) \to e_X(x, y)) = e_X(x_0, y).$$

(2) x_1 is a meet of A iff $\bigwedge_{x\in X}(A(x) \to e_X(y, x)) = e_X(y, x_1)$.

(3) If x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

Remark 1.9. [8, 9] Let (L^X, e_{L^X}) be a fuzzy partially ordered and $\Phi \in L^{L^X}$.

(1) Since

$$\bigwedge_{A\in L^X} (\Phi(A) \to e_{L^X}(A, B)) = e_{L^X}(\bigvee_{A\in L^X} (\Phi(A) \odot A), B) = e_{L^X}(\sqcup\Phi, B),$$

then $\sqcup\Phi = \bigvee_{A\in L^X}(\Phi(A) \odot A)$.

(2) We have $\sqcap\Phi = \bigwedge_{A\in L^X}(\Phi(A) \to A)$ because

$$\begin{aligned} &\textstyle\bigwedge_{A\in L^X}(\Phi(A) \to e_{L^X}(B, A)) \\ &= \textstyle\bigwedge_{A\in L^X} e_{L^X}(B, (\Phi(A) \to A)) \\ &= e_{L^X}(B, \textstyle\bigwedge_{A\in L^X}(\Phi(A) \to A)). \end{aligned}$$

2. Some Properties of Alexandrov Topologies

Theorem 2.1. (1) A subset $\tau \subset L^X$ is an Alexandrov topology on X iff for each $\Phi : \tau \to L$, $\sqcup\Phi \in \tau$ and $\sqcap\Phi \in \tau$.

(2) τ is an Alexandrov topology on X iff $\tau^* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on X.

Proof. (1) ($\Rightarrow$) For each $\Phi : \tau \to L$, we define

$$P = \bigvee_{A\in\tau} (\Phi(A) \odot A).$$

Since τ is an Alexandrov topology on X, $(\Phi(A) \odot A) \in \tau$. Thus $P \in \tau$. Then $P = \sqcup\Phi$ from:

$$\begin{aligned} e_\tau(P, B) &= e_\tau(\textstyle\bigvee_{A\in\tau}(\Phi(A) \odot A), B) \\ &= \textstyle\bigwedge_{A\in\tau}(\Phi(A) \to e_\tau(A, B)) \\ &= e_\tau(\sqcup\Phi, B). \end{aligned}$$

For each $\Phi : \tau \to L$, we define $Q = \bigwedge_{A\in\tau}(\Phi(A) \to A)$. Since τ is an Alexandrov topology on X, $(\Phi(A) \to A) \in \tau$. Thus $Q \in \tau$. Then $Q = \sqcap\Phi$ from:

$$\begin{aligned} e_\tau(B, Q) &= e_\tau(B, \textstyle\bigwedge_{A\in\tau}(\Phi(A) \to A)) \\ &= \textstyle\bigwedge_{A\in\tau}(\Phi(A) \to e_\tau(B, A)) \\ &= e_\tau(B, \sqcap\Phi). \end{aligned}$$

($\Rightarrow$) (T1) For $\Phi(A) = \bot$ for all $A \in \tau$, $\sqcup\Phi = \bigvee_{A\in\tau}(\Phi(A)\odot A) = \bot_X \in \tau$ and $\sqcap\Phi = \bigwedge_{A\in\tau}(\Phi(A) \to A) = \top_X \in \tau$.

(T2) Let $\Phi(A_i) = \top$ for all $\{A_i \mid i \in \Gamma\} \subset \tau$, otherwise $\Phi(A) = \bot$. We have

$$\sqcup\Phi = \bigvee_{A\in\tau} (\Phi(A) \odot A) = \bigvee_{i\in\Gamma} A_i \in \tau$$

$$\sqcap\Phi = \bigwedge_{A\in\tau} (\Phi(A) \to A) = \bigwedge_{i\in\Gamma} A_i \in \tau.$$

(T3) Let $\Phi(A) = \bot$ for $A = B \in \tau$, otherwise $\Phi(A) = \alpha$ if $A \neq B$. We have

$$\sqcup\Phi = \bigvee_{A\in\tau} (\Phi(A) \odot A) = \alpha \odot B \in \tau$$

$$\sqcap\Phi = \bigwedge_{A\in\tau} (\Phi(A) \to A) = \alpha \to B \in \tau.$$

(2) Let $A^* \in \tau^*$ for $A \in \tau$. Since $\alpha \odot A^* = (\alpha \to A)^*$ and $\alpha \to A^* = (\alpha \odot A)^*$, τ^* is an Alexandrov topology on X.

