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ANALYTIC SOLUTIONS OF THE CAUCHY PROBLEM FOR THE GENERALIZED TWO-COMPONENT HUNTER-SAXTON SYSTEM

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Abstract. In this paper we consider the periodic Cauchy problem for the generalized two-component Hunter-Saxton system with analytic initial data and we prove a Cauchy-Kowalevski type theorem for the generalized two-component Hunter-Saxton system, that establishes the existence and uniqueness of real analytic solutions.

1. Introduction

In this paper, we consider the periodic Cauchy problem of the following generalized two-component Hunter-Saxton system [18]

$$
(1.1)\qquad \begin{cases} u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + Au_x = 0, \\ \rho_t + (\rho u)_x = 0, \\ u(0,x) = u_0(x), \\ \rho(0,x) = \rho_0(x), \end{cases}
$$

where $t \in \mathbb{R},\ x \in \mathbb{T},\ \mathbb{T} \equiv \mathbb{R}/\mathbb{Z}$, $\sigma \in \mathbb{R}$ is the new free parameter, and $A \geq 0$. System (1.1) is the short-wave (or high-frequency) limit

$$
(t,x) \mapsto (\epsilon t, \epsilon x), \qquad \epsilon \to 0
$$

of the generalized two-component Camassa-Holm system (gCH2) [6]

$$
\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0. \\ \rho_t + (\rho u)_x = 0, \end{cases}
$$

derived from shallow water theory with nonzero constant vorticity by using Ivanov's modeling approach [14]. System (1.1) was recently studied in [18]. The authors proved that the local well-posedness, wave-breaking

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criterion, wave-breaking data and sufficient condition for the global solution to (1.1) in the periodic setting.

For $(\sigma, A) = (1, 0)$, (1.1) becomes the two-component Hunter-Saxton system(HS2)

$$\begin{cases} u_{txx} + 2u_x u_{xx} + u u_{xxx} - \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases}$$

It arises in the short-wave (or high-frequency) limit of the two-component Camassa-Holm system(CH2) [7] derived from the Green-Naghdi equations, which are approximations to the governing equations for water waves. The main motivation for seeking and studying such a system lies in capturing nonlinear phenomena such as wave-breaking and traveling waves which are not exhibited by small-amplitude models [8, 15]. The two-component Hunter-Saxton system is a particular case of the Gurevich-Zybin system describing the dynamics in a model of non-dissipative dark matter (see [23] and the references therein). This two-component HS system is formally integrable with a Lax pair and a bi-Hamiltonian formulation [7, 21]- it can be written as a compatibility condition of two linear system (Lax pair) with a spectral parameter ζ:

$$\begin{aligned} \Psi_{xx} &= \big(- \zeta^2 \rho^2 + \zeta m \big) \Psi \\ \Psi_t &= \big(\frac{1}{2\zeta} - u \big) \Psi_x + \frac{1}{2} u_x \Psi, \quad m = -u_{xx}. \end{aligned}$$

Moreover, it was shown in [7] that peakon solutions exist for the system, given explicitly by

$$\begin{aligned} m(x,t) &= \sum_{k=1}^{N} m_k(t) \delta(x - x_k(t)), \\ u(x,t) &= -\frac{1}{2} \sum_{k=1}^{N} |x - x_k(t)|, \\ \rho(x,t) &= \sum_{k=1}^{N} \rho_k(t) \theta(x - x_k(t)), \end{aligned}$$

where θ is the Heaviside function and the coefficients are subject to the condition

$$\sum_{j=1}^{N} m_j = \sum_{j=1}^{N} \rho_j = 0.$$

Furthermore, mathematical properties of this system have been also studied further in many works, for example [10, 16, 17, 27, 28, 29, 30].

Its local well-posedness, global existence and blow up phenomena were discussed recently in [28]. Lenells-Lechtenfeld [16] showed that it can be interpreted as the Euler equation on the superconformal algebra of contact vector fields on the 1|2-dimensional supercircle. Moreover, Wu-Wunsch [27], and Liu-Yin [17] gave sufficient conditions for the global existence of strong solutions to the Hunter-Saxton system. On the other hand, Escher [10] gives geometric meaning to the two-component Hunter-Saxton system, which is used by Wunsch [30] to show that there are global conservative solutions. Finally, Wunsch [29] proved that there are global dissipative solutions to the two-component Hunter-Saxton system on $\mathbb{R}$.

