Honam Mathematical J. **37** (2015), No. 1, pp. 65–97
http://dx.doi.org/10.5831/HMJ.2015.37.1.65

SUM AND PRODUCT THEOREMS OF RELATIVE TYPE AND RELATIVE WEAK TYPE OF ENTIRE FUNCTIONS

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Abstract. Orders and types of entire functions have been actively investigated by many authors. In this paper, we aim at investigating some basic properties in connection with sum and product of relative type and relative weak type of entire functions.

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ is defined as follows:

$$M_f(r) := \max_{|z|=r} |f(z)|,$$

which is known as maximum modulus function corresponding to f.

It is noted that, if f is non-constant, then $M_f(r)$ is strictly increasing and continuous, and its inverse ${M_f}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and satisfies $\lim_{s\to\infty} {M_f}^{-1}(s) = \infty$.

On the other hand, the Nevanlinna's characteristic function of f denoted by $T_f(r)$ is defined as

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for all $x > 0$.

We begin by recalling the following definitions.

Received November 26, 2014. Accepted December 16, 2014.
2010 Mathematics Subject Classification: 30D20, 30D30, 30D35.
Key words and phrases: Entire functions; Relative order (relative lower order); Relative type (relative lower type); Relative weak type; Regular relative growth.

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Definition 1. *The* order ρ_f *and* lower order λ_f *of an entire function* f *are defined as*

$$\rho_f := \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \textit{and} \quad \lambda_f := \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \ .$$

An entire function whose order *and* lower order *are the same is said to be of* regular growth. *Entire functions which are not of* regular growth *are said to be of* irregular growth.

Definition 2. *The* type σ_f *and* lower type $\overline{\sigma}_f$ *of an entire function* f *are defined as*

$$\sigma_f := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \quad \textit{and} \quad \overline{\sigma}_f := \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \quad (0 < \rho_f < \infty).$$

Datta and Jha [3] introduced to define *weak type* of an entire function of finite positive *lower order* in the following way:

Definition 3. *The* weak type τ_f *and the growth indicator* $\overline{\tau}_f$ *of an entire function* f *of finite* positive lower order λ_f *are defined by*

$$\overline{\tau}_f := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \quad \textit{and} \quad \tau_f := \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}} \quad (0 < \lambda_f < \infty)\,.$$

For any two given entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum modulii. From Definition 1, it is seen that the *order* of an entire function f which is generally used for computational purpose is defined in terms of the growth of f with respect to the *exponential* function as follows:

$$\rho_f := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp(z)}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

Bernal [1, 2] introduced to define *relative order* of an entire function g with respect to an entire function f denoted by $\rho_f(g)$ to avoid comparing growth with just the exponential function $\exp(z)$ as follows:

$$\begin{aligned} \rho_f(g) :&= \inf\{\mu > 0 : M_g(r) < M_f(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log M_f^{-1} M_g(r)}{\log r}. \end{aligned}$$

It is easy to see that the above definition coincides with the classical one if $f(z) = \exp(z)$ (*cf.* [16]).

Similarly one can define the *relative lower order* of g with respect to f, denoted by $\lambda_f(g)$, as follows:

$$\lambda_f(g) := \liminf_{r\to\infty} \frac{\log M_f^{-1} M_g(r)}{\log r}.$$

An entire function g is said to be of *regular relative growth* with respect to f if its *relative order* with respect to f coincides with its *relative lower order* with respect to the same function f.

To compare the relative growth of two entire functions having same nonzero finite *relative order* with respect to another entire function, Roy [15] recently introduced the notion of *relative type* of two entire functions in the following manner.

Definition 4. *Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the* relative type $\sigma_g(f)$ *of f with respect to g is defined as follows:*

$$\begin{aligned}&\sigma_g(f)\\ &:= \inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\}\\ &= \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}}.\end{aligned}$$

Likewise one can define the relative lower type *of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows:*

$$\overline{\sigma}_g(f) := \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}} \quad (0 < \rho_g(f) < \infty).$$

Analogously to determine the relative growth of two entire functions having same nonzero finite *relative lower order* with respect to another entire function, Datta and Biswas [5] introduced to define *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way.

Definition 5. *The* relative weak type $\tau_g(f)$ *of an entire function f with respect to another entire function g having finite positive* relative lower order $\lambda_g(f)$ *is defined as follows:*

$$\tau_g(f) := \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}}.$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way:

$$\overline{\tau}_g(f) := \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} \quad (0 < \lambda_g(f) < \infty).$$

Choosing $g(z) = \exp(z)$, one may easily verify that Definition 4 and Definition 5 coincide with the classical definitions of *type* (*lower type*) and *weak type*, respectively.

In this connection, the following definition is introduced (see [2]).

Definition 6. *A non-constant entire function f is said to have* Property (A)
if for any $\sigma > 1$ and for all large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may refer to [2].

Here, in this paper, we aim at investigating some basic properties of *relative type* and *relative weak type* of entire functions under somewhat different conditions. Throughout this paper, for entire functions f_i and g_k $(i,\, k = 1,\, 2)$, we assume that $\sigma_{f_i}(g_k)$, $\overline{\sigma}_{f_i}(g_k)$, $\tau_{f_i}(g_k)$ and $\overline{\tau}_{f_i}(g_k)$ are all nonzero finite.

It is also remarked in passing that the standard definitions and notations in the theory of entire functions, for which one may refer to [17], are not given here.

2. Some Known and New Results

Determination of the *order* and *type* of entire functions are very important to study the basic growth properties in the value distribution theory. In this regard, during the past decades, many researchers have made close investigations on this research subject to yield many results, for example, some of which are recalled here.

