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THE COMPLETENESS OF SOME METRICS ON LORENTZIAN WARPED PRODUCT MANIFOLDS WITH FIBER MANIFOLD OF CLASS (B)

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Abstract. In this paper, we prove the existence of warping functions on Lorentzian warped product manifolds and the completeness of the resulting metrics with some prescribed scalar curvatures.

1. Introduction

In [9, 10], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. In this paper, we study also the existence and nonexistence of Lorentzian warped metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner ([6, 7, 8]), if N is a compact Riemannian n-manifold without boundary, $n \geq 3$, then N belongs to one of the following three categories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

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This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [6, 7, 8], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

In [9, 10], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [4], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [4], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [4] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. And we prove the existence of warping functions on Lorentzian warped product manifolds and the completeness of the resulting metrics with some prescribed scalar curvatures. The results of this paper are extensions of the results of Theorem 2.6, Corollary 2.7 and Corollary 3.7 in [5].

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. The Lorentzian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t, x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in N$. (For details, cf. [3] or [4])

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1} u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is a nonempty, connected and compact Riemannian n-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [4], we have the following proposition.

Proposition 2.1. If the scalar curvature of the fiber manifold N is an arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.

Proposition 2.1 implies that in Lorentzian warped product there is no obstruction of the existence of metric with positive scalar curvature. However, the results of [6] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 2.2. Theorem 5.5 in [11] implies that all timelike geodesics are future (resp. past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) for some t_0 and Remark 2.58 in [1] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) for some t_0 (cf. Theorem 4.1 and Remark 4.2 in [2]. In this reference, the warped product metric is $g' = -dt^2 + v(t)g$).

If N admits a Riemannian metric of zero scalar curvature, then we let $u(t) = t^\alpha$ in equation (2.3), where $\alpha \in (0,1)$ is a constant, and we have

$$R(t,x) \leq -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > a.$$

Therefore, from the above fact, Remark 2.2 implies the following:

Theorem 2.3. For $n \geq 3$, let $M = [a, \infty) \times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (B), then on M there is a future geodesically complete Lorentzian metric of negative scalar curvature outside a compact set.

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = t^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

If $R(t,x)$ is the function of only t-variable, then we have the following proposition whose proof is smilar to that of Lemma 1.8 in [10].

Proposition 2.4. If $R(g) = 0$, then there is no positive solution to equation (2.3) with

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. See Proposition 2.4 in [5].

In particular, if $R(g) = 0$, then using Lorentzian warped product it is impossible to obtain a Lorentzian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

Proposition 2.5. Suppose that $R(g) = 0$ and $R(t,x) = R(t) \in C^\infty([a,\infty))$. Assume that for $t > t_0$, there exist an (weak) upper solution $u_+(t)$ and a (weak) lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a (weak) solution $u(t)$ of equation (2.3) such that for $t > t_0, \quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. See Theorem 2.5 in [5].

Theorem 2.6. Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} < R(t) \leq \frac{4n}{n+1} b \frac{1}{t^2} \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a$, $0 < c < 1$ and $0 < b < \frac{(n+1)(n+3)}{4}$ are constants. Then equation (2.3) has a positive solution on $[a, \infty)$ and on M the resulting Lorentzian warped product metric is a future geodesically complete metric.

Proof. Since $R(g) = 0$, put $u_+(t) = t^{\frac{1}{2}}$. Then $u''_+(t) = \frac{-1}{4}t^{\frac{1}{2}-2}$. Hence

$$\frac{4n}{n+1}u''_+(t) - R(t)u_+(t) = \frac{4n}{n+1}\frac{-1}{4}t^{\frac{1}{2}-2} - R(t)t^{\frac{1}{2}}$$
$$= \frac{4n}{n+1}t^{\frac{1}{2}}[\frac{-1}{4}t^{-2} - \frac{n+1}{4n}R(t)] \le \frac{4n}{n+1}\frac{1}{4}t^{\frac{1}{2}-2}[-1+c] \le 0.$$

Therefore $u_+(t)$ is our (weak) upper solution. Since $R(g) = 0$ and $R(t) \le \frac{4n}{n+1}b\frac{1}{t^2}$, we take the lower solution $u_-(t) = t^{-\beta}$ where the constant β‘$(0 < \beta < \frac{n+1}{2})$ will be determined later. Then $u''_-(t) = \beta(\beta+1)t^{-\beta-2}$. Hence

$$\frac{4n}{n+1}u''_-(t) - R(t)u_-(t) \ge \frac{4n}{n+1}\beta(\beta+1)t^{-\beta-2} - \frac{4n}{n+1}b\frac{1}{t^2}t^{-\beta}$$
$$= \frac{4n}{n+1}t^{-\beta-2}[\beta(\beta+1) - b] \ge 0$$

if β is sufficiently close to $\frac{n+1}{2}$. Thus $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$ for large t. Hence Proposition 2.5 implies that equation (2.3) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$ for large t. And since β is sufficiently close to $\frac{n+1}{2}$, $-\frac{2\beta}{n+1} + 1 > 0$. Therefore

