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GENERALIZATION OF EXTENDED APPELL'S AND LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

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Abstract. Recently, Liu and Wang generalized Appell's and Lauricella's hypergeometric functions. Motivated by the work of Liu and Wang, the main object of this paper is to present new generalizations of Appell's and Lauricella's hypergeometric functions. Some integral representations, transformation formulae, differential formulae and recurrence relations are obtained for these new generalized Appell's and Lauricella's functions.

1. Introduction

Very recently, Parmar [10] introduced and investigated some fundamental properties and characteristics of more generalized beta type function $B_p^{(\alpha,\beta;m)}$ defined by (see [10])

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\;{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt \tag{1.1}$$

$$(\Re(p)>0,\ \Re(x)>0,\ \Re(y)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(m)>0).$$

When $m=1$, (1.1) reduces to the well-known generalized beta type function given in [1]:

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\;{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt \tag{1.2}$$

$$(\Re(p)>0,\ \Re(x)>0,\ \Re(y)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0).$$

For $\alpha=\beta$, (1.2) reduces to

$$B_p(x,y)=B_p^{(\alpha,\alpha)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(\frac{-p}{t(1-t)}\right)dt\ (\Re(p)>0), \tag{1.3}$$

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which was firstly introduced by Chaudhry ital [6]. Clearly, the classical beta function $B(x, y)$ is given by

$$B(x, y) = B_0(x, y) = B_0^{(\alpha,\beta)}(x, y).$$

Using (1.3), Chaudhry ital [7] extended the Gauss hypergeometric function as follows:

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \; B_p(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!} \tag{1.4}$$

$$(p \geq 0; \; |z| < 1; \; \Re(c) > \Re(b) > 0).$$

By appealing to $B_p^{(\alpha,\beta)}(x, y)$, Özergin ital [1] further extended Gauss hypergeometric function by

$$F_p^{(\alpha,\beta)}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \; B_p^{(\alpha,\beta)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!} \tag{1.5}$$

$$(p \geq 0; \; |z| < 1; \; \Re(c) > \Re(b) > 0, \; \Re(\alpha) > 0, \; \Re(\beta) > 0).$$

Very recently, by using $B_p^{(\alpha,\beta;m)}(x, y)$, Parmar [10] defined a new generalization of the extended Gauss hypergeometric function as follows:

$$F_p^{(\alpha,\beta;m)}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n \; B_p^{(\alpha,\beta;m)}(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!} \tag{1.6}$$

$$(p \geq 0; \; |z| < 1; \; \Re(c) > \Re(b) > 0, \; \Re(\alpha) > 0, \; \Re(\beta) > 0, \; \Re(m) > 0).$$

It is obvious that

$$F_p(a, b; c; z) = F_p^{(\alpha,\alpha)}(a, b; c; z) = F_p^{(\alpha,\alpha;1)}(a, b; c; z).$$

In 2010, Özarslan and Özergin [8] used $B_p(x, y)$ to extend the Appell's hypergeometric functions of two variables, $F_1(a, b, c; d; x, y; p)$ and $F_2(a, b, c; d, e; x, y; p)$, and Lauricella's hypergeometric function of three variables, $F_{D,p}^3(a, b, c, d; e; x, y, z)$. Very recently, Liu and Wang [3] generalized these Appell's and Lauricella's functions by using the extended beta function $B_p^{(\alpha,\beta)}(x, y)$. Motivated by the above-mentioned works, we present new generalizations of Appell's hypergeometric functions and Lauricella's hypergeometric function in terms of the generalized extended beta function $B_p^{(\alpha,\beta;m)}(x, y)$.

