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COMPARISON AMONG SEVERAL ADJACENCY PROPERTIES FOR A DIGITAL PRODUCT

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Abstract. Owing to the notion of a normal adjacency for a digital product in [8], the study of product properties of digital topological properties has been substantially done. To explain a normal adjacency of a digital product more efficiently, the recent paper [22] proposed an S-compatible adjacency of a digital product. Using an S-compatible adjacency of a digital product, we also study product properties of digital topological properties, which improves the presentations of a normal adjacency of a digital product in [8]. Besides, the paper [16] studied the product property of two digital covering maps in terms of the L_S- and the L_C-property of a digital product which plays an important role in studying digital covering and digital homotopy theory. Further, by using HS- and HC-properties of digital products, the paper [18] studied multiplicative properties of a digital fundamental group. The present paper compares among several kinds of adjacency relations for digital products and proposes their own merits and further, deals with the problem: consider a Cartesian product of two simple closed k_i-curves with l_i elements in $\mathbf{Z}^{n_i}, i \in \{1,2\}$ denoted by $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$. Since a normal adjacency for this product and the L_C-property are different from each other, the present paper address the problem: for the digital product does it have both a normal k-adjacency of $\mathbf{Z}^{n_1+n_2}$ and another adjacency satisfying the L_C-property? This research plays an important role in studying product properties of digital topological properties.

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1. Introduction

The study of product (or multiplicative) properties of a certain digital topological property is very important in digital topology. For instance, the papers [8, 11, 16, 18, 22, 24, 25] introduced the notions of a normal adjacency, an S-compatible adjacency, L_C-, L_S-properties and HC-, HS-properties for studying digital products, and further, dealt with several cases for investigating multiplicative properties of a digital fundamental group. Indeed, a digital image (X, k) can be considered to be a set $X \subset \mathbf{Z}^n$ with a k-adjacency relation on $\mathbf{Z}^n$ (or an adjacency graph), where $\mathbf{Z}^n$ is the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbf{N}$ and $\mathbf{N}$ is the set of natural numbers. To study digital topological properties of a digital image (X, k), we have used various tools such as a digital fundamental group [4, 13, 18], digital covering spaces [7, 8, 14, 15, 16, 19, 20], digital homotopy equivalences [6, 17, 26] and digital k-surface structures [2, 3, 5, 10, 21].

In graph theory it is well known that both a *normal* adjacency and a *Cartesian* adjacency for a Cartesian product of graphs in [1, 27] have substantially contributed to the study of a Cartesian product of graphs. Motivated by this approach, in relation to the study of product properties of a certain digital property, since we have often used several kinds of adjacency relations for a digital product, the present paper compares them, proposes some merits of their own approaches and deals with the problem: for $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$, are there both its normal k-adjacency of $\mathbf{Z}^{n_1+n_2}$ and another adjacency satisfying the L_C-property?

The paper is organized as follows: Section 2 provides basic notions. Section 3 studies product properties related to the study of digital products. Section 4 compares several kinds of adjacency relations related to digital products and proposes their own merits, which plays an important role in investigating digital topological properties of a digital product. Besides, for $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$, it deals with an existence problem of both its normal k-adjacency of $\mathbf{Z}^{n_1+n_2}$ and another adjacency satisfying the L_C-property for the digital product. This research plays an important role in studying multiplicative properties of digital topological properties. Finally, Section 5 concludes the paper with a summary.