Theorem 2.2. Let τ be an Alexandrov topology on X. Define $\mathcal{I}_\tau : L^X \to L^X$ as follows:

$$\mathcal{I}_\tau(A) = \bigvee_{B\in\tau} (e_{L^X}(B, A) \odot B).$$

Then the following properties hold.

(1) $e_{L^X}(A, B) \le e_{L^X}(\mathcal{I}_\tau(A), \mathcal{I}_\tau(B))$, for $A, B \in L^X$.

(2) $\mathcal{I}_\tau(A) \le A$ for all $A \in L^X$.

(3) $\mathcal{I}_\tau(\mathcal{I}_\tau(A)) = \mathcal{I}_\tau(A)$ for all $A \in L^X$.

(4) $\mathcal{I}_\tau(\alpha \to A) = \alpha \to \mathcal{I}_\tau(A)$ for all $\alpha \in L, A \in L^X$.

(5) $\mathcal{I}_\tau(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{I}_\tau(A_i)$ for all $A_i \in L^X$.

(6) $\mathcal{I}_\tau(\sqcap\Phi) = \sqcap\mathcal{I}_\tau^{\to}(\Phi)$ for each $\Phi : L^X \to L$ where $\mathcal{I}_\tau^{\to} : L^{L^X} \to L^{L^X}$ defined as $\mathcal{I}_\tau^{\to}(\Phi)(B) = \bigvee_{\mathcal{I}_\tau(A)=B} \Phi(A)$.

(7) $I_\tau(A) = \bigvee\{B \in L^X \mid B \le A, B \in \tau\}$.

(8) Define $\tau_{I_\tau} = \{A \mid A = \mathcal{I}_\tau(A)\}$. Then $\tau = \tau_{I_\tau}$.

(9) There exists a fuzzy preorder $e_X : X \times X \to L$ such that

$$\mathcal{I}_\tau(A)(y) = \bigwedge_{x\in X} (e_X(x, y) \to A(x)).$$

Proof. (1) By Lemma 1.2 (8,10,14), we have

$$\begin{aligned} & e_{L^X}(\mathcal{I}_\tau(A), \mathcal{I}_\tau(B)) \\ &= \textstyle\bigwedge_{x\in X}(\bigvee_{C\in\tau}(e_{L^X}(C, A) \odot C(x)) \\ &\to \textstyle\bigvee_{D\in\tau}(e_{L^X}(D, B) \odot D(x)) \\ &\ge \textstyle\bigwedge_{x\in X} \bigwedge_{C\in\tau}((e_{L^X}(C, A) \odot C(x)) \\ &\to (e_{L^X}(C, B) \odot C(x)) \\ &\ge \textstyle\bigwedge_{C\in\tau}((e_{L^X}(C, A) \to (e_{L^X}(C, B)) \\ &\ge e_{L^X}(A, B) \end{aligned}$$

(2) Since $e_{L^X}(C, A)\odot C \le A$ from Lemma 1.2 (9), $\mathcal{I}_\tau(A) \le A$.

(3) Since $\mathcal{I}_\tau(A) \in \tau$, then

$$\mathcal{I}_\tau(\mathcal{I}_\tau(A)) \ge e_{L^X}(\mathcal{I}_\tau(A), \mathcal{I}_\tau(A)) \odot \mathcal{I}_\tau(A) = \mathcal{I}_\tau(A).$$

By (2), $\mathcal{I}_\tau(\mathcal{I}_\tau(A)) = \mathcal{I}_\tau(A)$.