For $(\sigma, A) = (1, 0)$ and $\rho \equiv 0$, (1.1) reduces to the Hunter-Saxton equation [12]

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0,$$

modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal [3, 12, 34]. In the Hunter-Saxton equation [12], x is the space variable in a reference frame moving with the linearized wave velocity, t is a slow-time variable, and $u(t, x)$ is a measure of the average orientation of the medium locally around x at time t. More precisely, the orientation of the molecules is described by the field of unit vectors $(\cos u(t, x), \sin u(t, x))$ [34]. The Hunter-Saxton equation also arises in a different physical context as the high-frequency limit [13] of the Camassa-Holm equation for shallow water waves [5] and a re-expression of the geodesic flow on the diffeomorphism group of the circle [9] with a bi-Hamiltonian structure [12, 21] which is completely integrable [3, 13]. The initial value problem for the Hunter-Saxton equation on the line (nonperiodic case) and on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ were studied by Hunter-Saxton in [12] using the method of characteristics and by Yin in [34] using the Kato semigroup method. Moreover, the two classes of admissible weak solutions, dissipative and conservative solutions, and their stability were studied in [4].

Recently, the analyticity of the Cauchy problem for two-component shallow water systems has been proved in [32] and the analyticity of solutions to two-component μ-Hunter-Saxton system has also been established in [33]. Moreover, the analytic regularity (i.e., existence and uniqueness of analytic solutions for analytic initial data) of the Cauchy problem for Novikov's equation has been obtained in [24].

However, the goal of the present paper is to prove the analyticity of its solutions to system (1.1) in both variables, with x in $\mathbb{T}$ and t in an interval around zero, provided that the initial data is analytic on $\mathbb{T}$.

Note that the classical Cauchy-Kowalevski theorem does not apply to (1.1) since the initial line $t = 0$ is characteristic. Hence, this analytic result can be viewed as an extended Cauchy-Kowalevski theorem for the nonlinear case (1.1). It is well known that the solutions of the KdV equation are analytic in the space variable for all time [26] but are not analytic in the time variable [15]. In contrast with the KdV equation, the solutions to the Hunter-Saxton (HS) and Camassa-Holm equations are analytic in both space and time variables for a short time [11]. Like the CH and HS equations, we will show that solutions of the Cauchy problem (1.1) are analytic in both space and time variables.

This paper is structured as follows. In Section 2, some preliminary properties, which will be used later, are presented. Section 3 is devoted to the study of the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Such an approach to analytic regularity of solutions to Cauchy problem (1.1) was initiated by Ovsjannikov [22] as an abstract Cauchy-Kowalevski theorem and later further developed by Treves [25], Nirenberg [19], Nishida [20], and Baouendi-Goulaouic [2], among others [31] and subsequently applied to the Euler and Navier-Stokes equations.

2. Preliminaries

In this section, we first introduce a new space, whose properties are studied. In order to verify the analyticity of solutions to (1.1), the abstract Cauchy-Kowalevski theorem for identifying analyticity of the Cauchy problem is presented and the needed results to pursue our goal is briefly given.

In order to apply the contraction argument to the proof of the analytic regularity of the Cauchy problem (1.1), we need a suitable scale of Banach spaces which we proceed to describe below. For any $s > 0$, we set

$$E_s \equiv \Big\{ u \in C^\infty(\mathbb{T}) : |||u|||_s = \sup_{k\in\mathbb{N}_0} \frac{s^k \|\partial_x^k u\|_{H^r(\mathbb{T})}}{k!/(k+1)^2} < \infty \Big\},$$

where $r > \frac{1}{2}$ is any fixed real number and $\mathbb{N}_0$ is the set of nonnegative integers. It is not difficult to check that E_s equipped the norm $|||\cdot|||_s$ is a Banach space by the completeness of $H^r(\mathbb{T})$ and the closedness of the differential operator ∂_x, and that, for any $0 < s' < s$, E_s is continuously included in E_s' with $|||u|||_{s'} < |||u|||_s$.

Proposition 2.1. [32] $E_s \in C^\omega(\mathbb{T})$. *This implies any* $u \in E_s$ *is real analytic on* $\mathbb{T}$.