2. Preliminaries

A (binary) digital image (X,k) can be considered as a subset $X \subset \mathbf{Z}^n$ with a k-adjacency relation of $\mathbf{Z}^n$. For $a,b \in \mathbf{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ is called a digital interval [29]. Further, let us recall the following: let $p := (p_i)_{i\in[1,n]_{\mathbf{Z}}}$ be a point of $\mathbf{Z}^n$ and m an integer in $[1,n]_{\mathbf{Z}}$. Consider all points $q := (q_i)_{i\in[1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ satisfying the property of (2.1) [8] such that $p \neq q$

$$\left\{ \begin{array}{l} \bullet \text{ there are at most } m \text{ indices } i \text{ such that } |p_i - q_i| = 1 \text{ and} \\ \bullet \text{ for all other indices } i,\ p_i = q_i. \end{array} \right\} \tag{2.1}$$

The number of such points is [15] (for more details, see [17])

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\ i!}. \tag{2.2}$$

As a generalization of the typical "4-adjacent" and "8-adjacent" well established in the context of 2-dimensional integer grids, we will say that two points $p,q \in \mathbf{Z}^n$ are *k-adjacent* if they satisfy the condition (2.1), where $k := k(m,n)$ of (2.2) [8] (see also [15, 17]).

Owing to the phrase "*at most m*" from the first bullet of (2.1), we can obviously see that the points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n) \in \mathbf{Z}^n$ may differ in as many as m coordinates. Thus, in general, we obtain that [18] if two points $x,y \in \mathbf{Z}^n$ are $k(m,n)$-adjacent, then they are obviously $k(m',n)$-adjacent, where $m \leq m'$ [24]. This observation will be often used in Section 4.

Using the k-adjacency relations of $\mathbf{Z}^n$ in (2.2), we study digital topological properties of a set $X \subset \mathbf{Z}^n$ with a k-adjacency, $n \in \mathbf{N}$. This has been often used for representing digital continuity, a digital isomorphism, a digital homotopy, a digital k-surface structure, etc. Owing to the digital k-connectivity paradox in [29], we have often used a binary digital images such as $(X,k,\bar{k},\mathbf{Z}^n)$, where the adjacency k (resp. $\bar{k}$) is concerned with the set X (resp. $\mathbf{Z}^n \setminus X$) we remind the reader that $k \neq \bar{k}$ except the case $(\mathbf{Z},2,2,X)$. However, in this paper we are not concerned with $\bar{k}$-adjacency between two points in $\mathbf{Z}^n \setminus X$.

We say that a set $X \subset \mathbf{Z}^n$ is k-connected if it is not a union of two disjoint non-empty subsets of X that are not k-adjacent to each other [29]. For an adjacency relation k of $\mathbf{Z}^n$, a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$

[29]. If $x_0 = x$ and $x_l = y$, then we say that the length of the simple k-path is l. A simple closed k-curve with l elements in $\mathbf{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [8] (see also [10]), is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i+1(mod\, l)$ or $i = j+1(mod\, l)$ [29]. Besides, for $\mathbf{Z}^n$ we remind the following [29]:

$$\left\{ \begin{array}{l} N_k(x) := \{x' \mid x \text{ is } \ k\text{-adjacent to } x' \text{ in } \mathbf{Z}^n\} \text{ and} \\ N_k^*(x) := N_k(x) \cup \{x\}. \end{array} \right\} \qquad (2.3)$$

As a generalization of $N_k^*(x)$ in $\mathbf{Z}^n$, for a multi-dimensional digital image (X, k) and a point $x \in X \subset \mathbf{Z}^n$, the notion of a (digital) k-neighborhood of a point x with radius $\varepsilon \in \mathbf{N}$ was established [8], as follows.

$$N_k(x_0, \varepsilon) := \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \qquad (2.4)$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x in X. Besides, we need to remind that while $N_k(x)$ in (2.3) does not contain the point x, the set $N_k(x_0, \varepsilon)$ has the point x_0. This difference should be reminded in Section 4. If a point x in a digital image (X, k) is isolated, then for any $\varepsilon \in \mathbf{N}$ $N_k(x, \varepsilon)$ is a singleton $\{x\}$. The (digital) k-neighborhood of (2.4) will be often used for establishing several compatible adjacency relations for a Cartesian product of two digital images (see Section 3) and digital continuity.

The original version of digital continuity was developed in [31]: let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $f : (X, k_0) \to (Y, k_1)$ be a function. We say that f is (k_0, k_1)-continuous if the image under f of every k_0-connected subset of X is k_1-connected [31].