Theorem A ([9]). *Let f and g be any two entire functions of order* ρ_f and ρ_g *respectively. Then*

$$\rho_{f+g} = \rho_g \ \text{ when } \rho_f < \rho_g \text{ and } \rho_{f\cdot g} \leq \rho_g \ \text{ when } \rho_f \leq \rho_g\ .$$

Theorem B ([12]). *Let f and g be any two entire functions with order ρ_f, ρ_g, and type σ_f, σ_g, respectively. Then*

$$\rho_{f+g} \leq \max\{\rho_f, \rho_g\}\,,\ \rho_{f\cdot g} \leq \max\{\rho_f, \rho_g\}$$

and

$$\sigma_{f+g} \leq \max\{\sigma_f, \sigma_g\}\,,\ \sigma_{f\cdot g} \leq \sigma_f + \sigma_g\ .$$

Detailed investigations on the properties of relative order of entire functions have been made in [2], [8], [10] and [11]. In this connection we state the following two theorems.

Theorem C ([2]). *Let f_1, g_1 and g_2 be any three entire functions. If $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k, i = 1, 2\}$, then*

$$\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_i) \quad (i = 1, 2),$$

whose equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Theorem D ([2, 14]). *Let f_1, g_1 and g_2 be any three entire functions. If $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k, i = 1, 2\}$, then*

$$\rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_i) \quad (i = 1, 2),$$

whose equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Similar results hold for the quotient $\frac{g_1}{g_2}$ provided $\frac{g_1}{g_2}$ is entire.

Datta *et al.* [4] proved the following two theorems for *relative lower order.*

Theorem E ([4]). *Let f_1, f_2 and g_1 be any three entire functions.* If $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k, i = 1, 2\}$, then

$$\lambda_{f_1 \pm f_2}(g_1) \geq \lambda_{f_i}(g_1) \quad (i = 1, 2),$$

whose equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$.

Theorem F ([4]). *Let f_1, f_2 and g_1 be any three entire functions. If $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k, i = 1, 2\}$ and g_1 has the* Property (A), *then*

$$\lambda_{f_1 \cdot f_2}(g_1) \geq \lambda_{f_i}(g_1) \quad (i = 1, 2),$$

whose equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$.

Similar results hold for the quotient $\frac{f_1}{f_2}$ provided $\frac{f_1}{f_2}$ is entire.

Extending the results, Datta *et al.* [6] established the following theorems under somewhat different conditions.

Theorem G ([6]). *Let f_1, f_2, g_1 and g_2 be any four entire functions.*

(i) *If $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k, i = 1, 2\}$ and g_1 is of regular relative growth with respect to at least any one of f_1* or *f_2, then*

$$\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_i}(g_1) \quad (i = 1, 2),$$

whose equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$.

(ii) *If* $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k, i = 1, 2\}$ *and at least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1, *then*

$$\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i) \quad (i = 1, 2),$$

whose equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Theorem H ([6]). *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions.*

(i) *If* $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k, i = 1, 2\}$, g_1 *has the* Property (A) *and is of regular relative growth with respect to at least any one of* f_1 or f_2, *then*

$$\rho_{f_1 \cdot f_2}(g_1) \geq \rho_{f_i}(g_1) \quad (i = 1, 2),$$

whose equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$.

Similar results hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

(ii) *If* $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k, i = 1, 2\}$, f_1 *has the* Property (A) *and at least* g_1 or g_2 *is of regular relative growth with respect to* f_1, *then*

$$\lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_i) \quad (i = 1, 2),$$

whose equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Similar results hold for the quotient $\frac{g_1}{g_2}$ *provided* $\frac{g_1}{g_2}$ *is entire.*

Theorem I ([6]). *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions.*

(i) *If* $(g_1) \neq \rho_{f_2}(g_1)$, $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ *and* g_1 *and* g_1 *are both of regular relative growth with respect to at least any one of* f_1 or f_2, *then*

$$\rho_{f_1 \pm f_2}(g_1 \pm g_2)$$
$$\leq \max\left[\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}, \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}\right],$$

whose equality holds when

$$\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\} \neq \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}.$$

(ii) *If* $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$, $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$ *and at least* g_1 or g_2 *is of regular relative growth with respect to* f_1 *and* f_2, *respectively, then*

$$\lambda_{f_1 \pm f_2}(g_1 \pm g_2)$$
$$\geq \min\left[\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\}, \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}\right],$$

whose equality holds when

$$\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\} \neq \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}.$$

Theorem J ([6]). *Let* f_1, f_2, g_1 and g_2 *be any four entire functions.*

(i) *If* (*a*) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$, (*b*) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ (*c*) $f_1 \cdot f_2$, g_1 *and* g_2 *have the* Property (A) *and* (*d*) g_1 *and* g_1 *are both of regular relative growth with respect to at least any one of* f_1 *or* f_2, *then*

$$\rho_{f_1 \cdot f_2}(g_1 \cdot g_2) \leq \max\left[\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}, \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}\right]$$

and

$$\rho_{f_1/f_2}(g_1/g_2) \leq \max\left[\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}, \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}\right],$$

whose equality holds when

$$\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\} \neq \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}.$$

(ii) *If* (*a*) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$, (*b*) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$, (*c*) $g_1 \cdot g_2$, f_1 *and* f_2 *have the* Property (A) *and* (*d*) *at least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1 *and* f_2, *respectively, then*

$$\lambda_{f_1 \cdot f_2}(g_1 \cdot g_2) \geq \min\left[\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\}, \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}\right]$$

and

$$\lambda_{f_1/f_2}(g_1/g_2) \geq \min\left[\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\}, \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}\right],$$

whose equality holds when

$$\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\} \neq \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}.$$

In the cases of *relative type* and *relative weak type*, it therefore seems natural to make parallel investigations of their basic properties. In this connection, Roy [15] proved only the following theorem.