(4) Since $\alpha \to \mathcal{I}_\tau(A) \le \alpha \to A$ and $\alpha \to \mathcal{I}_\tau(A) \in \tau$,

$$\begin{aligned} & \mathcal{I}_\tau(\alpha \to A) \\ &\ge e_{L^X}(\alpha \to \mathcal{I}_\tau(A), \alpha \to A) \odot (\alpha \to \mathcal{I}_\tau(A)) \\ &= \alpha \to \mathcal{I}_\tau(A) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_\tau(\alpha \to A) &= \textstyle\bigvee_{B\in\tau}(e_{L^X}(B, \alpha \to A) \odot B) \\ &= \textstyle\bigvee_{B\in\tau}((\alpha \to e_{L^X}(B, A)) \odot B) \\ &\le \alpha \to \textstyle\bigvee_{B\in\tau}(e_{L^X}(B, A)) \odot B) \text{ (by Lemma 1.2 (15))} \\ &= \alpha \to \mathcal{I}_\tau(A). \end{aligned}$$

(5) By (1), since $\mathcal{I}_\tau(A) \le \mathcal{I}_\tau(B)$ for $A \le B$, $\mathcal{I}_\tau(\bigwedge_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma} \mathcal{I}_\tau(A_i)$. Since $\bigwedge_{i\in\Gamma} \mathcal{I}_\tau(A_i) \le \bigwedge_{i\in\Gamma} A_i$ and $\bigwedge_{i\in\Gamma} \mathcal{I}_\tau(A_i)$

$\in \tau$, we have

$$\begin{aligned}&\mathcal{I}_\tau(\bigwedge_{i\in\Gamma} A_i)\\ &\geq e_{L^X}(\bigwedge_{i\in\Gamma}\mathcal{I}_\tau(A_i), \bigwedge_{i\in\Gamma} A_i)\odot\bigwedge_{i\in\Gamma}\mathcal{I}_\tau(A_i))\\ &= \bigwedge_{i\in\Gamma}\mathcal{I}_\tau(A_i).\end{aligned}$$

(6) For each $\Phi : L^X \to L$, put $Q = \sqcap\mathcal{I}_\tau^{\to}(\Phi)$. Since $\mathcal{I}_\tau^{\to}(\Phi) : L^X \to L$ is a map, we have

$$\sqcap\mathcal{I}_\tau^{\to}(\Phi) = \bigwedge_{C\in L^X}(\mathcal{I}_\tau^{\to}(\Phi)(C) \to C)$$

and $Q = \sqcap\mathcal{I}_\tau^{\to}(\Phi) = \mathcal{I}_\tau(\sqcap\Phi)$ from:

$$\begin{aligned}&e_{L^X}(B, Q) = \bigwedge_{C\in L^X}(\mathcal{I}_\tau^{\to}(\Phi)(C) \to e_{L^X}(B, C)\\ &= e_{L^X}(B, \bigwedge_{C\in L^X}(\mathcal{I}_\tau^{\to}(\Phi)(C) \to C))\\ &= e_{L^X}(B, \bigwedge_{C\in L^X}(\bigvee_{\mathcal{I}_\tau(A)=C}\Phi(A) \to C))\\ &= e_{L^X}(B, \bigwedge_{A\in L^X}(\Phi(A) \to \mathcal{I}_\tau(A)))\\ &= e_{L^X}(B, \mathcal{I}_\tau(\bigwedge_{A\in L^X}(\Phi(A) \to A)))\ \text{(by (4) and (5))}\\ &= e_{L^X}(B, \mathcal{I}_\tau(\sqcap\Phi)).\end{aligned}$$

(7) Put $I(A) = \bigvee\{B \in L^X \mid B \leq A, B \in \tau\}$. Since $I(A) \leq A$ and $I(A) \in \tau$, we have
$\mathcal{I}_\tau(A) = \bigvee_{B\in\tau}(e_{L^X}(B, A)\odot B) \geq e_{L^X}(I(A), A)\odot I(A) = I(A)$.
Since $\mathcal{I}_\tau(A) \leq A$ and $\mathcal{I}_\tau(A) \in \tau$, we have $I(A) \geq \mathcal{I}_\tau(A)$.
(8) It follows from $A \in \tau$ iff $\mathcal{I}_\tau(A) = A$ iff $A \in \tau_{\mathcal{I}_\tau}$.
(9) Since $A = \bigwedge_{x\in X}(A^* \to \top_x^*)$, by (4) and (5), $\mathcal{I}_\tau(A)(y) = \bigwedge_{x\in X}(A^* \to \mathcal{I}_\tau(\top_x^*)(y)) = \bigwedge_{x\in X}(\mathcal{I}_\tau^*(\top_x^*)(y) \to A(x))$. Put $e_X(x, y) = \mathcal{I}_\tau^*(\top_x^*)(y)$. Then