In view of the k-neighborhood of (2.4), since for each point x of a digital image (X, k) in $\mathbf{Z}^n$ there is always $N_k(x, 1) \subset (X, k)$, the notion of *digital continuity* in [31] can be represented in terms of the following form [8] (see also [14]), which has been efficiently used in studying product properties of digital topological invariants.

Definition 1. [8] *(see also* [14]*) Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

We have often used the following notion of a (k_0, k_1)-*isomorphism* instead of a (k_0, k_1)-homeomorphism in [4]: for two digital images (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* [30] (see also [9, 13]) if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous [9] (see also [14]), and we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism*.

3. Some properties related to the study of adjacency relations for digital products

In graph theory two compatible adjacency relations for a Cartesian product such as *normal adjacency* in [1] and *Cartesian product adjacency* in [27] play important roles in studying graphs. For given two digital images $(X_i, k(m_i, n_i)), i \in \{1, 2\}$, consider the Cartesian product $X_1 \times X_2 \subset \mathbf{Z}^{n_1+n_2}$ with a $k(m, n_1+n_2)$-adjacency of $\mathbf{Z}^{n_1+n_2}$ (see (2.2)) which is called a *digital product*. Thus the paper [8] developed the notion of a normal adjacency for a digital product to study a digital fundamental group of a digital product as follows:

Definition 2. [8] *For two digital images (X, k_1) in $\mathbf{Z}^{n_1}$, (Y, k_2) in $\mathbf{Z}^{n_2}$, consider the digital product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. Then we say that two points $(x, y) \in X \times Y$, $(x', y') \in X \times Y$ are normally k-adjacent to each other if and only if*

(1) x is k_1-adjacent to x' and $y = y'$;

(2) y is k_2-adjacent to y' and $x = x'$; or

(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'.

Unfortunately, the paper [8] misused an 8-adjacency on $MSC_4 \times MSC_4 := SC_4^{2,8} \times SC_4^{2,8} \subset \mathbf{Z}^4$ as a normal adjacency, where $MSC_4 := SC_4^{2,8}$ means a simple closed 4-curve with eight elements in $\mathbf{Z}^2$ (see Figure 1). Indeed, the product $SC_4^{2,8} \times SC_4^{2,8}$ does not have any normal k-adjacency [25], which invokes the development of another property for studying digital products such as the L_S- and the L_C-property in [16]. Finally, it turns out that this product with a $k(1, 4) :=$ 8-adjacency has the L_C-property (for more details, see [16]).

The following simple closed 4- and 8-curves in $\mathbf{Z}^2$ in [7, 8, 9] and simple closed 18- and 26-curves in $\mathbf{Z}^3$ in [8, 12] will be often used later in the paper (see Figure 1).

$$\left\{\begin{array}{l} SC_4^{2,8} \approx_4 ((0,0),(1,0),(2,0),(2,1),(2,2),(1,2),(0,2),(0,1)), \\ SC_8^{2,6} := MSC_8 \approx_8 ((0,0),(1,1),(1,2),(0,3),(-1,2),(-1,1)), \\ SC_8^{2,4} \approx_8 ((0,0),(1,1),(2,0),(1,-1)), \\ SC_8^{2,8} := ((0,0),(1,1),(2,2),(1,3),(0,4),(-1,3),(-2,2),(-1,1)), \\ MSC_{18} := ((0,0,0),(1,-1,0),(1,-1,1),(2,0,1),(1,1,1),(1,1,0)), \\ SC_{18}^{3,6} := ((0,0,0),(1,0,1),(1,1,2),(0,2,2),(-1,1,2),(-1,0,1)), \\ SC_{26}^{3,4} := ((0,0,0),(1,1,1),(0,2,2),(-1,1,1). \end{array}\right\} \tag{3.1}$$

We need to recall that the set $MSC_{18} := (c_i)_{i\in[0,5]_{\mathbf{Z}}}$ in Figure 1 is a simple closed 18-curve with six element in $\mathbf{Z}^3$ which is different from the set $SC_{18}^{3,6}$ suggested in (3.1) (see the elements c_1 and c_2 of MSC_{18} in Figure 1).