Theorem K ([15]). *Let* f_1, g_1 *and* g_2 *be any three entire functions. If* (*i*) $\rho_{f_1}(g_i) = \max\left\{\rho_{f_1}(g_k) \mid k, i = 1, 2\right\}$ *and* (*ii*) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$, *then*

$$\sigma_{f_1}(g_1 \pm g_2) = \sigma_{f_1}(g_i).$$

Here, under somewhat different conditions, we present the following theorems related to *relative type* (*relative lower type*) and *relative weak type* that extend the previous results in some sense.

Theorem 1. *Let* f_1, f_2, g_1 and g_2 *be any four entire functions such that* $\rho_{f_k}(g_k)$ $(k = 1, 2)$ *are non-zero finite.*

(**I**) *If* (A) $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k, i = 1, 2\}$ *and* (B) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$, *then*

$$\overline{\sigma}_{f_1}(g_1 \pm g_2) = \overline{\sigma}_{f_1}(g_i) \quad (i = 1, 2).$$

(**II**) *If* (A) $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k = 1, 2\}$, (B) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$ *and* (C) g_1 *is of regular relative growth with respect to at least any one of* f_1 *or* f_2, *then*
(i) $\sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_i}(g_1) \quad (i = 1, 2)$
and
(ii) $\overline{\sigma}_{f_1 \pm f_2}(g_1) = \overline{\sigma}_{f_i}(g_1) \quad (i = 1, 2)$.

(**III**) *Assume the functions* f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(A) $\rho_{f_i}(g_k) = \max\big\{\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}, \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}\big\}$;
(B) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$;
(C) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$;
(D) $\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\} \neq \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}$;
(E) g_1 *and* g_2 *are both of regular relative growth with respect to at least any one of* f_1 *or* f_2.
Then we have
(i) (i) $\sigma_{f_1 \pm f_2}(g_1 \pm g_2) = \sigma_{f_i}(g_k) \quad (i, k = 1, 2)$
and
(ii) $\overline{\sigma}_{f_1 \pm f_2}(g_1 \pm g_2) = \overline{\sigma}_{f_i}(g_k) \quad (i, k = 1, 2)$.

Theorem 2. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions such that* $\lambda_{f_k}(g_k)$ $(k = 1, 2)$ *are non-zero finite.*

(**I**) *The following conditions are assumed to be satisfied:*
(A) (A) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k = 1, 2\}$;
(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$;
(C) *At least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1.
Then we have
(i) $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i)$ $(i = 1, 2)$.
and
(ii) $\overline{\tau}_{f_1}(g_1 \pm g_2) = \overline{\tau}_{f_1}(g_i)$ $(i = 1, 2)$.

(**II**) *The following two conditions are assumed to be satisfied:*
(A) $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = 1, 2\}$
and
(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$.
Then we have
(i) $\tau_{f_1 \pm f_2}(g_1) = \tau_{f_i}(g_1)$ $(i = 1, 2)$

and

(ii) $\overline{\tau}_{f_1 \pm f_2}(g_1) = \overline{\tau}_{f_i}(g_1)$ $(i = 1, 2)$.

(**III**) *The following conditions are assumed to be satisfied:*

(A) $\lambda_{f_i}(g_k) = \min\left\{\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\}, \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}\right\}$;

(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$;

(C) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$;

(D) $\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}$;

(E) *At least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1 *and* f_2*, respectively.*

Then we have

(i) $\tau_{f_1 \pm f_2}(g_1 \pm g_2) = \tau_{f_i}(g_k)$ $(i, k = 1, 2)$

and

(ii) $\overline{\tau}_{f_1 \pm f_2}(g_1 \pm g_2) = \overline{\tau}_{f_i}(g_k)$ $(i, k = 1, 2)$.

Theorem 3. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions such that* $\rho_{f_k}(g_k)$ $(k = 1, 2)$ *are non-zero finite.*

(**I**) *The following conditions are assumed to be satisfied:*

(A) $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k, i = 1, 2\}$;

(B) $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$;

(C) f_1 *has the* Property (A).

Then we have

(i) $\sigma_{f_1}(g_1 \cdot g_2) \leq \sigma_{f_1}(g_i)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\rho_{f_1}(g_i)} \leq 1$.

(ii) $\overline{\sigma}_{f_1}(g_1 \cdot g_2) \leq \overline{\sigma}_{f_1}(g_i)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\rho_{f_1}(g_i)} \leq 1$.

(**II**) *The following conditions are assumed to be satisfied:*

(A) $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k = 1, 2\}$;

(B) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$;

(C) g_1 *has the* Property (A) *and also* g_1 *is of regular relative growth with respect to at least any one of* f_1 *or* f_2.

Then we have

(i) $\sigma_{f_1 \cdot f_2}(g_1) \geq \sigma_{f_i}(g_1)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\rho_{f_i}(g_1)} \geq 1$.

and

(ii) $\overline{\sigma}_{f_1 \cdot f_2}(g_1) \geq \overline{\sigma}_{f_i}(g_1)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\rho_{f_i}(g_1)} \geq 1$.