Next, we restate the Cauchy problem (1.1) in a more convenient form. Integrating both side of the first equation of (1.1) with respect to x, we have

$$u_{tx} + \frac{\sigma}{2}{u_x}^2 + \sigma u u_{xx} - \frac{1}{2}\rho^2 + Au = g(t),$$

where $g(t) : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function, denoted by $g(t) \in C(\mathbb{R})$. Integrating once more with respect to x, we obtain

$$u_t + \sigma u u_x = \partial_x^{-1}\Big(\frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 - Au + g(t)\Big) + h(t),$$

where $\partial_x^{-1} f(x) := \int_0^x f(y)dy$ and $h(t) : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function, denoted by $h(t) \in C(\mathbb{R})$. Thus the problem (1.1) can be written as follows :

$$\text{(2.1)} \quad \begin{cases} \partial_t u = -\frac{\sigma}{2}\partial_x u^2 + \partial_x^{-1}\Big(\frac{\sigma}{2}(\partial_x u)^2 + \frac{1}{2}\rho^2 - Au + g(t)\Big) + h(t), \\ \partial_t \rho = -\partial_x(u\rho), \\ u(0,x) = u_0(x), \\ \rho(0,x) = \rho_0(x), \end{cases}$$

where $t \in \mathbb{R},\ x \in \mathbb{T},\ u_0, \rho_0 \in C^\omega(\mathbb{T})$. Letting

$$f(x) = x^2, \quad P_1 = \partial_x, \quad P_2 = \partial_x^{-1},$$

we rewrite equation (2.1) in the following form :

$$\text{(2.2)} \quad \partial_t u = -\frac{\sigma}{2}P_1 f(u) + P_2\Big(\frac{\sigma}{2}f(P_1 u) + \frac{1}{2}f(\rho) - Au + g(t)\Big) + h(t),$$

$$\text{(2.3)} \quad \partial_t \rho = -P_1(u\rho).$$

To transform (2.2) and (2.3) into a new system, we set

$$u_1 = u,\ u_2 = P_1 u = \partial_x u_1,\ u_3 = P_1 u_2 = \partial_x u_2,$$

$$u_4 = \rho,\ \text{and}\ u_5 = P_1\rho = \partial_x u_4.$$

Then

$$\begin{aligned} \partial_t u_1 &= -\sigma u_1 u_2 + P_2\Big(\frac{\sigma}{2}f(u_2) + \frac{1}{2}f(u_4) - Au_1 + g(t)\Big) + h(t), \\ &\equiv F_1(u_1, u_2, u_3, u_4, u_5), \\ \partial_t u_2 &= P_1(\partial_t u_1) = -\sigma u_1 u_3 - \frac{\sigma}{2}f(u_2) + \frac{1}{2}f(u_4) - Au_1 + g(t), \\ &\equiv F_2(u_1, u_2, u_3, u_4, u_5), \end{aligned}$$

$$\begin{aligned}\partial_t u_3 &= P_1(\partial_t u_2) = P_1\Big(-\sigma u_1 u_3 - \frac{\sigma}{2}f(u_2) + \frac{1}{2}f(u_4) - Au_1\Big),\\ &\equiv F_3(u_1,u_2,u_3,u_4,u_5),\\ \partial_t u_4 &= P_1(-u_1u_4) \equiv F_4(u_1,u_2,u_3,u_4,u_5),\\ \partial_t u_5 &= P_1(-u_2u_4 - u_1u_5) \equiv F_5(u_1,u_2,u_3,u_4,u_5),\end{aligned}$$

where we used $P_1(u_1u_2) = u_1u_3 + f(u_2)$, $P_1(g(t)) = P_1(h(t)) = 0, P_1P_2 = Id$, and $P_1(u_1u_4) = u_2u_4 + u_1u_5$.

Therefore our initial value problem takes the following form :

$$\begin{cases}\partial_t u_1 = F_1(u_1,u_2,u_3,u_4,u_5), & u_1(0,x) = u(0,x) = u_0(x),\\ \partial_t u_2 = F_2(u_1,u_2,u_3,u_4,u_5), & u_2(0,x) = \partial_x u_1(0,x) = \partial_x u_0(x),\\ \partial_t u_3 = F_3(u_1,u_2,u_3,u_4,u_5), & u_3(0,x) = \partial_x u_2(0,x) = \partial_x^2 u_0(x),\\ \partial_t u_4 = F_4(u_1,u_2,u_3,u_4,u_5), & u_4(0,x) = \rho(0,x) = \rho_0(x),\\ \partial_t u_5 = F_5(u_1,u_2,u_3,u_4,u_5), & u_5(0,x) = \partial_x\rho(0,x) = \partial_x\rho_0(x).\end{cases}$$