2. Generalized extended Appell's and Lauricella's functions

In this section, we define the extensions of the Appell's and Lauricella's hypergeometric functions by

$$F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)=\sum_{n,m=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m\frac{x^n}{n!}\frac{y^m}{m!} \tag{2.1}$$

$$(\max\{|x|,\ |y|\}<1),$$

$$F_2{}^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)$$
$$=\sum_{n,m=0}^{\infty}\frac{(a)_{m+n}B_p^{(\alpha,\beta;s)}(b+n,d-b)\ B_p^{(\alpha',\beta';s')}(c+m,e-c)}{B(b,d-b)\ B(c,e-c)}\frac{x^n}{n!}\frac{y^m}{m!} \tag{2.2}$$

$$(|x|+|y|<1),$$

and

$$F_{D,p}{}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)$$
$$=\sum_{m,n,r=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+m+n+r,e-a)\ (b)_m\ (c)_n\ (d)_r}{B(a,e-a)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!} \tag{2.3}$$

$$(|x|<1,\ |y|<1,\ |z|<1),$$

respectively.

On setting $s=s'=1$, these functions reduce to the generalized extended Appell's and Lauricella's hypergeometric functions defined by Liu and Wang [3] and which further gives the extended Appell's and Lauricella's hypergeometric functions defined by Özarslan and Özergin [8] by taking $\alpha=\beta$ and $\alpha'=\beta'$.

3. Integral representations

In this section, we present the integral representations of the generalized Appell's and Lauricella's hypergeometric functions.

Theorem 3.1. For the generalized extended Appell's function $F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$, we have the following integral representation:

$$F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$$
$$= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt, \tag{3.1}$$

$$(p>0;\ p=0 \text{ and } |\arg(1-x)|<\pi,\ |\arg(1-y)|<\pi;\ \Re(d)>\Re(a)>0)$$
$$(\Re(b)>0,\ \Re(c)>0,\ \Re(s)>0).$$

Proof. Using the definition of generalized extended beta function (1.1) on the right hand side of (2.1), we get

$$F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$$
$$= \sum_{n,m=0}^{\infty}\left\{\int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt\right\}\frac{(b)_n(c)_m}{B(a,d-a)}\frac{x^n}{n!}\frac{y^m}{m!}.$$

Interchanging the order of summation and integration, we have

$$F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$$
$$= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)$$
$$\cdot\left(\sum_{n=0}^{\infty}\frac{(b)_n(xt)^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{(c)_m(yt)^m}{m!}\right)dt$$
$$= \frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt,$$

which gives the result.

Theorem 3.2. For the generalized extended Appell's function $F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)$, we have the following integral representation:

$$F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p) = \frac{1}{B(b,d-b)B(c,e-c)}$$
$$\int_0^1\int_0^1 t^{b-1}(1-t)^{d-b-1}u^{c-1}(1-u)^{e-c-1}(1-xt-yu)^{-a}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)$$
$$\times{}_1F_1\left(\alpha';\beta';\frac{-p}{u^{s'}(1-u)^{s'}}\right)dtdu, \tag{3.2}$$

$$(p>0;\ p=0 \text{ and } |x|+|y|<1;\ \Re(d)>\Re(b)>0,\ \Re(e)>\Re(c)>0)$$

$$(\Re(a) > 0,\ \Re(s) > 0,\ \Re(s') > 0).$$

Proof. Using the definition of generalized extended beta function (1.1) on the right hand side of (2.2), we get

$$\begin{aligned}
&F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)\\
&= \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}(x)^n(y)^m}{B(b,d-b)B(c,e-c)n!\,m!}\\
&\quad\times\left\{\int_0^1 t^{b+n-1}(1-t)^{d-b-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt\right\}\\
&\quad\times\left\{\int_0^1 u^{c+m-1}(1-u)^{e-c-1}{}_1F_1\left(\alpha';\beta';\frac{-p}{u^{s'}(1-u)^{s'}}\right)du\right\}.
\end{aligned}$$

Interchanging the order of summation and integration, we have

$$\begin{aligned}
&F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)\\
&= \frac{1}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 t^{b-1}(1-t)^{d-b-1}u^{c-1}(1-u)^{e-c-1}\\
&\quad\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right){}_1F_1\left(\alpha';\beta';\frac{-p}{u^{s'}(1-u)^{s'}}\right)\\
&\quad\times\left(\sum_{n,m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(yu)^m}{m!}\right)dtdu. \qquad (3.3)
\end{aligned}$$

Taking account of the summation formula (see [4, p.52]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!},$$

we have

$$\sum_{n,m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(yu)^m}{m!} = \sum_{N=0}^{\infty}(a)_N\frac{(xt+yu)^N}{N!} = (1-xt-yu)^{-a},$$

using this result in (3.3), we get the desired result.