$$\mathcal{I}_\tau(A)(y) = \bigwedge_{x\in X}(e_X(x, y) \to A(x)).$$

$$e_X(x, x) = \mathcal{I}_\tau^*(\top_x^*)(x) \geq \top_x(x) = \top$$

$$\begin{aligned}&\bigvee_{y\in X}(e_X(x, y)\odot e_X(y, z)) \leq e_X(x, z)\\ &\text{iff } \bigvee_{y\in X}(\mathcal{I}_\tau^*(\top_x^*)(y)\odot\mathcal{I}_\tau^*(\top_y^*)(z)) \leq \mathcal{I}_\tau^*(\top_x^*)(z)\\ &\text{iff } \bigwedge_{y\in X}(\mathcal{I}_\tau^*(\top_x^*)(y) \to \mathcal{I}_\tau(\top_y^*)(z)) \geq \mathcal{I}_\tau(\top_x^*)(z)\\ &\text{iff } \mathcal{I}_\tau(\bigwedge_{y\in X}(\mathcal{I}_\tau^*(\top_x^*)(y) \to \top_y^*))(z) \geq \mathcal{I}_\tau(\top_x^*)(z)\\ &\text{iff } \mathcal{I}_\tau(\mathcal{I}_\tau(\top_x^*))(z) \geq \mathcal{I}_\tau^*(\top_x^*)(z)\end{aligned}$$

Hence e_X is a fuzzy preorder.

Theorem 2.3. Let τ be an Alexandrov topology on X. Define $\mathcal{C}_\tau : L^X \to L^X$ as follows:

$$\mathcal{C}_\tau(A) = \bigwedge_{B\in\tau}(e_{L^X}(A, B) \to B).$$

Then the following properties hold.

(1) $e_{L^X}(A, B) \leq e_{L^X}(\mathcal{C}_\tau(A), \mathcal{C}_\tau(B))$, for all $A, B \in L^X$.
(2) $A \leq \mathcal{C}_\tau(A)$ for all $A \in L^X$.
(3) $\mathcal{C}_\tau(\mathcal{C}_\tau(A)) = \mathcal{C}_\tau(A)$ for all $A \in L^X$.
(4) $\mathcal{C}_\tau(\alpha\odot A) = \alpha\odot\mathcal{C}_\tau(A)$ for all $\alpha \in L, A \in L^X$.
(5) $\mathcal{C}_\tau(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma}\mathcal{C}_\tau(A_i)$ for all $A_i \in L^X$.
(6) $\mathcal{C}_\tau(\sqcup\Phi) = \sqcup\mathcal{C}_\tau^{\to}(\Phi)$ for each $\Phi : L^X \to L$ where $\mathcal{C}_\tau^{\to} : L^{L^X} \to L^{L^X}$ defined as $\mathcal{C}_\tau^{\to}(\Phi)(B) = \bigvee_{\mathcal{C}_\tau(A)=B}(\Phi(A))$.
(7) $C_\tau(A) = \bigwedge\{B \in L^X \mid A \leq B, B \in \tau\}$.
(8) Define $\tau_{C_\tau} = \{A \mid A = \mathcal{C}_\tau(A)\}$. Then $\tau = \tau_{C_\tau}$.
(9) $(C_\tau(A^*))^* = I_{\tau^*}(A)$ for all $A \in L^X$.
(10) There exists a fuzzy preorder $e_X : X \times X \to L$ such that

$$\mathcal{C}_\tau(A)(y) = \bigvee_{x\in X}(e_X(x, y)\odot A(x)),$$

$$\mathcal{I}_{\tau^*}(A)(y) = \bigwedge_{x\in X}(e_X(x, y) \to A(x)).$$

Proof. (1) By Lemma 1.2 (8,10), we have

$$\begin{aligned}&e_{L^X}(\mathcal{C}_\tau(A), \mathcal{C}_\tau(B))\\ &= \bigwedge_{x\in X}(\bigwedge_{C\in\tau}(e_{L^X}(A, C) \to C(x))\\ &\quad \to \bigwedge_{D\in\tau}(e_{L^X}(B, D) \to D(x))\\ &\geq \bigwedge_{x\in X}\bigwedge_{C\in\tau}((e_{L^X}(A, C) \to C(x))\\ &\quad \to (e_{L^X}(B, C) \to C(x))\\ &\geq \bigwedge_{C\in\tau}((e_{L^X}(B, C) \to (e_{L^X}(A, C))\\ &\geq e_{L^X}(A, B).\end{aligned}$$