Note that the initial data in the Cauchy problem (2.5) of the abstract Cauchy-Kowalevski theorem equals to zero, one can set

$$\bar{u}_1 = u_1 - u_0,\ \bar{u}_2 = u_2 - \partial_x u_0,\ \bar{u}_3 = u_3 - \partial_x^2 u_0,\ \bar{u}_4 = u_4 - \rho_0,\ \bar{u}_5 = u_5 - \partial_x\rho_0,$$

then the above problem is equivalent to

(2.4)

$$\begin{cases}\partial_t\bar{u}_1 = -\sigma(\bar{u}_1+u_0)(\bar{u}_2+\partial_x u_0)\\ \qquad\qquad + P_2\Big(\frac{\sigma}{2}f(\bar{u}_2+\partial_x u_0) + \frac{1}{2}f(\bar{u}_4+\rho_0) - A(\bar{u}_1+u_0) + g(t)\Big) + h(t)\\ \qquad = F_1(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5),\\ \partial_t\bar{u}_2 = -\sigma(\bar{u}_1+u_0)(\bar{u}_3+\partial_x^2 u_0) - \frac{\sigma}{2}f(\bar{u}_2+\partial_x u_0) + \frac{1}{2}f(\bar{u}_4+\rho_0)\\ \qquad\quad - A(\bar{u}_1+u_0) + g(t)\\ \qquad = F_2(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5),\\ \partial_t\bar{u}_3 = P_1\Big(-\sigma(\bar{u}_1+u_0)(\bar{u}_3+\partial_x^2 u_0) - \frac{\sigma}{2}f(\bar{u}_2+\partial_x u_0) + \frac{1}{2}f(\bar{u}_4+\rho_0)\\ \qquad\quad - A(\bar{u}_1+u_0)\Big)\\ \qquad = F_3(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5),\\ \partial_t\bar{u}_4 = P_1\big(-(\bar{u}_1+u_0)(\bar{u}_4+\rho_0)\big) = F_4(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5),\\ \partial_t\bar{u}_5 = P_1\big(-(\bar{u}_2+\partial_x u_0)(\bar{u}_4+\rho_0) - (\bar{u}_1+u_0)(\bar{u}_5+\partial_x\rho_0)\big)\\ \qquad = F_5(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5),\\ \bar{u}_1(0,x) = 0,\quad \bar{u}_2(0,x) = 0,\quad \bar{u}_3(0,x) = 0,\quad \bar{u}_4(0,x) = 0,\quad \bar{u}_5(0,x) = 0.\end{cases}$$

Define

$$\bar{U} \equiv (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)$$

and

$$F(\bar{U}) = F(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) \equiv (F_1, F_2, F_3, F_4, F_5)(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5).$$

Then we have

$$\begin{cases} \frac{\partial \bar{U}(t)}{\partial t} = F(t, \bar{U}(t)), \\ \bar{U}(0) = 0. \end{cases}$$

The following Proposition comes from [1, 2, 11].

Proposition 2.2. [1, 2, 11] *Let $\{X_s, |||\cdot|||_s\}_{0<s\leq 1}$ be a scale of decreasing Banach spaces, so that for any $0 < s' < s$ we have $X_s \subset X_{s'}$ with $|||\cdot|||_{s'} \leq |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t, u(t)) \\ u(0) = 0. \end{cases} \tag{2.5}$$

Let T, R and C be positive numbers and suppose that F satisfies the following conditions:

(i) If for any $0 < s' < s < 1$, the function $t \mapsto u(t)$ is holomorphic on $|t| < T$ and continuous on $|t| \leq T$ with values in X_s and

$$\sup_{|t|\leq T} |||u(t)|||_s < R,$$

then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{s'}$.

(ii) For any $0 < s' < s \leq 1$ and any $u, v \in B(0, R) \subset X_s$, that is, $|||u|||_s < R$, $|||v|||_s < R$, we have

$$\sup_{|t|\leq T} |||F(t, u) - F(t, v)|||_{s'} \leq \frac{C}{s - s'} |||u - v|||_s.$$

(iii) There exists a $M > 0$, such that for any $0 < s < 1$,

$$\sup_{|t|\leq T} |||F(t, 0)|||_s \leq \frac{M}{1 - s}.$$

Then there exists a $T_0 \in (0, T)$ and a unique function $u(t)$, which for every $s \in (0, 1)$ is holomorphic in $|t| < (1 - s)T_0$ with values in X_s, and is a solution to the initial value problem (2.5).