Theorem 3.3. For the generalized extended Lauricella's function of three variables $F_{D,p}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)$, we have the following integral representation:

$$F_{D,p}{}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)$$
$$= \frac{1}{B(a,e-a)} \int_0^1 t^{a-1}(1-t)^{e-a-1}(1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d}$$
$$\times\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right) dt, \tag{3.4}$$

$$(p>0;\ p=0 \text{ and } |\arg(1-x)|<\pi,\ |\arg(1-y)|<\pi,\ |\arg(1-z)|<\pi;\ \Re(e)>\Re(a)>0)$$

$$(\Re(b)>0,\ \Re(c)>0, \Re(d)>0,\ \Re(s)>0).$$

The above theorem can be established by a similar argument as in the proof of Theorem 3.1.

4. Transformations

We have the following transformations for the generalized Appell's and Lauricella's hypergeometric functions:

(i)
$$F_1{}^{(\alpha,\beta;s)}(a,b,c;d;x,y;p) = (1-x)^{-b}(1-y)^{-c}\ F_1^{(\alpha,\beta;s)}(d-a,b,c;d;\frac{x}{x-1},\frac{y}{y-1};p).$$

(ii)
$$F_2{}^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p) = (1-x)^{-a}\ F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,d-b,c;d,e;\frac{x}{x-1},\frac{y}{1-x};p).$$

(iii)
$$F_2{}^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p) = (1-y)^{-a}\ F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,e-c;d,e;\frac{x}{1-y},\frac{y}{y-1};p).$$

(iv)
$$F_{D,p}{}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z) = (1-x)^{-b}(1-y)^{-c}(1-z)^{-d} \times F_{D,p}^{(3;\alpha,\beta;s)}(e-a,b,c,d;e;\frac{x}{x-1},\frac{y}{y-1},\frac{z}{z-1}).$$

The above transformations can be established by taking $t=1-v$ and $u=1-w$ in (3.1), (3.2) and (3.4).

5. Mellin transforms

In this section, we obtain the Mellin transforms of
$F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$, $F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)$
and $F_{D,p}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)$.

Recall that the Mellin transform of function $f(x)$ is given by (see [2], [9])

$$M\{f(x)\}(r)=\varphi(r)=\int_0^\infty x^{r-1}f(x)dx, \tag{5.1}$$

and its inverse transform is

$$f(x)=M^{-1}\{\varphi(r)\}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}x^{-r}\varphi(r)dr. \tag{5.2}$$

Theorem 5.1. For the new generalized Appell's function $F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$, we have the following Mellin transform representation:

$$M\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}(r) = \frac{\Gamma(r)B(\alpha-r,r)B(a+sr,d-a+sr)}{B(\beta-r,r)B(a,d-a)}F_1(a+sr,b,c;d+2sr;x,y). \tag{5.3}$$

Proof. Using the definition of Mellin transform (5.1) and integral representation (3.1), we get

$$\begin{aligned}
&M\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}(r)\\
&=\frac{1}{B(a,d-a)}\int_0^\infty p^{r-1}\left(\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\right.\\
&\qquad\qquad\left.\times\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt\right)dp\\
&=\frac{1}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\\
&\qquad\qquad\times\left(\int_0^\infty p^{r-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dp\right)dt.
\end{aligned}$$

Substitution of $u=\frac{p}{t^s(1-t)^s}$ in the integral leads to

$$\int_0^\infty p^{r-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dp=\int_0^\infty u^{r-1}\; t^{sr}(1-t)^{sr}{}_1F_1(\alpha;\beta;-u)du$$

$$= t^{sr}(1-t)^{sr}\int_0^\infty u^{r-1}\ {}_1F_1(\alpha;\beta;-u)du.$$

Using the result (see [5, p.48, Eq.(3.3.14)]):