(**III**) *The following conditions are assumed to be satisfied:*

(A) $\rho_{f_i}(g_k) = \max\left\{\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}, \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}\right\}$;

(B) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$;

(C) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$;
(D) $\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\} \neq \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}$;
(E) $f_1 \cdot f_2$, g_1 and g_2 *have the* Property (A);
(F) g_1 *and* g_2 *are both of regular relative growth with respect to at least any one of* f_1 *or* f_2;
(G) $2^{\rho_{f_1 \cdot f_2}(g_k)} \leq 1$ *and* $2^{\rho_{f_k}(g_k)} \geq 1$.

Then we have

(i) $\sigma_{f_1 \cdot f_2}(g_1 \cdot g_2) = \sigma_{f_i}(g_k)$ $(i, k = 1, 2)$

and

(ii) $\overline{\sigma}_{f_1 \cdot f_2}(g_1 \cdot g_2) = \overline{\sigma}_{f_i}(g_k)$ $(i, k = 1, 2)$.

Similar results for the above three cases hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

Theorem 4. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions such that* $\rho_{f_k}(g_k)$ $(k = 1, 2)$ *are non-zero finite.*

(**I**) *The following conditions are assumed to be satisfied:*
(A) $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k, i = 1, 2\}$;
(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$;
(C) f_1 *has the* Property (A) *and at least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1.

Then we have

(i) $\tau_{f_1}(g_1 \cdot g_2) \leq \tau_{f_1}(g_i)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\lambda_{f_1}(g_i)} \leq 1$.

and

(ii) $\overline{\tau}_{f_1}(g_1 \cdot g_2) \leq \overline{\tau}_{f_1}(g_i)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\lambda_{f_1}(g_i)} \leq 1$.

(**II**) *The following conditions are assumed to be satisfied:*
(A) $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = 1, 2\}$;
(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$;
(C) g_1 *has the* Property (A).

Then we have

(i) $\tau_{f_1 \cdot f_2}(g_1) \geq \tau_{f_i}(g_1)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\lambda_{f_i}(g_1)} \geq 1$.

(ii) $\overline{\tau}_{f_1 \cdot f_2}(g_1) \geq \overline{\tau}_{f_i}(g_1)$ $(i = 1, 2)$, *whose equality holds only when* $2^{\lambda_{f_i}(g_1)} \geq 1$.

(**III**) *The following conditions are assumed to be satisfied:*
(A) $\lambda f_i(g_k) = \min\left\{\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\}, \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}\right\}$;
(B) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$;

(C) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$;
(D) $\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}$;
(E) $g_1 \cdot g_2$, f_1 *and* f_2 *have the* Property (A);
(F) *At least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1 *and* f_2, *respectively;*
(G) $2^{\lambda_{f_1 \cdot f_2}(g_k)} \leq 1$ *and* $2^{\lambda_{f_k}(g_k)} \geq 1$.

Then we have

(i) $\tau_{f_1 \cdot f_2}(g_1 \cdot g_2) = \tau_{f_i}(g_k)$ $(i, k = 1, 2)$

and

(ii) $\overline{\tau}_{f_1 \cdot f_2}(g_1 \cdot g_2) = \overline{\tau}_{f_i}(g_k)$ $(i, k = 1, 2)$.

Similar results for the above three cases hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

Here we reconsider the equalities in Theorem C to Theorem H under somewhat different conditions and give our assertions as in following four theorems.

Theorem 5. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions.*

(**I**) *If either* $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$ *or* $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_1}(g_2)$ *holds, then*

$$\rho_{f_1}(g_1 \pm g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2).$$

(**II**) *The following two conditions are assumed to be satisfied:*

(A) *Either* $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$ *or* $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1)$ *holds;*
(B) g_1 *is of regular relative growth with respect to at least any one of* f_1 *or* f_2.

Then we have

$$\rho_{f_1 \pm f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1).$$

Theorem 6. *Let* f_1, f_2, g_1 *and* g_2 *be any four entire functions.*

(**I**) *The following conditions are assumed to be satisfied:*

(A) *Either* $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$ *or* $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$ *holds;*
(B) *At least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1.

Then we have

$$\lambda_{f_1}(g_1 \pm g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2).$$

(**II**) *If either* $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$ *or* $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_2}(g_1)$ *holds, then*

$$\lambda_{f_1 \pm f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1).$$

Theorem 7. *Let f_1, f_2, g_1 and g_2 be any four entire functions.*

(**I**) *The following conditions are assumed to be satisfied:*

(A) *Either* $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$ *or* $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_1}(g_2)$ *holds;*

(B) f_1 *has the* Property (A)*;*

(C) $2^{\rho_{f_1}(g_1)} \geq 1$.

Then we have

$$\rho_{f_1}(g_1 \cdot g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2).$$

(**II**) *The following conditions are assumed to be satisfied:*

(A) *Either* $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$ *or* $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1)$ *holds;*

(B) g_1 *has the* Property (A) *and is of regular relative growth with respect to at least any one of* f_1 *or* f_2*;*

(C) $2^{\rho_{f_i}(g_1)} \geq 1$.

Then we have

$$\rho_{f_1 \cdot f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1).$$

Similar results for the above two cases hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

Theorem 8. *Let f_1, f_2, g_1 and g_2 be any four entire functions.*

(**I**) *The following conditions are assumed to be satisfied:*

(A) *Either* $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$ *or* $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$ *holds;*

(B) f_1 *has the* Property (A) *and at least* g_1 *or* g_2 *is of regular relative growth with respect to* f_1*;*

(C) $2^{\lambda_{f_1}(g_i)} \leq 1$.