The following two useful lemmas will facilitate the required computations.

Lemma 2.3. [32] *Let $s > 0$. There is a constant $C > 0$, independent of s, such that for any u and v in E_s, we have*

$$|||uv|||_s \leq C|||u|||_s|||v|||_s,$$

where $C = C(r)$ depends only on r. In particular, for any $s > 0$, we have

$$|||f(u) - f(v)|||_s = |||u^2 - v^2|||_s \leq C|||u + v|||_s|||u - v|||_s,$$

for any $u, v \in E_s$.

Lemma 2.4. [32, 33] *For any $0 < s' < s \leq 1$, we have*

$$|||P_1 u|||_{s'} \leq \frac{1}{s - s'}|||u|||_s \quad \text{and} \quad |||P_2 u||_s \leq C|||u|||_s,$$

where C is a positive constant depending only on r.

3. Analyticity of solutions

In this section, we will show the existence and uniqueness of analytic solutions to the system (1.1) on the circle $\mathbb{T}$.

Consider system (2.4) as the abstract form of the Cauchy problem in (2.5).

$$\begin{cases} \partial_t \bar{u}_1 = -\sigma(\bar{u}_1 + u_0)(\bar{u}_2 + \partial_x u_0) \\ \qquad\qquad + P_2\Big(\frac{\sigma}{2} f(\bar{u}_2 + \partial_x u_0) + \frac{1}{2} f(\bar{u}_4 + \rho_0) - A(\bar{u}_1 + u_0) + g(t)\Big) + h(t) \\ \qquad = F_1(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5), \\ \partial_t \bar{u}_2 = -\sigma(\bar{u}_1 + u_0)(\bar{u}_3 + \partial_x^2 u_0) - \frac{\sigma}{2} f(\bar{u}_2 + \partial_x u_0) + \frac{1}{2} f(\bar{u}_4 + \rho_0) \\ \qquad\qquad - A(\bar{u}_1 + u_0) + g(t) \\ \qquad = F_2(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5), \\ \partial_t \bar{u}_3 = P_1\Big(- \sigma(\bar{u}_1 + u_0)(\bar{u}_3 + \partial_x^2 u_0) - \frac{\sigma}{2} f(\bar{u}_2 + \partial_x u_0) + \frac{1}{2} f(\bar{u}_4 + \rho_0) \\ \qquad\qquad - A(\bar{u}_1 + u_0)\Big) \\ \qquad = F_3(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5), \\ \partial_t \bar{u}_4 = P_1\big(- (\bar{u}_1 + u_0)(\bar{u}_4 + \rho_0)\big) = F_4(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5), \\ \partial_t \bar{u}_5 = P_1\big(- (\bar{u}_2 + \partial_x u_0)(\bar{u}_4 + \rho_0) - (\bar{u}_1 + u_0)(\bar{u}_5 + \partial_x \rho_0)\big) \\ \qquad = F_5(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5), \\ \bar{u}_1(0, x) = 0, \quad \bar{u}_2(0, x) = 0, \quad \bar{u}_3(0, x) = 0, \quad \bar{u}_4(0, x) = 0, \quad \bar{u}_5(0, x) = 0. \end{cases}$$

Now we have the following analyticity result :

Theorem 3.1. *Let* $\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}$ *be a real analytic function on* $\mathbb{T}$. *There exists an* $\epsilon > 0$ *and a unique solution* $\begin{pmatrix} u \\ \rho \end{pmatrix}$ *of the Cauchy problem* (1.1) *that is analytic on* $(-\epsilon, \epsilon) \times \mathbb{T}$.

Define

$$|||(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)|||_s \equiv \sum_{i=1}^{5} |||\bar{u}_i|||_s$$

and

$$|||F(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)|||_s \equiv \sum_{i=1}^{5} |||F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)|||_s.$$

Theorem 3.1 then is a straightforward consequence of the Proposition 2.2. To prove this Theorem 3.1, it is enough to show that all three conditions of the abstract version of the Cauchy-Kowalevski theorem hold.