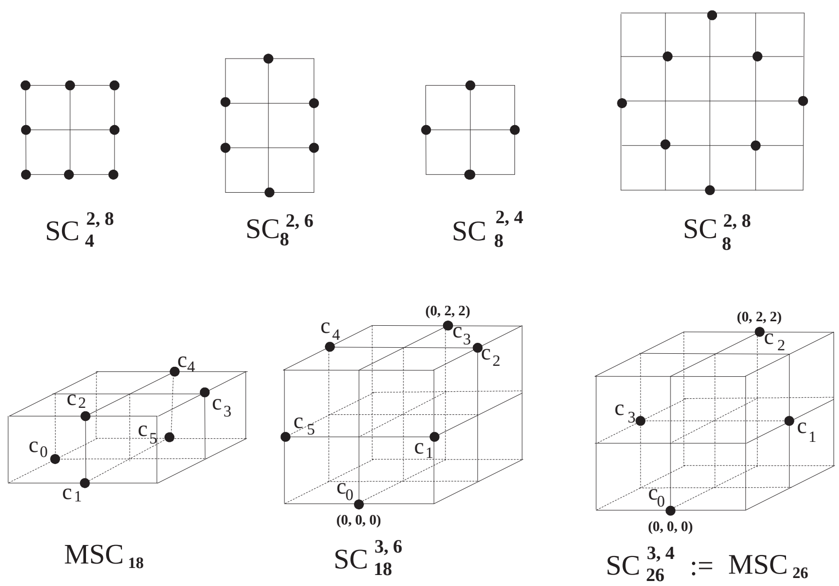

FIGURE 1. Various kinds of simple closed k-curves [8, 13, 15].

To represent a digital product as a matrix, we use the following notation:

$SC_{k_1}^{n_1,l_1} := (a_i)_{i\in[1,l_1]_{\mathbf{Z}}}$ and $SC_{k_2}^{n_2,l_2} := (b_j)_{j\in[1,l_2]_{\mathbf{Z}}}$ (as examples, see (3.1)).

Then we can obtain their products in $\mathbf{Z}^{n_1+n_2}$ as sets without any k-adjacency at the moment as follows:

$$SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} := (c_{i\,j})_{(i,j)\in[1,l_1]_{\mathbf{Z}}\times[1,l_2]_{\mathbf{Z}}}, \text{ as a matrix} \tag{3.2}$$

where $c_{i\,j} := (a_i, b_j)$.

Let us now consider the natural projection maps p_1 and p_2 defined by

$$\left\{\begin{array}{l} p_1 : SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \to SC_{k_1}^{n_1,l_1} \text{ and} \\ p_2 : SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \to SC_{k_2}^{n_2,l_2} \text{ such that} \\ p_1(c_{i\,j}) = a_i, i \in [1,l_1]_{\mathbf{Z}}, \text{ and } p_2(c_{i\,j}) = b_j, j \in [1,l_2]_{\mathbf{Z}}. \end{array}\right\} \tag{3.3}$$

Indeed, we observe that each of the maps p_1 and p_2 need not be a digitally continuous map (see [16]).

In view of the above discussion about the Cartesian product property of given digitally continuous maps, we obtain the following:

Remark 3.1. [16, 22] *Let (X, k_0), (Y, k_1), (Z, k_2), and (W, k_3) be digital images in $\mathbf{Z}^{n_0}$, $\mathbf{Z}^{n_1}$, $\mathbf{Z}^{n_2}$, and $\mathbf{Z}^{n_3}$, respectively. Let $f : X \to Y$ and $g : Z \to W$ be (k_0, k_1)- and (k_2, k_3)-continuous maps, respectively. Then, not every k-adjacency for $X \times Z \subset \mathbf{Z}^{n_0+n_2}$ and k'-adjacency for $Y \times W \subset \mathbf{Z}^{n_1+n_3}$ makes the Cartesian product map $f \times g : X \times Z \to Y \times W$ given by $f \times g(x, z) = (f(x), g(z))$ be (k, k')-continuous.*