Then we have

$$\lambda_{f_1}(g_1 \cdot g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2).$$

(**II**) *The following conditions are assumed to be satisfied:*

(A) *Either* $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$ *or* $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$ *holds;*

(B) g_1 *has the* Property (A)*;*

(C) $2^{\lambda_{f_i}(g_1)} \geq 1$.

Then we have

$$\lambda_{f_1 \cdot f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1).$$

Similar results for the above three cases hold for the quotient $\frac{f_1}{f_2}$ *provided* $\frac{f_1}{f_2}$ *is entire.*

3. Required Known Properties

Here we recall some known properties, which will be required in the next section, as in the following lemmas. For Lemmas 1 and 2, see [2]. For Lemma 3 and Lemma 4 , one may refer to [12] and [7, p.18], respectively.

Lemma 1. *Suppose f be an entire function and α, β be such that $\alpha > 1$ and $0 < \beta < \alpha$. Then*

$$M_f(\alpha r) > \beta M_f(r).$$

Lemma 2. *Let f be an entire function satisfying the* Property (A). *Then for any positive integer n and for all sufficiently large r,*

$$[M_f(r)]^n \leq M_f\left(r^{\delta}\right)$$

holds for $\delta > 1$.

Lemma 3. *Every entire function f satisfying the* Property (A) *is transcendental.*

Lemma 4. *Let f be an entire function. Then, for all sufficiently large values of r, we have*

$$T_f(r) \leq \log M_f(r) \leq 3T_f(2r).$$

4. Proofs

Here we prove our main results.

Proof of Theorem 1. From the definition of relative type and relative lower type of entire function, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left[\left(\sigma_{f_k}(g_k) + \varepsilon\right) r^{\rho_{f_k}(g_k)}\right], \tag{1}$$

$$\begin{aligned} M_{g_k}(r) &\geq M_{f_k}\left[\left(\overline{\sigma}_{f_k}(g_k) - \varepsilon\right) r^{\rho_{f_k}(g_k)}\right] \\ i.e.,\ M_{f_k}(r) &\leq M_{g_k}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_k}(g_k) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_k}(g_k)}}\right], \end{aligned} \tag{2}$$

and also for a sequence $\{r_n\}$ of values of r tending to infinity, we get

$$M_{g_k}(r) \geq M_{f_k}\left[(\sigma_{f_k}(g_k)-\varepsilon)\, r_n^{\rho_{f_k}(g_k)}\right]$$

$$\text{(3)} \qquad i.e.,\ M_{f_k}(r) \leq M_{g_k}\left[\left(\frac{r_n}{(\sigma_{f_k}(g_k)-\varepsilon)}\right)^{\frac{1}{\rho_{f_k}(g_k)}}\right],$$

$$\text{(4)} \qquad M_{g_k}(r) \leq M_{f_k}\left[(\overline{\sigma}_{f_k}(g_k)+\varepsilon)\, r_n^{\rho_{f_k}(g_k)}\right],$$

where $\varepsilon > 0$ is any arbitrary positive number and $k = 1, 2$.

Case I. Let $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$ where $k, i = 1, 2$ with $g_k \neq g_i$ for $k \neq i$.

Now from (1) and (4) we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\text{(5)} \qquad M_{g_1 \pm g_2}(r_n) < M_{g_1}(r_n) + M_{g_2}(r_n),$$

which implies that

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\sigma_{f_1}(g_k)+\varepsilon)\, r_n^{\rho_{f_1}(g_k)}\right] + M_{f_1}\left[(\overline{\sigma}_{f_1}(g_i)+\varepsilon)\, r_n^{\rho_{f_1}(g_i)}\right].$$

So we have

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\overline{\sigma}_{f_1}(g_i)+\varepsilon)\, r_n^{\rho_{f_1}(g_i)}\right]\left[1 + \frac{M_{f_1}\left[(\sigma_{f_1}(g_k)+\varepsilon)\, r_n^{\rho_{f_1}(g_k)}\right]}{M_{f_1}\left[(\overline{\sigma}_{f_1}(g_i)+\varepsilon)\, r_n^{\rho_{f_1}(g_i)}\right]}\right].$$

Since $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$, one can make the term $\frac{M_{f_1}\left[(\sigma_{f_1}(g_k)+\varepsilon) r_n^{\rho_{f_1}(g_k)}\right]}{M_{f_1}\left[(\overline{\sigma}_{f_1}(g_i)+\varepsilon) r_n^{\rho_{f_1}(g_i)}\right]}$ sufficiently small by taking n sufficiently large. Therefore in view of Lemma 1 and the above inequality, we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[(\overline{\sigma}_{f_1}(g_i)+\varepsilon)\, r_n^{\rho_{f_1}(g_i)}\right](1+\varepsilon_1).$$

That is,

$$M_{g_1 \pm g_2}(r_n) < M_{f_1}\left[\alpha\,(\overline{\sigma}_{f_1}(g_i)+\varepsilon)\, r_n^{\rho_{f_1}(g_i)}\right],$$

where $\alpha > (1+\varepsilon_1)$.