(2) Since $e_{L^X}(A, B)\odot A \leq B$ iff $A \leq e_{L^X}(A, B) \to B$, then $A \leq \mathcal{C}_\tau(A)$.
(3) Since $\mathcal{C}_\tau(A) \in \tau$, then $\mathcal{C}_\tau(\mathcal{C}_\tau(A)) \leq e_{L^X}(\mathcal{C}_\tau(A), \mathcal{C}_\tau(A)) \to \mathcal{C}_\tau(A) = \mathcal{C}_\tau(A)$. By (2), $\mathcal{C}_\tau(\mathcal{C}_\tau(A)) = \mathcal{C}_\tau(A)$.
(4) Since $\alpha\odot A \leq \alpha\odot\mathcal{C}_\tau(A)$ and $\alpha\odot\mathcal{C}_\tau(A) \in \tau$,

$$\begin{aligned}\mathcal{C}_\tau(\alpha\odot A) &\leq e_{L^X}(\alpha\odot A, \alpha\odot\mathcal{C}_\tau(A)) \to \alpha\odot\mathcal{C}_\tau(A)\\ &= \alpha\odot\mathcal{C}_\tau(A).\end{aligned}$$

$$\begin{aligned}&\mathcal{C}_\tau(\alpha\odot A) = \bigwedge_{B\in\tau}(e_{L^X}(\alpha\odot A, B) \to B)\\ &= \bigwedge_{B\in\tau}((\alpha \to e_{L^X}(A, B)) \to B)\\ &\geq \bigwedge_{B\in\tau}(\alpha\odot(e_{L^X}(A, B) \to B))\\ &\ \text{(by Lemmma 1.2(15))}\\ &\geq \alpha\odot\bigwedge_{B\in\tau}(e_{L^X}(A, B) \to B)\\ &= \alpha\odot\mathcal{C}_\tau(A).\end{aligned}$$

(5) By (1), since $\mathcal{C}_\tau(A) \leq \mathcal{C}_\tau(B)$ for $A \leq B$, $\bigvee_{i\in\Gamma}\mathcal{C}_\tau(A_i) \leq \mathcal{C}_\tau(\bigvee_{i\in\Gamma} A_i)$. Since

$$\bigvee_{i\in\Gamma} A_i \leq \bigvee_{i\in\Gamma}\mathcal{C}_\tau(A_i)$$

and

$$\bigvee_{i\in\Gamma} \mathcal{C}_\tau(A_i) \in \tau,$$

we have

$$\begin{aligned}&\mathcal{C}_\tau(\bigvee_{i\in\Gamma} A_i)\\ &\le e_{L^X}(\bigvee_{i\in\Gamma} A_i, \bigvee_{i\in\Gamma} \mathcal{C}_\tau(A_i)) \to \bigvee_{i\in\Gamma} \mathcal{C}_\tau(A_i)\\ &= \bigvee_{i\in\Gamma} \mathcal{C}_\tau(A_i).\end{aligned}$$

(6) For each $\Phi : L^X \to L$, put $P = \sqcap\mathcal{C}_\tau^{\to}(\Phi)$. Since $\mathcal{C}_\tau^{\to}(\Phi) : L^X \to L$ is a map, we have

$$\sqcup\mathcal{C}_\tau^{\to}(\Phi) = \bigvee_{C\in\tau} (\mathcal{C}_\tau^{\to}(\Phi)(C) \odot C)$$

and $P = \sqcup\mathcal{C}_\tau^{\to}(\Phi) = \mathcal{C}_\tau(\sqcup\Phi)$ from:

$$\begin{aligned}&e_{L^X}(P, B) = \bigwedge_{C\in L^X} (\mathcal{C}_\tau^{\to}(\Phi)(C) \to e_{L^X}(C, B)\\ &= \bigwedge_{C\in L^X} e_{L^X}(\mathcal{C}_\tau^{\to}(\Phi)(C) \odot C, B)\\ &= e_{L^X}(\bigvee_{C\in L^X} (\mathcal{C}_\tau^{\to}(\Phi)(C) \odot C), B)\\ &= e_{L^X}(\bigvee_{A\in L^X} (\Phi(A) \odot \mathcal{C}_\tau(A)), B)\\ &= e_{L^X}(\mathcal{C}_\tau(\bigvee_{A\in L^X} (\Phi(A) \odot A)), B) \ \text{(by (4) and (5))}\\ &= e_{L^X}(\mathcal{C}_\tau(\sqcup\Phi), B)\end{aligned}$$

(7) Put $C(A) = \bigwedge\{B \in L^X \mid A \le B, B \in \tau\}$. Since $A \le C(A)$ and $C(A) \in \tau$, we have

$$\begin{aligned}&\mathcal{C}_\tau(A) = \bigwedge_{B\in\tau}(e_{L^X}(A, B) \to B)\\ &\le e_{L^X}(A, C(A)) \to C(A) = C(A).\end{aligned}$$

Since $A \le \mathcal{C}_\tau(A)$ and $\mathcal{C}_\tau(A) \in \tau$, we have $C(A) \le \mathcal{C}_\tau(A)$.

(8) It follows from $A \in \tau$ iff $\mathcal{C}_\tau(A) = A$ iff $A \in \tau_{\mathcal{C}_\tau}$.

(9) $$\begin{aligned}(C_\tau(A^*))^* &= (\bigwedge_{B\in\tau}(e_{L^X}(A^*, B) \to B))^*\\ &= \bigvee_{B\in\tau}(e_{L^X}(B^*, A) \odot B^*)\\ &= \bigvee_{B^*\in\tau^*}(e_{L^X}(B^*, A) \odot B^*)\\ &= I_{\tau^*}(A).\end{aligned}$$

(10) Since $A = \bigvee_{x\in X}(A \odot \top_x)$, by (4) and (5), $\mathcal{C}_\tau(A)(y) = \bigvee_{x\in X}(A \odot \mathcal{C}_\tau(\top_x)(y))$. Put $e_X(x, y) = \mathcal{C}_\tau(\top_x)(y)$. Then

$$\mathcal{C}_\tau(A)(y) = \bigvee_{x\in X} (e_X(x, y) \odot A(x)).$$

$$e_X(x, x) = \mathcal{C}_\tau(\top_x)(x) \ge \top_x(x) = \top$$

$$\begin{aligned}&\bigvee_{y\in X}(e_X(x, y) \odot e_X(y, z)) \le e_X(x, z)\\ &\text{iff } \bigvee_{y\in X}(\mathcal{C}_\tau(\top_x)(y) \odot \mathcal{C}_\tau(\top_y)(z)) \le \mathcal{C}_\tau(\top_x)(z)\\ &\text{iff } \mathcal{C}_\tau(\bigvee_{y\in X}(\mathcal{C}_\tau(\top_x)(y) \odot \top_y))(z) \le \mathcal{C}_\tau(\top_x)(z)\\ &\text{iff } \mathcal{C}_\tau(\mathcal{C}_\tau(\top_x))(z) \le \mathcal{C}_\tau(\top_x)(z)\end{aligned}$$

Hence e_X is a fuzzy preorder. Since $e_X(x, y) = \mathcal{C}_\tau(\top_x)(y) = \mathcal{I}^*_{\tau^*}(\top^*_x)(y)$, by Theorem 2.2(9),

$$\mathcal{I}_{\tau^*}(A)(y) = \bigwedge_{x\in X} (e_X(x, y) \to A(x)).$$

Example 2.4. Let $(L = [0, 1], \odot, \to, ^*)$ be a complete residuated lattice with a negation defined by

$$x\odot y = (x+y-1)\vee 0, \ \ x \to y = (1-x+y)\wedge 1, \ x^* = 1-x.$$

Let $X = \{x, y, z\}$ be a set and $A_1 = (1, 0.8, 0.6), A_2 = (0.7, 1, 0.7), A_3 = (0.5, 0.7, 1)$.