4. Comparison among normal k-adjacency, S-compatible adjacency, L_S-, and L_C-properties for digital products

To study product properties of two digital coverings, the L_S-property of a digital product $(X_1 \times X_2, k)$ was established as follows:

Definition 3. [16] *For digital images (X_i, k_i) in $\mathbf{Z}^{n_i}, i \in \{1, 2\}$, we say that the digital product $(X_1 \times X_2, k)$ has the L_S-property (relative to (X_i, k_i)) if each point $(c_i, d_j) \in X_1 \times X_2$ has $N_k((c_i, d_j), 1) \subset X_1 \times X_2$ which is $(k, 8)$-isomorphic with $N_8((0, 0), 1)$ in $(\mathbf{Z}^2, 8)$.*

In view of Remark 3.1, to study a product problem of a digital isomorphism, we need a compatible adjacency of a digital product such as *an S-compatible k-adjacency* (see Definition 4) which is used for studying both a product problem of two digital isomorphisms and the multiplicative property of a digital fundamental group. Thus, motivated by the strong product in graph theory [1], let us now recall an S-compatible adjacency for a digital product as follows.

Definition 4. [22] *For two digital images (X, k_1) in $\mathbf{Z}^{n_1}$ and (Y, k_2) in $\mathbf{Z}^{n_2}$, consider a Cartesian product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. We say that a k-adjacency on $X \times Y$ is S-compatible with the k_i-adjacency on X and Y, $i \in \{1, 2\}$ if every point (x, y) in $X \times Y$ satisfies the following property: for two distinct points (x, y) and (x', y') in $X \times Y$*

$$(x', y') \in N_k((x, y), 1) \Leftrightarrow x' \in N_{k_1}(x, 1), y' \in N_{k_2}(y, 1).$$

For instance, we obtain the following S-compatible adjacencies for the given sets: $([a, b]_{\mathbf{Z}} \times [c, d]_{\mathbf{Z}}, 8)$ and $(SC_8^{2,6} \times [a, b]_{\mathbf{Z}}, 26)$.

Besides, the following adjacency is often used in studying digital products.

Definition 5. [16] *For digital images (X_i, k_i) in $\mathbf{Z}^{n_i}, i \in \{1, 2\}$, we say that $(X_1 \times X_2, k)$ has the L_C-property if for each point $(c_i, d_j) \in X_1 \times X_2$ the set $N_k((c_i, d_j), 1) \subset X_1 \times X_2$ is $(k, 4)$-isomorphic with $N_4((0, 0), 1) \subset (\mathbf{Z}^2, 4)$.*

Remark 4.1 (Properties and merits of a normal adjacency, an S-compatible adjacency, L_S- and L_C-properties)**.** *Comparing among a normal adjacency of a digital product, an S-compatible adjacency, the properties L_S and L_C, we observe that they have their own features and some utilities. Assume $k_1 := k(m_1, n_1)$, where $m_1 = n_1$ and $k_2 := k(m_2, n_2)$, where $m_2 = n_2$. In view of Definition 4, it is obvious that an S-compatible $k(m, n_1 + n_2)$-adjacency on the digital product is not smaller than $3^{n_1+n_2} - 1$, where $m = n_1 + n_2$ and $k := k(m, n_1 + n_2)$. In general, for the $k_i := k(m_i, n_i)$-adjacency of (X, k_1) and (Y, k_2), $i \in \{1, 2\}$, an S-compatible $k := k(m, n_1 + n_2)$-adjacency of the Cartesian product $X \times Y$ should at least have the number $m \in [m_1 + m_2, n_1 + n_2]_{\mathbf{Z}}$* [24]*. However, not every number $m \in [m_1 + m_2, n_1 + n_2]_{\mathbf{Z}}$ formulates an S-compatible $k := k(m, n_1 + n_2)$-adjacency on the digital product* [22]*. Namely, for $SC_{k_i}^{n_i,l_i}, i \in \{1, 2\}$ the existence of an S-compatible $k := k(m, n_1 + n_2)$-adjacency on $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ depends on the situation. Thus, depending on the situation, we use the tools such as a normal adjacency, an S-compatible adjacency, L_S- and L_C-properties.*