Proof of Theorem 3.1. Let

$$\bar{U} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) \quad \text{and} \quad F = (F_1, F_2, F_3, F_4, F_5)$$

in (2.4) and X_s be a decreasing scale of Banach spaces. Then we only need to verify the first two conditions of the abstract Cauchy-Kowalevski theorem since the map $F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)$, $i = 1, 2, 3, 4, 5$ does not depend on t explicitly.

Clearly, $t \mapsto F(t, \bar{U}(t)) = (F_1, F_2, F_3, F_4, F_5)(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)$ is holomorphic if $t \mapsto \bar{u}_i(t)$, $i = 1, 2, 3, 4, 5$ is holomorphic. Therefore, to verify the first condition of the abstract theorem, we only need to show that for $0 < s' < s \leq 1$, $F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) \in E_{s'}$ if $\bar{u}_i \in E_s$, $i = 1, 2, 3, 4, 5$.

Note that $\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix}$ is analytic by the assumption of Theorem 3.1, we can deduce that $|||u_0|||_s$, $|||\partial_x u_0|||_{s'}$, $|||\partial_x^2 u_0|||_{s'}$, $|||\rho_0|||_s$, and $|||\partial_x \rho_0|||_{s'}$ are bounded. Without loss of generality, we assume that there exist constants M_0, M_1, M_2, L_0, $L_1 > 0$ such that $|||u_0|||_s \leq M_0$, $|||\partial_x u_0|||_s \leq M_1$, $|||\partial_x^2 u_0|||_s \leq M_2$, $|||\rho_0|||_s \leq L_0$, $|||\partial_x \rho_0|||_s \leq L_1$, and so

$$|||\partial_x u_0|||_{s'} \leq \frac{M_0}{s - s'}, \quad |||\partial_x^2 u_0|||_{s'} \leq \frac{M_1}{s - s'}, \quad |||\partial_x \rho_0|||_{s'} \leq \frac{L_0}{s - s'}.$$

By Lemma 2.1 and 2.2, we can estimate as follows.

$$
\begin{aligned}
&|||F_1(\bar u_1,\bar u_2,\bar u_3,\bar u_4,\bar u_5)|||_{s'}\\
&\le |\sigma|\cdot C(r)|||\bar u_1+u_0|||_{s'}|||\bar u_2+\partial_x u_0|||_{s'}+\frac{|\sigma|}{2}|||f(\bar u_2+\partial_x u_0)|||_{s'}\\
&\qquad +\frac{1}{2}|||f(\bar u_4+\rho_0)|||_{s'}+A|||\bar u_1+u_0|||_{s'}+|||g(t)|||_{s'}+|||h(t)|||_{s'}\\
&\le C(|\sigma|,r,A)\big[(R+M_0)(R+\frac{M_0}{s-s'})+(R+\frac{M_0}{s-s'})^2+(R+L_0)^2+(R+M_0)\big]\\
&\quad +\sup\big[||g(t)||_{H^r(\mathbb{T})}+||h(t)||_{H^r(\mathbb{T})}\big],
\end{aligned}
$$

$$
\begin{aligned}
&|||F_2(\bar u_1,\bar u_2,\bar u_3,\bar u_4,\bar u_5)|||_{s'}\\
&\le |\sigma|\cdot C(r)|||\bar u_1+u_0|||_{s}|||\bar u_3+\partial_x^2 u_0|||_{s'}+\frac{|\sigma|}{2}\cdot C(r)|||\bar u_2+\partial_x u_0|||_{s'}^2\\
&\qquad +\frac{C(r)}{2}|||\bar u_4+\rho_0|||_{s'}^2+A|||\bar u_1+u_0|||_{s}+|||g(t)|||_{s}\\
&\le C(|\sigma|,r,A)\big[(R+M_0)(R+\frac{M_1}{s-s'})+(R+\frac{M_0}{s-s'})^2+(R+L_0)^2+(R+M_0)\big]\\
&\qquad +\sup||g(t)||_{H^r(\mathbb{T})},
\end{aligned}
$$