$$\int_0^\infty u^{r-1}\ {}_1F_1(\alpha;\beta;-u)du = \frac{\Gamma(\beta)\Gamma(\alpha-r)\Gamma(r)}{\Gamma(\alpha)\Gamma(\beta-r)},$$

we get

$$\begin{aligned}
&M\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}(r)\\
&\quad= \frac{\Gamma(\beta)\Gamma(\alpha-r)\Gamma(r)}{B(a,d-a)\Gamma(\alpha)\Gamma(\beta-r)}\int_0^1 t^{a+sr-1}(1-t)^{d-a+sr-1}(1-xt)^{-b}(1-yt)^{-c}dt\\
&\quad= \frac{\Gamma(r)B(\alpha-r,r)B(a+sr,d-a+sr)}{B(\beta-r,r)B(a,d-a)}F_1(a+sr,b,c;d+2sr;x,y),
\end{aligned}$$

which is the required result.

Corollary 5.2. By the Mellin inversion formula, we have the following complex integral representation for $F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$:

$$\begin{aligned}
&M\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}(r)\\
&\quad= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(r)B(\alpha-r,r)B(a+sr,d-a+sr)}{B(\beta-r,r)B(a,d-a)}\\
&\qquad\times F_1(a+sr,b,c;d+2sr;x,y)p^{-r}dr. \qquad (5.4)
\end{aligned}$$

Theorem 5.3. For the new generalized Appell's function $F_2^{(\alpha,\beta;s,\alpha',\alpha';s')}(a,b,c;d,e;x,y;p)$, we have the following Mellin transform representation:

$$\begin{aligned}
&M\{F_2^{(\alpha,\beta;s,\alpha',\alpha';s')}(a,b,c;d,e;x,y;p)\}(r)\\
&\quad= \Gamma(r)\sum_{n=0}^{\infty}\frac{B(c+rs'+ns',e-c+rs'+ns')}{B(b,d-b)\ B(c,e-c)}\\
&\qquad\times\frac{B(b-ns,d-b-ns)\ (\alpha)_n(r)_n(-1)^n}{(\beta)_n\ n!}\\
&\qquad\times F_2(a,b-ns,c+rs'+ns';d-2ns,e+2rs'+2ns';x,y).
\end{aligned}$$

Proof. Using the definition of Mellin transform (5.1), we get

$$M\{F_2^{(\alpha,\beta;s,\alpha',\alpha';s')}(a,b,c;d,e;x,y;p)\}(r)$$
$$= \int_0^\infty p^{r-1} F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)dp$$
$$= \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}u^{c-1}(1-u)^{e-c-1}(1-xt-yu)^{-a}}{B(b,d-b)B(c,e-c)}$$
$$\times \left(\int_0^\infty p^{r-1} \exp\left(\frac{-p}{u^{s'}(1-u)^{s'}} \right) {}_1F_1\left(\alpha;\beta; \frac{-p}{t^s(1-t)^s} \right) dp \right) dtdu.$$

On setting $l = \frac{p}{t^s(1-t)^s}$, we get

$$\int_0^\infty p^{r-1} \exp\left(\frac{-p}{u^{s'}(1-u)^{s'}} \right) {}_1F_1\left(\alpha;\beta; \frac{-p}{t^s(1-t)^s} \right) dp$$

$$= t^{sr}(1-t)^{sr} \int_0^\infty l^{r-1} \exp\left(\frac{-t^s(1-t)^s}{u^{s'}(1-u)^{s'}} l \right) \ {}_1F_1(\alpha;\beta;-l)dl.$$

Using the result (see [5, p.43, Eq.(3.2.16)]):

$$\int_0^\infty t^{r-1}e^{-ct} \ {}_1F_1(a;b;-t)dt = c^{-r}\Gamma(r) \ {}_2F_1\left(a,r;b;-\frac{1}{c} \right),$$