Example 4.2. *By using the simple closed k-curves in Figure 1, we obtain S-compatible k-adjacencies of the following several digital products* [22]*:*

$$(SC_{18}^{3,6} \times SC_8^{2,4}, 242), (SC_8^{2,6} \times SC_{18}^{3,6}, 242),$$

where $242 = k(5, 5)$.

Remark 4.3. *Even though the existence of an* S*-compatible* k*-adjacency of a digital product does not always hold, if an* S*-compatible* k*-adjacency of a digital product exists, then it plays an important role in studying digital products. Indeed, the papers* [21, 24] *referred to a relation between a normal* k*-adjacency of a digital product and the* L_S*-property in Definition 3.*

Let us now recall a relation between a normal k-adjacency on the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ and the L_S-property of this product with some k-adjacency (for more detail, see the papers [21, 22, 24]).

Proposition 4.4. [21] *For* $SC_{k_i}^{n_i,l_i}, i \in \{1, 2\}$ *there is a normal* k*-adjacency on the digital product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ *if and only if the digital product has the* L_S*-property.*

Remark 4.5. *(1) A normal* k*-adjacency on the digital product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ *is equivalent to an* S*-compatible adjacency for this digital product* [22].

(2) Not every normal k*-adjacency of* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ *satisfies the* L_C*-property.*

(3) For a digital product, none of a normal adjacency and the L_C*-property implies the other.*

Proof: The proof of (1) is straightforward.

(2) As a counter example, consider the digital product $SC_8^{2,6} \times SC_8^{2,6} \subset \mathbf{Z}^4$. Even though this has a normal $k(4, 4) := 80$-adjacency, it cannot have the L_C-property.

(3) In view of Definitions 4 and 5, a normal k-adjacency of a digital product cannot be a k-adjacency satisfying the L_C-property and vice versa. □

In view of Remark 4.5, we may now pose the following question: For $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ does it have both a normal k-adjacency of $\mathbf{Z}^{n_1+n_2}$ and another adjacency satisfying the L_C-property?

In relation to the above query, we have the following.

Theorem 4.6. *For some* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$*, there are both a normal* k*-adjacency on this product and another adjacency satisfying the* L_C*-property.*

Proof: Before proving this theorem, as mentioned in Remark 4.5(3), we need to refer that an existence of a normal adjacency for a digital product need not imply an existence of an adjacency satisfying the

L_C-property and vice versa. For instance, consider the digital product $SC_8^{2,6} \times SC_8^{2,6} \subset \mathbf{Z}^4$. Then it has a normal $80 = k(4,4)$-adjacency. But it cannot have any adjacency satisfying the L_C-property. Besides, consider the digital product $SC_4^{2,8} \times SC_4^{2,8} \subset \mathbf{Z}^4$. Even though it has an $k(1,4)$(or 8)-adjacency satisfying the L_C-property, it cannot have any normal adjacency.

In order to prove the assertion, we need to recall the following properties. For $x_i \in SC_k^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ $N^*_{3^n-1}(x_i) \cap SC_k^{n,l}$ need not have the only three points $x_{i-1(mod\, l_i)}, x_i$ and $x_{i+1(mod\, l_i)}$ (for more details, see [22]). For instance, consider the sets $SC_4^{2,8}$ and MSC_{18}. Besides, by Remark 4.1, if $k_i = 3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$), then $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a normal k-adjacency, where $k := k(m_0, n_1 + n_2) = 3^{n_1+n_2} - 1$ (or $m_0 \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$). In terms of this approach, the paper [24] investigated several cases as follows: consider $SC_{k_i}^{n_i,l_i}, n_i \geq 2$, where $k_i := k(m_i, n_i)$ in (2.2) and $i \in \{1, 2\}$.