and

$$
\begin{aligned}
&|||F_3(\bar u_1,\bar u_2,\bar u_3,\bar u_4,\bar u_5)|||_{s'}\\
\le\ & \frac{1}{s-s'}\Big[|\sigma|\cdot C(r)|||\bar u_1+u_0|||_{s}|||\bar u_3+\partial_x^2 u_0|||_{s}+\frac{|\sigma|\cdot C(r)}{2}|||\bar u_2+\partial_x u_0|||_{s}^2\\
&\qquad +\frac{C(r)}{2}|||\bar u_4+\rho_0|||_{s}^2+A|||\bar u_1+u_0|||_{s}\Big]\\
\le\ & \frac{C(|\sigma|,r,A)}{s-s'}\Big[(R+M_0)(R+M_2)+(R+M_1)^2+(R+L_0)^2+(R+M_0)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&|||F_4(\bar u_1,\bar u_2,\bar u_3,\bar u_4,\bar u_5)|||_{s'}\\
\le\ & \frac{C(r)}{s-s'}|||\bar u_1+u_0|||_{s}|||\bar u_4+\rho_0|||_{s}\le\frac{C(r)}{s-s'}(R+M_0)(R+L_0),
\end{aligned}
$$

$$
\begin{aligned}
&|||F_5(\bar u_1,\bar u_2,\bar u_3,\bar u_4,\bar u_5)|||_{s'}\\
\le\ & \frac{C(r)}{s-s'}(|||\bar u_2+\partial_x u_0|||_{s}|||\bar u_4+\rho_0|||_{s}+|||\bar u_1+u_0|||_{s}|||\bar u_5+\partial_x\rho_0|||_{s})\\
\le\ & \frac{C(r)}{s-s'}\Big[(R+M_1)(R+L_0)+(R+M_0)(R+L_1)\Big],
\end{aligned}
$$

hence, condition $i)$ holds.

Note that to verify the second condition it suffices to show estimate

$$|||F(\bar U)-F(\bar V)|||_{s'}\le\frac{C}{s-s'}|||\bar U-\bar V|||_{s},$$

where $\bar{U} \equiv (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)$ and $\bar{V} \equiv (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)$.

For any $\bar{u}_j, \bar{v}_j \in B(0, R) \subset E_s,\ j = 1, 2, 3, 4, 5,$ we have

$$\begin{aligned}
|||F(\bar{U}) - F(\bar{V})|||_{s'} &= |||F(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)|||_{s'} \\
&= \sum_{i=1}^{5} |||F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)|||_{s'} \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Note that

$$\begin{aligned}
&|||F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)|||_{s'} \\
&\quad \le |||F_i(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)|||_{s'} \\
&\qquad + |||F_i(\bar{v}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5)|||_{s'} \\
&\qquad\quad + |||F_i(\bar{v}_1, \bar{v}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{u}_4, \bar{u}_5)|||_{s'} \\
&\qquad\qquad + |||F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{u}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{u}_5)|||_{s'} \\
&\qquad\qquad\quad + |||F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{u}_5) - F_i(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5)|||_{s'}.
\end{aligned}$$

Using this together with Lemma 2.1 and 2.2, each term of the above equation can be estimated as follows.