we get

$$M\{F_2^{(\alpha,\beta;s,\alpha',\alpha';s')}(a,b,c;d,e;x,y;p)\}(r)$$
$$= \Gamma(r) \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}u^{c+rs'-1}(1-u)^{e-c+rs'-1}}{(1-xt-yu)^a \ B(b,d-b)B(c,e-c)}$$
$$\times \ {}_2F_1\left(\alpha, r; \beta; -\frac{u^{s'}(1-u)^{s'}}{t^s(1-t)^s} \right) dtdu$$
$$= \frac{\Gamma(r)}{B(b,d-b) \ B(c,e-c)} \sum_{n=0}^\infty \frac{(\alpha)_n (r)_n (-1)^n}{(\beta)_n \ n!}$$
$$\times \Bigg[\int_0^1 \int_0^1 t^{b-ns-1}(1-t)^{d-b-ns-1}u^{c+rs'+ns'-1}(1-u)^{e-c+rs'+ns'-1}$$
$$(1-xt-yu)^{-a} dt \ du \Bigg].$$

By the integral representation of Appell's function F_2, we get the required result.

Theorem 5.4. For the function $F_{D,p}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)$, we have the following Mellin transform representation:

$$M\{F_{D,p}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)\}(r) = \frac{\Gamma(r)B(\alpha-r,r)B(a+sr,e-a+sr)}{B(\beta-r,r)B(a,e-a)}F_D^{(3)}(a+sr,b,c,d;e+2sr;x,y,z). \quad (5.6)$$

The above theorem can be established by a similar argument as in the proof of Theorem 5.1.

6. Differentiation formulae

In this section, we define the differentiation formulae for these generalized extended Appell's and Lauricella's hypergeometric function as follows:

(i). $\frac{d^{m+n}}{dx^m dy^n}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$

$$= \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\ F_1^{(\alpha,\beta;s)}(a+m+n,b+m,c+n;d+m+n;x,y;p). \quad (6.1)$$

(ii). $\frac{d^{m+n}}{dx^m dy^n}\{F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c;d,e;x,y;p)\}$

$$= \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\ F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a+m+n,b+m,c+n;d+m,e+n;x,y;p). \quad (6.2)$$

(iii). $\frac{d^{m+n+r}}{dx^m dy^n dz^r}\{F_{D,p}^{(3;\alpha,\beta;s)}(a,b,c,d;e;x,y,z)\}$

$$= \frac{(a)_{m+n+r}(b)_m(c)_n(d)_r}{(e)_{m+n+r}}\ F_{D,p}^{(3;\alpha,\beta;s)}(a+m+n+r,b+m,c+n,d+r;e+m+n+r;x,y,z). \quad (6.3)$$

Proof (i). Differentiating $F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)$ with respect to x and y, we get

$$\frac{d^2}{dxdy}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m\frac{x^{n-1}}{(n-1)!}\frac{y^{m-1}}{(m-1)!}.$$

Replacing $n\to n+1$, $m\to m+1$ and using the result $(a)_{n+1}=a(a+1)_n$, we get

$$\frac{d^2}{dxdy}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$$
$$=bc\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+2+m+n,d-a)}{B(a,d-a)}(b+1)_n(c+1)_m\frac{x^n}{n!}\frac{y^m}{m!}.$$

Now using the result $B(a,d-a)=\frac{d(d+1)}{a(a+1)}\ B(a+2,d-a)$ yield

$$\frac{d^2}{dxdy}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$$
$$=\frac{a(a+1)}{d(d+1)}bc\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+2+m+n,d-a)}{B(a+2,d-a)}(b+1)_n(c+1)_m\frac{x^n}{n!}\frac{y^m}{m!} \quad (6.4)$$
$$=\frac{a(a+1)}{d(d+1)}bc\ F_1^{(\alpha,\beta;s)}(a+2,b+1,c+1;d+2;x,y;p).$$

Further, differentiating (6.4) with respect to x, we get

$$\frac{d^3}{dx^2dy}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$$
$$=\frac{a(a+1)}{d(d+1)}bc\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+3+m+n,d-a)}{B(a+2,d-a)}(b+1)_{n+1}\ (c+1)_m\frac{x^n}{n!}\frac{y^m}{m!}$$
$$=\frac{a(a+1)(a+2)b(b+1)c}{d(d+1)(d+2)}$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_p^{(\alpha,\beta;s)}(a+3+m+n,d-a)}{B(a+3,d-a)}(b+2)_n\ (c+1)_m\frac{x^n}{n!}\frac{y^m}{m!}$$

$$\frac{d^3}{dx^2dy}\{F_1^{(\alpha,\beta;s)}(a,b,c;d;x,y;p)\}$$
$$=\frac{a(a+1)(a+2)b(b+1)c}{d(d+1)(d+2)}\ F_1^{(\alpha,\beta;s)}(a+3,b+2,c+1;d+3;x,y;p).$$

Thus, by induction, we can obtain the desired result.