(Case 1) Let $k_i := k_i(m_i, n_i)$ be equal to $3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$), $i \in \{1, 2\}$. Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a normal $k(m_0, n_1 + n_2)$-adjacency, where $m_0 = n_1 + n_2$.

(Case 2) Let $k_1 := k_1(m_1, n_1)$ be equal to $3^{n_1} - 1$ (or $m_1 \notin [1, n_1 - 1]_{\mathbf{Z}}$) and assume that the number m_2 taken from the $k_2(m_2, n_2)$-adjacency is equal to $n_2 - 1$ instead of n_2. Further, assume that for each element $x_i \in SC_{k_2}^{n_2,l_2} := (x_i)_{i\in[0,l_2-1]_{\mathbf{Z}}}, n_2 \geq 3$, the set $N^*_{3^{n_2}-1}(x_i) \cap SC_{k_2}^{n_2,l_2}$ has the only three points $x_{i-1(mod\, l_2)}, x_i$ and $x_{i+1(mod\, l_2)}$. Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a normal $k(m_0, n_1 + n_2)$-adjacency, where $m_0 \in [n_1 + n_2 - 1, n_1 + n_2]_{\mathbf{Z}}$.

(Case 3) Let $k_i := k_i(m_i, n_i)$ be not equal to $3^{n_i} - 1, i \in \{1, 2\}$. Assume that for each element $x_i \in SC_{k_i}^{n_i,l_i} := (x_i)_{i\in[0,l_i-1]_{\mathbf{Z}}}, n_i \geq 3$ the set $N^*_{3^{n_i}-1}(x_i) \cap SC_{k_i}^{n_i,l_i}$ has the only three points $x_{i-1(mod\, l_i)}, x_i$ and $x_{i+1(mod\, l_i)}$. If the number m_i taken from the $k_i := k(m_i, n_i)$-adjacency is equal to $n_i - 1$ instead of $n_i, i \in \{1, 2\}$, then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a normal $k(m_0, n_1 + n_2)$-adjacency, where $m_0 \in [n_1 + n_2 - 2, n_1 + n_2]_{\mathbf{Z}}$.

Let us now move into an examination of which a digital digital product has both a normal adjacency and another adjacency satisfying the L_C-property. For instance, $SC_8^{2,6} \times SC_{18}^{3,6} := (d_{i\,j})_{(i,j)\in[1,6]_{\mathbf{Z}}\times[1,6]_{\mathbf{Z}}}$ is represented via the following matrix (see (4.1)), where for each $(i, j) \in$

$[1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}$ $d_{ij} := (a_{i+1}, b_{j+1})$ and $SC_8^{2,6} := (a_i)_{i\in[0,5]_{\mathbf{Z}}}$ and $SC_{18}^{3,6} := (b_i)_{i\in[0,5]_{\mathbf{Z}}}$ in Figure 1.

$$\begin{pmatrix} (0,0,0,0,0) & (0,0,1,0,1) & (0,0,1,1,2) & (0,0,0,2,2) & (0,0,-1,1,2), & (0,0,-1,0,1) \\ (1,1,0,0,0) & (1,1,1,0,1) & (1,1,1,1,2) & (1,1,0,2,2) & (1,1,-1,1,2) & (1,1,-1,0,1) \\ (1,2,0,0,0) & (1,2,1,0,1) & (1,2,1,1,2) & (1,2,0,2,2) & (1,2,-1,1,2) & (1,2,-1,0,1) \\ (0,3,0,0,0) & (0,3,1,0,1) & (0,3,1,1,2) & (0,3,0,2,2) & (0,3,-1,1,2) & (0,3,-1,0,1) \\ (-1,2,0,0,0) & (-1,2,1,0,1) & (-1,2,1,1,2) & (-1,2,0,2,2) & (-1,2,-1,1,2) & (-1,2,-1,0,1) \\ (-1,1,0,0,0) & (-1,1,1,0,1) & (-1,1,1,1,2) & (-1,1,0,2,2) & (-1,1,-1,1,2) & (-1,1,-1,0,1) \end{pmatrix} \tag{4.1}$$

Then we see that this product $SC_8^{2,6} \times SC_{18}^{3,6}$ has a normal $k(m_0,5)$-adjacency, where $m_0 \in \{4,5\}$ and further, $(SC_8^{2,6} \times SC_{18}^{3,6}, k(2,5))$ has the L_C-property.