(1) We define

$$\begin{aligned}\tau &= \{\vee_{i=1}^3(a_i \odot A_i) \mid A = (a_1, a_2, a_3) \in L^X\}\\ &= \{e_X(A)(y) = \bigvee_{x\in X}(e_X(x, y) \odot A(x)) \mid A \in L^X\}\end{aligned}$$

where

$$e_X = \begin{pmatrix} 1 & 0.8 & 0.6\\ 0.7 & 1 & 0.7\\ 0.5 & 0.7 & 1\end{pmatrix}.$$

(T1) For $\bot_X \in L^X$, $e_X(\bot_X) = \bot_X \in \tau$. For $\top_X \in L^X$, $e_X(\top_X) = \top_X \in \tau$.

(T2) For $e_X(A_i) \in \tau$ for each $i \in \Gamma$, $\bigvee_{i\in\Gamma} e_X(A_i) = e_X(\bigvee_{i\in\Gamma} A_i) \in \tau$. Moreover, since $e_X(A)(x) \ge e_X(x, x) \odot A(x) = A(x)$ and $e_X(e_X(A)) = e_X(A)$,

$$\bigwedge_{i\in\Gamma} e_X(A_i) \le e_X(\bigwedge_{i\in\Gamma} e_X(A_i)) \le \bigwedge_{i\in\Gamma} e_X(e_X(A_i)).$$

Hence $\bigwedge_{i\in\Gamma} e_X(A_i) = e_X(\bigwedge_{i\in\Gamma} A_i) \in \tau$.

(T3) For $e_X(A) \in \tau$, $\alpha \odot e_X(A) = e_X(\alpha \odot A) \in \tau$.

(T4) Since $\alpha \odot e_X(\alpha \to e_X(A)) \le e_X(e_X(A)) = e_X(A)$, we have

$$\alpha \to e_X(A) \le e_X(\alpha \to e_X(A)) \le \alpha \to e_X(A)$$

Hence, for $e_X(A) \in \tau$, $\alpha \to e_X(A) = e_X(\alpha \to e_X(A)) \in \tau$. Hence τ is an Alexandrov topology on X.

(2) For $B_1 = (0.7, 0.3, 0.6), B_1 = (0.5, 0.9, 0.3)$, we obtain

$$\mathcal{I}_\tau(B_1) = (0.5, 0.3, 0.6), \mathcal{I}_\tau(B_2) = (0.5, 0.6, 0.3),$$
$$\mathcal{C}_\tau(B_1) = (0.7, 0.5, 0.6), \mathcal{C}_\tau(B_2) = (0.6, 0.9, 0.6).$$

Let $\Phi : L^X \to L$ as follows

$$\Phi(B) = \begin{cases} 0.9, & \text{if } B = B_1,\\ 0.8, & \text{if } B = B_2,\\ 0, & \text{otherwise.}\end{cases}$$

$$\begin{aligned}\sqcap\Phi &= \bigwedge_{A\in L^X}(\Phi(A)\to A)\\ &= (\Phi(B_1)\to B_1)\wedge(\Phi(B_2)\to B_2)\\ &= (0.9\to(0.7,0.3,0.6))\wedge(0.8\to(0.5,0.9,0.3))\\ &= (0.7,0.4,0.5)\end{aligned}$$

$$\begin{aligned}\sqcap\mathcal{I}_\tau^{\to}(\Phi) &= \bigwedge_{A\in L^X}(\Phi(A)\to\mathcal{I}_\tau(A))\\ &= (\Phi(B_1)\to\mathcal{I}_\tau(B_1))\wedge(\Phi(B_2)\to\mathcal{I}_\tau(B_2))\\ &= (0.9\to(0.5,0.3,0.6))\wedge(0.8\to(0.5,0.6,0.3))\\ &= (0.6,0.4,0.5)\end{aligned}$$

Thus, $\mathcal{I}(\sqcap\Phi)=\sqcap\mathcal{I}_\tau^{\to}(\Phi)$.