Similarly, we can establish (6.2) and (6.3).

7. Recurrence relations

In this section, we define some recurrence relations for the Appell's and Lauricella's hypergeometric functions as follows:

(i). $(\alpha-\beta)F_1^{(\alpha-1,\beta;s)}+\alpha F_1^{(\alpha+1,\beta;s)}+(\beta-2\alpha)F_1^{(\alpha,\beta;s)}$

$$+\frac{p\ B(a-s,d-a-s)}{B(a,d-a)}\ F_1^{(\alpha,\beta;s)}(a-s,b,c;d-2s;x,y;p)=0. \quad (7.1)$$

(ii). $(\alpha-\beta)F_2^{(\alpha-1,\beta;s,\alpha',\beta';s')}+\alpha F_2^{(\alpha+1,\beta;s,\alpha',\beta';s')}+(\beta-2\alpha)F_2^{(\alpha,\beta;s,\alpha',\beta';s')}$

$$+\frac{p\ B(b-s,d-b-s)}{B(b,d-b)}\ F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b-s,c;d-2s,e;x,y;p)=0. \quad (7.2)$$

(iii). $(\alpha'-\beta')F_2^{(\alpha,\beta;s,\alpha'-1,\beta';s')}+\alpha' F_2^{(\alpha,\beta;s,\alpha'+1,\beta';s')}+(\beta'-2\alpha')F_2^{(\alpha,\beta;s,\alpha',\beta';s')}$

$$+\frac{p\ B(c-s',e-c-s')}{B(c,e-c)}\ F_2^{(\alpha,\beta;s,\alpha',\beta';s')}(a,b,c-s';d,e-2s';x,y;p)=0. \quad (7.3)$$

(iv). $(\alpha-\beta)F_{D,p}^{(3;\alpha-1,\beta;s)}+\alpha F_{D,p}^{(3;\alpha+1,\beta;s)}+(\beta-2\alpha)F_{D,p}^{(3;\alpha,\beta;s)}$

$$+\frac{p\ B(a-s,e-a-s)}{B(a,e-a)}\ F_{D,p}^{(3;\alpha,\beta;s)}(a-s,b,c,d;e-2s;x,y,z)=0. \quad (7.4)$$

Here, for the sake of making the contiguous expressions more transparent, we omit the parameters.

Proof (i). We have the following recurrence relation of the confluent hypergeometric function ${}_1F_1$ (see [5, p.19, Eq.(2.2.11)]):

$$(a-b){}_1F_1(a-1;b;z)+a{}_1F_1(a+1;b;z)+(b-2a-z){}_1F_1(a;b;z)=0.$$

With the help of above relation, we derive

$$\frac{(\alpha-\beta)}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha-1;\beta;\frac{-p}{t^s(1-t)^s}\right)dt$$

$$+\frac{\alpha}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha+1;\beta;\frac{-p}{t^s(1-t)^s}\right)dt$$
$$+\frac{(\beta-2\alpha)}{B(a,d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt$$
$$+\frac{p}{B(a,d-a)}\int_0^1 t^{a-s-1}(1-t)^{d-a-s-1}(1-xt)^{-b}(1-yt)^{-c}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^s(1-t)^s}\right)dt.$$
$$=0$$

By using the integral representation of $F_1^{(\alpha,\beta;s)}$ in the above expression, we get the result (7.1). Similarly, we can establish (7.2), (7.3) and (7.4). Also, it is noticed that, by using the other recurrence relations of the confluent hypergeometric function ${}_1F_1$ (see [5]), we can establish some more recurrence relations for these new generalized Appell's and Lauricella's hypergeometric functions.

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