Besides, consider the product $SC_{18}^{3,6} \times SC_{26}^{3,4} \subset \mathbf{Z}^6$ (see (4.2)).

$$\begin{pmatrix} (0,0,0,0,0,0) & (0,0,0,1,1,1) & (0,0,0,0,2,2) & (0,0,0,-1,1,1) \\ (1,0,1,0,0,0) & (1,0,1,1,1,1) & (1,0,1,0,2,2) & (1,0,1,-1,1,1) \\ (1,1,2,0,0,0) & (1,1,2,1,1,1) & (1,1,2,0,2,2) & (1,1,2,-1,1,1) \\ (0,2,2,0,0,0) & (0,2,2,1,1,1) & (0,2,2,0,2,2) & (0,2,2,-1,1,1) \\ (-1,1,2,0,0,0) & (-1,1,2,1,1,1) & (-1,1,2,0,2,2) & (-1,1,2,-1,1,1) \\ (-1,0,1,0,0,0) & (-1,0,1,1,1,1) & (-1,0,1,0,2,2) & (-1,0,1,-1,1,1) \end{pmatrix} \tag{4.2}$$

Then we observe that it has a normal $k := k(m,6)$-adjacency, where $m \in \{5,6\}$. Besides, $(SC_{18}^{3,6} \times SC_{26}^{3,4}, k(m_0,6))$ has the L_C-property, where $m_0 \in \{3,4\}$. Thus we conclude that $SC_{18}^{3,6} \times SC_{26}^{3,4}$ has both a normal $k := k(m,6)$-adjacency and $k(m_0,6)$-adjacency satisfying the L_C-property.

Remark 4.7. *Using Theorem 4.6, we obtain a lot of advantages to calculate a digital k-fundamental group of a given digital product such as* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbf{Z}^{n_1+n_2}$. *More precisely, in relation to the calculation of* $\pi^{k(m_0,6)}(SC_{18}^{3,6} \times SC_{26}^{3,4}, (a_0,b_0))$, $m_0 \in \{3,4\}$, *we may take two methods as follows: indeed, the calculation is related to the product property of the following two digital covering maps* $p_1 : (\mathbf{Z},0) \to (SC_{18}^{3,6}, a_0)$ *and* $p_2 : (\mathbf{Z},0) \to (SC_{26}^{3,4}, b_0)$. *Owing to Theorem 4.6, we obtain that the product map*

$$p_1 \times p_2 : \mathbf{Z}^2 \to SC_{18}^{3,6} \times SC_{26}^{3,4} \tag{4.3}$$

is both a $(4, k(m_0,6))$*-covering map and an* $(8, k(m,6))$*-covering map, where* $m \in \{5,6\}$. *Hence we use two methods of calculating the digital* $k(t,6)$*-fundamental group of* $SC_{18}^{3,6} \times SC_{26}^{3,4}$ *and its discrete deck transformation group of the covering map in (4.3),* $t \in \{3,4,5,6\}$. *Indeed, we obtain that* $\pi^{k(5,6)}(SC_{18}^{3,6} \times SC_{26}^{3,4}, (a_0,b_0))$ *is isomorphic to the infinite cyclic group such as* $(\mathbf{Z},+)$*, more precisely,* $(6\mathbf{Z},+)$.

5. Summary

The paper have studied several kinds of adjacency relations for studying digital products. As a further work, based on Theorem 4.6, we need to study a more generalized case which have contributed to the study of multiplicative properties of a digital topological properties.

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