$$\begin{aligned}\sqcup\Phi &= \bigvee_{A\in L^X}(\Phi(A)\odot A)\\ &= (\Phi(B_1)\odot B_1)\vee(\Phi(B_2)\odot B_2)\\ &= (0.9\odot(0.7,0.3,0.6))\vee(0.8\odot(0.5,0.9,0.3))\\ &= (0.6,0.7,0.5)\end{aligned}$$

$$\begin{aligned}\sqcup\mathcal{C}_\tau^{\to}(\Phi) &= \bigvee_{A\in L^X}(\Phi(A)\odot\mathcal{C}_\tau(A))\\ &= (\Phi(B_1)\odot\mathcal{C}_\tau(B_1))\vee(\Phi(B_2)\odot\mathcal{C}_\tau(B_2))\\ &= (0.9\odot(0.7,0.5,0.6))\vee(0.8\odot(0.6,0.9,0.6))\\ &= (0.6,0.7,0.5)\end{aligned}$$

Thus, $\mathcal{C}_\tau(\sqcup\Phi)=\sqcup\mathcal{C}_\tau^{\to}(\Phi)$.

(3) We define

$$\begin{aligned}\tau^* &= \{\wedge_{i=1}^3(a_i\to A_i^*)\mid A=(a_1,a_2,a_3)\in L^X\}\\ &= \{\bigwedge_{y\in X}(A(y)\to e_X^*(-,y))\mid A\in L^X\}\\ &= \{\bigwedge_{y\in X}(e_X(-,y)\to B(y))\mid B\in L^X\}.\end{aligned}$$

For B_1, B_2 and Φ in (2), we obtain

$$\begin{aligned}&\mathcal{I}_{\tau^*}(B_1)=\mathcal{C}_\tau^*(B_1^*)=(0.6,0.3,0.6),\\ &\mathcal{I}_{\tau^*}(B_2)=\mathcal{C}_\tau^*(B_2^*)=(0.5,0.6,0.3),\\ &\mathcal{C}_{\tau^*}(B_1)=(0.7,0.4,0.6),\mathcal{C}_{\tau^*}(B_2)=(0.7,0.9,0.6).\end{aligned}$$

Since $\sqcap\Phi=(0.7,0.4,0.5)$ and

$$\begin{aligned}\sqcap\mathcal{I}_{\tau^*}^{\to}(\Phi) &= \bigwedge_{A\in L^X}(\Phi(A)\to\mathcal{I}_{\tau^*}(A))\\ &= (\Phi(B_1)\to\mathcal{I}_{\tau^*}(B_1))\wedge(\Phi(B_2)\to\mathcal{I}_{\tau^*}(B_2))\\ &= (0.9\to(0.6,0.3,0.6))\wedge(0.8\to(0.5,0.6,0.3))\\ &= (0.7,0.4,0.5)\end{aligned}$$

we have $\mathcal{I}_{\tau^*}(\sqcap\Phi)=\sqcap\mathcal{I}_{\tau^*}^{\to}(\Phi)$.

Since $\sqcup\Phi=(0.6,0.7,0.5)$ and

$$\begin{aligned}\sqcup\mathcal{C}_{\tau^*}^{\to}(\Phi) &= \bigvee_{A\in L^X}(\Phi(A)\odot\mathcal{C}_{\tau^*}(A))\\ &= (\Phi(B_1)\odot\mathcal{C}_{\tau^*}(B_1))\vee(\Phi(B_2)\odot\mathcal{C}_{\tau^*}(B_2))\\ &= (0.9\odot(0.7,0.4,0.6))\vee(0.8\odot(0.7,0.9,0.6))\\ &= (0.6,0.7,0.5),\end{aligned}$$

then $\mathcal{C}_{\tau^*}(\sqcup\Phi)=\sqcup\mathcal{C}_{\tau^*}^{\to}(\Phi)$.

3. Conclusions

The fuzzy complete lattice is defined with join and meet operators on fuzzy partially ordered sets. Alexandrov topologies are the extensions of fuzzy topology and strong topology.

Several properties of join and meet operators induced by Alexandrov topologies in complete residuated lattices have been elicited and proved. In addition, with the concepts of Zhang's completeness, some extensions of interior and closure operators are investigated in the sense of Pawlak's rough set theory on complete residuated lattices. It is expected to find some interesting functorial relationships between Alexandrov topologies and two operators.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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