Now making $\alpha \to 1+$, we obtain from Theorem C for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n) < (\overline{\sigma}_{f_1}(g_i) + \varepsilon)\, r_n^{\rho_{f_1}(g_i)}$$

and so

$$\frac{M_{f_1}^{-1} M_{g_1 \pm g_2}(r_n)}{r_n^{\rho_{f_1}(g_1 \pm g_2)}} < (\overline{\sigma}_{f_1}(g_i) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\overline{\sigma}_{f_1}(g_1 \pm g_2) \leq \overline{\sigma}_{f_1}(g_i).$$

Further without any loss of generality, let $\rho_{f_1}(g_1) < \rho_{f_1}(g_2)$ and $g = g_1 \pm g_2$. Then $\overline{\sigma}_{f_1}(g) \leq \overline{\sigma}_{f_1}(g_2)$. Also let $g_2 = \pm(g - g_1)$ and in this case we obtain from Theorem C that $\rho_{f_1}(g_1) < \rho_{f_1}(g)$. So $\overline{\sigma}_{f_1}(g_2) \leq \overline{\sigma}_{f1}(g)$. Hence $\overline{\sigma}_{f_1}(g) = \overline{\sigma}_{f_1}(g_2) \Rightarrow \overline{\sigma}_{f_1}(g_1 \pm g_2) = \overline{\sigma}_{f_1}(g_2)$. Thus, $\overline{\sigma}_{f_1}(g_1 \pm g_2) = \overline{\sigma}_{f_1}(g_i)$ $(i = 1,\, 2)$ where $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k,\, i = 1, 2\}$ and $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$ which is the first part of the theorem.

Case II. Now suppose that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ where $k,\, i = 1,\, 2$ with $f_i \neq f_k$ $(i \neq k)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Therefore, in view of (2) and (3), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{f_1 \pm f_2}(r_n) < M_{f_1}(r_n) + M_{f_2}(r_n). \tag{6}$$

Thus we have

$$\begin{aligned} & M_{f_1 \pm f_2}(r_n) \\ < \;& M_{g_1}\left[\left(\frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right] + M_{g_1}\left[\left(\frac{r_n}{(\overline{\sigma}_{f_k}(g_1) - \varepsilon)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right] \end{aligned}$$

and so

$$\begin{aligned} & M_{f_1 \pm f_2}(r_n) \\ < \;& M_{g_1}\left[\left(\frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]\left[1 + \frac{M_{g_1}\left[\left(\frac{r_n}{(\overline{\sigma}_{f_k}(g_1) - \varepsilon)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r_n}{(\sigma_{f_i}(g_1) - \varepsilon)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]}\right]. \end{aligned}$$

Since $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$, we can make the term $\dfrac{M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_k}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]}$ sufficiently small by taking n sufficiently large. Hence in view of Lemma 1 and the above inequality we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\begin{aligned} M_{f_1 \pm f_2}(r_n) &< M_{g_1}\left[\left(\frac{r_n}{(\sigma_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right](1+\varepsilon_1) \\ &< M_{g_1}\left[\alpha\left(\frac{r_n}{(\sigma_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right], \end{aligned}$$

where $\alpha > (1+\varepsilon_1)$.

Hence, making $\alpha \to 1+$, we obtain the first part of Theorem G for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\begin{aligned} (\sigma_{f_i}(g_1)-\varepsilon)\, r_n^{\rho_{f_i}(g_1)} &< M_{f_1 \pm f_2}^{-1} M_{g_1}(r_n) \\ \textit{i.e.,}\ (\sigma_{f_i}(g_1)-\varepsilon) &< \frac{M_{f_1 \pm f_2}^{-1} M_{g_1}(r_n)}{r_n^{\rho_{f_1 \pm f_2}(g_1)}}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we find

$$\sigma_{f_1 \pm f_2}(g_1) \geq \sigma_{f_i}(g_1).$$

Now without loss of generality, we may consider that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$ and $f = f_1 \pm f_2$. Then $\sigma_f(g_1) \geq \sigma_{f_1}(g_1)$. Further let $f_1 = (f \pm f_2)$. Therefore in view of the first part of Theorem G, $\rho_f(g_1) < \rho_{f_2}(g_1)$ and accordingly $\sigma_{f_1}(g_1) \geq \sigma_f(g_1)$. Hence $\sigma_f(g_1) = \sigma_{f_1}(g_1) \Rightarrow \sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_1}(g_1)$. So, $\sigma_{f_1 \pm f_2}(g_1) = \sigma_{f_i}(g_1)$ $(i = 1, 2)$ where $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k, i = 1, 2\}$ provided $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Case III. In this case, one can clearly assume that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ where k, $i = 1, 2$ with $f_i \neq f_k$ $(i \neq k)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Then, in view of (2), we obtain for all sufficiently large values of r that

$$M_{f_1 \pm f_2}(r) < M_{f_1}(r) + M_{f_2}(r). \tag{7}$$

That is, we have

$$\begin{aligned} & M_{f_1 \pm f_2}(r) \\ < \; & M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right] + M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_k}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]. \end{aligned}$$

And so

$$\begin{aligned} & M_{f_1 \pm f_2}(r) \\ & < M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]\left[1 + \frac{M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_k}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]}\right]. \end{aligned} \tag{8}$$

As $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$, we can make the term $\dfrac{M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_k}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]}$ sufficiently small by taking r sufficiently large and therefore using the similar technique for all sufficiently large values of r as executed in the proof of Case II we get from (8) that $\overline{\sigma}_{f_1 \pm f_2}(g_1) = \overline{\sigma}_{f_i}(g_1)$ $(i = 1, 2)$ where $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k,\, i = 1, 2\}$ provided $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Thus combining Case II and Case III we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem I (i), Theorem K and the first part and second part of the theorem. Hence its proof is omitted.