$$\begin{aligned}\int_{t_0}^{+\infty}\left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}}dt &= \int_{t_0}^{+\infty}\frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}}dt \\ &\ge \int_{t_0}^{+\infty}\frac{u_-(t)^{\frac{2}{n+1}}}{\sqrt{1+u_-(t)^{\frac{4}{n+1}}}}dt \\ &= \int_{t_0}^{+\infty}\frac{t^{-\frac{2\beta}{n+1}}}{\sqrt{1+t^{-\frac{4\beta}{n+1}}}}dt \\ &\ge \frac{1}{\sqrt{2}}\int_{t_0}^{+\infty}t^{-\frac{2\beta}{n+1}}dt = +\infty\end{aligned}$$

and

$$\int_{t_0}^{+\infty} f(t)dt = \int_{t_0}^{+\infty} u(t)^{\frac{2}{n+1}} dt$$
$$\geq \int_{t_0}^{+\infty} u_-(t)^{\frac{2}{n+1}} dt = \int_{t_0}^{+\infty} t^{-\frac{2\beta}{n+1}} dt = +\infty,$$

which, by Remark 2.2, implies that the resulting warped product metric is a future geodesically complete one.

Theorem 2.7. Suppose that $R(g) = 0$. Assume that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a function such that

$$\frac{4n}{n+1} bt^{-2} < R(t) < \frac{4n}{n+1} dt^s,$$

where b, d are positive constants. If $b > \frac{(n+1)(n+3)}{4}$, then equation (2.3) has a positive solution on $[a, \infty)$ and on M the resulting Lorentzian warped product metric is not a future geodesically complete metric.

Proof. Put $u_-(t) = e^{-t^\alpha}$, where $\alpha > \frac{s+2}{2}$ is a positive constant. Then $u_-''(t) = e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}]$. Hence

$$\begin{aligned}
&\frac{4n}{n+1} u_-''(t) - R(t)u_-(t) \\
&= \frac{4n}{n+1} e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2}] - R(t)e^{-t^\alpha} \\
&\geq \frac{4n}{n+1} e^{-t^\alpha}[\alpha^2 t^{2\alpha-2} - \alpha(\alpha-1)t^{\alpha-2} - dt^s] \geq 0
\end{aligned}$$

for large t and α such that $\alpha > \frac{s+2}{2}$. Thus, for large t, $u_-(t)$ is a (weak) lower solution.

Since $R(g) = 0$ and $R(t) \geq \frac{4n}{n+1} b \frac{1}{t^2}$, we take the upper solution $u_+(t) = t^{-\delta}$ where the constant δ‘($\delta > \frac{n+1}{2}$) will be determined later. Then $u_+''(t) = \delta(\delta+1)t^{-\delta-2}$. Hence

$$\begin{aligned}
&\frac{4n}{n+1} u_+''(t) - R(t)u_+(t) \leq \frac{4n}{n+1}\delta(\delta+1)t^{-\delta-2} - \frac{4n}{n+1} b \frac{1}{t^2} t^{-\delta} \\
= \quad &\frac{4n}{n+1} t^{-\delta-2}[\delta(\delta+1) - b] \leq 0
\end{aligned}$$

if δ is sufficiently close to $\frac{n+1}{2}$. Thus $u_+(t)$ is a (weak) upper solution and $0 < u_-(t) < u_+(t)$ for large t. Hence Proposition 2.5 implies that equation (2.3) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) < u(t) < u_+(t)$ for large t. And since δ is sufficiently close to $\frac{n+1}{2}$, $-\frac{2\delta}{n+1} + 1 < 0$. Therefore

$$\begin{aligned}\int_{t_0}^{+\infty} \left(\frac{f(t)^2}{1+f(t)^2}\right)^{\frac{1}{2}} dt &= \int_{t_0}^{+\infty} \frac{u(t)^{\frac{2}{n+1}}}{\sqrt{1+u(t)^{\frac{4}{n+1}}}} dt \\ &\leq \int_{t_0}^{+\infty} u_+(t)^{\frac{2}{n+1}} dt \\ &= \int_{t_0}^{+\infty} t^{-\frac{2\delta}{n+1}} dt < +\infty\end{aligned}$$

and

$$\begin{aligned}&\int_{t_0}^{+\infty} f(t)dt = \int_{t_0}^{+\infty} u(t)^{\frac{2}{n+1}} dt \\ &\leq \int_{t_0}^{+\infty} u_+(t)^{\frac{2}{n+1}} dt = \int_{t_0}^{+\infty} t^{-\frac{2\delta}{n+1}} dt < +\infty,\end{aligned}$$

which, by Remark 2.2, implies that the resulting warped product metric is not a future geodesically complete one.

Remark 2.8 In case that $R(g) = 0$, we see that the function $R(t) = \frac{(n+1)(n+3)}{4}\frac{1}{t^2}$ is a fiducial point whether the resulting warped product metric is geodesically complete or not. Note that $u(t) = t^{-\frac{n+1}{2}}$ is a solution of equation (2.3) when $R(t) = \frac{(n+1)(n+3)}{4}\frac{1}{t^2}$. In case that $u(t) = t^{-\frac{n+1}{2}}$, we know that the resulting warped product metric is a geodesically complete one.

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