$$\begin{aligned}
I_1 &\le |\sigma| |||(\bar{u}_1 + u_0)(\bar{u}_2 + \partial_x u_0) - (\bar{v}_1 + u_0)(\bar{u}_2 + \partial_x u_0)|||_{s'} \\
&\quad + A|||P_2[(\bar{u}_1 + u_0) - (\bar{v}_1 + u_0)]|||_{s'} \\
&\quad + |\sigma| |||(\bar{u}_1 + u_0)(\bar{u}_2 + \partial_x u_0) - (\bar{u}_1 + u_0)(\bar{v}_2 + \partial_x u_0)|||_{s'} \\
&\quad + \frac{|\sigma|}{2}|||P_2[f(\bar{u}_2 + \partial_x u_0) - f(\bar{v}_2 + \partial_x u_0)]|||_{s'} \\
&\quad + \frac{1}{2}|||P_2[f(\bar{u}_4 + \rho_0) - f(\bar{v}_4 + \rho_0)]|||_{s'} \\
&\le \Big[|\sigma| \cdot C(r)|||\bar{u}_2 + \partial_x u_0|||_{s'} + A \cdot C(r) + |\sigma| \cdot C(r)|||\bar{u}_1 + u_0|||_{s'} \\
&\quad + \frac{1}{2} \cdot C^2(r)|||\bar{u}_2 + \bar{v}_2 + 2\partial_x u_0|||_{s'} \\
&\quad + \frac{1}{2} \cdot C^2(r)|||\bar{u}_4 + \bar{v}_4 + 2\rho_0|||_{s'}\Big]|||\bar{U} - \bar{V}|||_{s'} \\
&\le \frac{1}{s - s'} \cdot \Big[2|\sigma| \cdot C(r) \cdot (R + M_0) + A \cdot C(r) + |\sigma| \cdot C^2(r) \cdot (R + M_0) \\
&\quad + C^2(r) \cdot (R + L_0)\Big] \cdot |||\bar{U} - \bar{V}|||_s,
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le |\sigma| |||(\bar{u}_1+u_0)(\bar{u}_3+\partial_x^2 u_0)-(\bar{v}_1+u_0)(\bar{u}_3+\partial_x^2 u_0)|||_{s'} + A|||\bar{u}_1-\bar{v}_1]|||_{s'} \\
&\quad + |\sigma| |||(\bar{u}_1+u_0)(\bar{u}_3+\partial_x^2 u_0)-(\bar{u}_1+u_0)(\bar{v}_3+\partial_x^2 u_0)|||_{s'} \\
&\quad + \frac{|\sigma|}{2}|||f(\bar{u}_2+\partial_x u_0)-f(\bar{v}_2+\partial_x u_0)]|||_{s'} + \frac{1}{2}|||f(\bar{u}_4+\rho_0)-f(\bar{v}_4+\rho_0)|||_{s'} \\
&\le \Big[|\sigma|\cdot C(r)|||\bar{u}_3+\partial_x^2 u_0|||_{s'} + A + \frac{|\sigma|}{2}\cdot C(r)|||\bar{u}_2+\bar{v}_2+2\partial_x u_0|||_{s'} \\
&\quad + |\sigma|\cdot C(r)|||\bar{u}_1+u_0|||_{s'} + \frac{1}{2}\cdot C(r)|||\bar{u}_4+\bar{v}_4+2\rho_0|||_{s'}\Big]|||\bar{U}-\bar{V}|||_{s'} \\
&\le \frac{1}{s-s'}\cdot\Big[2|\sigma|\cdot C(r)\cdot(R+M_0)+A+|\sigma|\cdot C(r)\cdot(R+M_1) \\
&\quad + C(r)\cdot(R+L_0)\Big]\cdot|||\bar{U}-\bar{V}|||_s,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\le \frac{1}{s-s'}\cdot\Big[|\sigma|\cdot C(r)|||\bar{u}_3+\partial_x^2 u_0|||_s + A + \frac{|\sigma|}{2}\cdot C(r)|||\bar{u}_2+\bar{v}_2+2\partial_x u_0|||_s \\
&\quad + |\sigma|\cdot C(r)|||\bar{u}_1+u_0|||_s + \frac{1}{2}\cdot C(r)|||\bar{u}_4+\bar{v}_4+2\rho_0|||_s\Big]\cdot|||\bar{U}-\bar{V}|||_s \\
&\le \frac{|\sigma|\cdot C(r)\cdot(3R+M_0+M_1+M_2)+A+C(r)\cdot(R+L_0)}{s-s'}|||\bar{U}-\bar{V}|||_s,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &\le \frac{C(r)}{s-s'}|||\bar{u}_4+\rho_0|||_s|||\bar{u}_1-\bar{v}_1|||_s + \frac{C(r)}{s-s'}|||\bar{u}_1+u_0|||_s|||\bar{u}_4-\bar{v}_4|||_s \\
&\le \frac{C(r)(2R+L_0+M_0)}{s-s'}|||\bar{U}-\bar{V}|||_s,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &\le \frac{C(r)}{s-s'}\cdot\Big[|||\bar{u}_5+\partial_x\rho_0|||_s|||\bar{u}_1-\bar{v}_1|||_s + |||\bar{u}_4+\rho_0|||_s|||\bar{u}_2-\bar{v}_2|||_s \\
&\quad + |||\bar{u}_2+\partial_x u_0|||_s|||\bar{u}_4-\bar{v}_4|||_s + |||\bar{u}_1+u_0|||_s|||\bar{u}_5-\bar{v}_5|||_s\Big] \\
&\le \frac{C(r)(4R+M_0+M_1+L_0+L_1)}{s-s'}|||\bar{U}-\bar{V}|||_s.
\end{aligned}
$$

This implies that the condition ii) also holds. Therefore, this completes the proof of Theorem 3.1.

□

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