Proof of Theorem 2. For any arbitrary positive number $\varepsilon > 0$, we have from Definition 5 for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left[\left(\overline{\tau}_{f_k}(g_k) + \varepsilon\right) r^{\lambda_{f_k}(g_k)}\right], \tag{9}$$

$$\begin{aligned} M_{g_k}(r) & \geq M_{f_k}\left[\left(\tau_{f_k}(g_k) - \varepsilon\right) r^{\lambda_{f_k}(g_k)}\right] \\ \textit{i.e.},\; M_{f_k}(r) & \leq M_{g_k}\left[\left(\frac{r}{\left(\tau_{f_k}(g_k) - \varepsilon\right)}\right)^{\frac{1}{\lambda_{f_k}(g_k)}}\right], \end{aligned} \tag{10}$$

and for a sequence $\{r_n\}$ of values of r tending to infinity we have

$$M_{g_k}(r) \geq M_{f_k}\left[(\overline{\tau}_{f_k}(g_k) - \varepsilon)\, r_n^{\lambda_{f_k}(g_k)}\right]$$

$$\text{(11)} \qquad i.e.,\ M_{f_k}(r) \leq M_{g_k}\left[\left(\frac{r_n}{(\overline{\tau}_{f_k}(g_k) - \varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_k)}}\right],$$

$$\text{(12)} \qquad M_{g_k}(r) \leq M_{f_k}\left[(\tau_{f_k}(g_k) + \varepsilon)\, r_n^{\lambda_{f_k}(g_k)}\right]$$

where $k = 1, 2$.

Case I. Let us consider $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ where $k, i = 1, 2$ with $g_k \neq g_i$ $(k \neq i)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1.

Therefore from (5), (9) and (12) we get for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\begin{aligned} & M_{g_1 \pm g_2}(r_n) \\ < \ & M_{f_1}\left[(\overline{\tau}_{f_1}(g_k) + \varepsilon)\, r_n^{\lambda_{f_1}(g_k)}\right] + M_{f_1}\left[(\tau_{f_1}(g_i) + \varepsilon)\, r_n^{\lambda_{f_1}(g_i)}\right]. \end{aligned}$$

That is, we have

$$\begin{aligned} & M_{g_1 \pm g_2}(r_n) \\ \text{(13)} \quad & < M_{f_1}\left[(\tau_{f_1}(g_i) + \varepsilon) r_n^{\lambda_{f_1}(g_i)}\right]\left[1 + \frac{M_{f_1}\left[(\overline{\tau}_{f_1}(g_k) + \varepsilon) r_n^{\lambda_{f_1}(g_k)}\right]}{M_{f_1}\left[(\tau_{f_1}(g_i) + \varepsilon) r_n^{\lambda_{f_1}(g_i)}\right]}\right]. \end{aligned}$$

Since $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, we can make the term $\frac{M_{f_1}\left[(\overline{\tau}_{f_1}(g_k)+\varepsilon) r_n^{\lambda_{f_1}(g_k)}\right]}{M_{f_1}\left[(\tau_{f_1}(g_i)+\varepsilon) r_n^{\lambda_{f_1}(g_i)}\right]}$ sufficiently small by taking n sufficiently large. So with the help of Lemma 1 and the second part of Theorem G and using the similar technique of Case I of Theorem 1, we get from (13) that

$$\tau_{f_1}(g_1 \pm g_2) \leq \tau_{f_1}(g_i).$$

Now without loss of generality, let us suppose that $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2)$ and $g = g_1 \pm g_2$. So $\tau_{f_1}(g) \leq \tau_{f_1}(g_2)$. Also let $g_2 = \pm(g - g_1)$ and in this case we have from Theorem E that $\lambda_{f_1}(g_1) < \lambda_{f_1}(g)$. Therefore $\tau_{f_1}(g_2) \leq \tau_{f1}(g)$. Hence $\tau_{f_1}(g) = \tau_{f_1}(g_2) \Rightarrow \tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_2)$. Thus, $\tau_{f_1}(g_1 \pm g_2) = \tau_{f_1}(g_i)$ $(i = 1, 2)$ where $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k, i = 1, 2\}$ and $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

Case II. Let us consider that $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ where $k = i = 1, 2$ with $g_k \neq g_i$. Now, in view of (9), we get for all sufficiently large values of r that

$$M_{g_1 \pm g_2}(r) < M_{g_1}(r) + M_{g_2}(r).$$

That is, we have

$$\begin{aligned} & M_{g_1 \pm g_2}(r) \\ < \ & M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_k) + \varepsilon\right) r^{\lambda_{f_1}(g_k)}\right] + M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_i) + \varepsilon\right) r^{\lambda_{f_1}(g_i)}\right]. \end{aligned}$$

And so

$$\begin{aligned} & M_{g_1 \pm g_2}(r_n) \\ & < M_{f_1}\left[(\overline{\tau}_{f_1}(g_i) + \varepsilon) r^{\lambda_{f_1}(g_i)}\right] \left[1 + \frac{M_{f_1}\left[(\overline{\tau}_{f_1}(g_k) + \varepsilon) r^{\lambda_{f_1}(g_k)}\right]}{M_{f_1}\left[(\overline{\tau}_{f_1}(g_i) + \varepsilon) r^{\lambda_{f_1}(g_i)}\right]}\right]. \end{aligned} \tag{14}$$

As $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, by taking r sufficiently large one can make the term

$$\frac{M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_k) + \varepsilon\right) r^{\lambda_{f_1}(g_k)}\right]}{M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_i) + \varepsilon\right) r^{\lambda_{f_1}(g_i)}\right]}$$

sufficiently small and therefore for similar reasoning of Case-I we get that $\overline{\tau}_{f_1}(g_1 \pm g_2) = \overline{\tau}_{f_1}(g_i) \mid i = 1, 2$ where $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k = i = 1, 2\}$ and $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$ and hence details of its proof are omitted.

Thus the first part of the theorem follows from Case I and Case II.

Case III. Now suppose that $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$.

Now in view of (7) and (10) we have for all sufficiently large values of r that

$$\begin{aligned} & M_{f_1 \pm f_2}(r) \\ < \ & M_{g_1}\left[\left(\frac{r}{(\tau_{f_i}(g_1) - \varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right] + M_{g_1}\left[\left(\frac{r}{(\tau_{f_k}(g_1) - \varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]. \end{aligned}$$

We thus have

$$M_{f_1 \pm f_2}(r) < M_{g_1}\left[\left(\frac{r}{(\tau_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]\left[1+\frac{M_{g_1}\left[\left(\frac{r}{(\tau_{f_k}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r}{(\tau_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]}\right].$$

Since $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$, one can make the term $\dfrac{M_{g_1}\left[\left(\frac{r}{(\tau_{f_k}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r}{(\tau_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]}$ sufficiently small by taking r sufficiently large. Therefore using the similar technique as executed in the proof of Case III of Theorem 1, it follows from above arguments and Theorem E that

$$\tau_{f_1 \pm f_2}(g_1) \geq \tau_{f_i}(g_1).$$

At this time without loss of generality, we may consider that $\lambda_{f_1}(g_1) < \lambda_{f_2}(g_1)$ and $f = f_1 \pm f_2$. Then $\tau_f(g_1) \geq \tau_{f_1}(g_1)$. Further let $f_1 = (f \pm f_2)$. Therefore, in view of Theorem C, $\lambda_f(g_1) < \lambda_{f_2}(g_1)$ and accordingly $\tau_{f_1}(g_1) \geq \tau_f(g_1)$. Hence $\tau_f(g_1) = \tau_{f_1}(g_1) \Rightarrow \tau_{f_1 \pm f_2}(g_1) = \tau_{f_1}(g_1)$. So, $\tau_{f_1 \pm f_2}(g_1) = \tau_{f_i}(g_1) \mid i = 1,2$ where $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = i = 1,2\}$ provided $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$.

Case IV. Now let us consider $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ where $k = i = 1$, 2 with $f_i \neq f_k$. Therefore in view of (6), (10) and (11) we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{f_1 \pm f_2}(r_n) < M_{g_1}\left[\left(\frac{r_n}{(\overline{\tau}_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right] + M_{g_1}\left[\left(\frac{r_n}{(\tau_{f_k}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right].$$

We thus have

$$M_{f_1 \pm f_2}(r_n) < M_{g_1}\left[\left(\frac{r_n}{(\overline{\tau}_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]\left[1+\frac{M_{g_1}\left[\left(\frac{r_n}{(\tau_{f_k}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r_n}{(\overline{\tau}_{f_i}(g_1)-\varepsilon)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]}\right]. \tag{15}$$

Since $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$, we can make the term $\dfrac{M_{g_1}\left[\left(\frac{r_n}{\left(\tau_{f_k}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]}{M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\tau}_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]}$ sufficiently small by taking n sufficiently large. Therefore using the similar technique of Case II of Theorem 1, we obtain the conclusion that $\overline{\tau}_{f_1 \pm f_2}(g_1) = \overline{\tau}_{f_i}(g_1) \mid i = 1, 2$ where $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = i = 1, 2\}$ provided $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$ from (15).

So the second part of the theorem follows from Case III and Case IV.

The proof of the third part of the theorem is omitted as it can be carried out in view of Theorem I (ii) and the above cases.

Proof of Theorem 3. **Case I.** By Lemma 3, f_1 is transcendental. Suppose that $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$ where $k = i = 1, 2$ with $g_k \neq g_i$. Now for any arbitrary $\varepsilon > 0$, we have from (1) for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) \leq M_{g_1}(r) \cdot M_{g_2}(r). \tag{16}$$

We thus have

$$M_{g_1 \cdot g_2}(r) \leq M_{f_1}\left[\left(\sigma_{f_1}(g_k) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_k)}\right] \cdot M_{f_1}\left[\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}\right].$$

Since $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$, we get for all sufficiently large values of r that $(\sigma_{f_1}(g_i) + \varepsilon) r^{\rho_{f_1}(g_i)} > (\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)}$. Therefore $M_{f_1}[(\sigma_{f_1}(g_i) + \varepsilon) r^{\rho_{f_1}(g_i)}] > M_{f_1}[(\sigma_{f_1}(g_k) + \varepsilon) r^{\rho_{f_1}(g_k)}]$ and from above arguments it follows for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}\right]^2. \tag{17}$$

Let us observe that

$$\delta_1 := \frac{\sigma_{f_1}(g_i) + \varepsilon}{\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}} > 1$$

$$\Rightarrow \quad \log\left(\sigma_{f_1}(g_i) + \varepsilon\right) r^{\rho_{f_1}(g_i)} > \log\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}$$

$$\Rightarrow \quad \frac{\log\left(\sigma_{f_1}(g_i) + \varepsilon\right) r^{\rho_{f_1}(g_i)}}{\log\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}} = \delta\,(\text{say}) > 1$$

$$\Rightarrow \quad \log\left(\sigma_{f_1}(g_i) + \varepsilon\right) r^{\rho_{f_1}(g_i)} = \delta \log\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}. \tag{18}$$

Since f_1 has the Property (A), in view of Lemma 2, Theorem D and (18) we obtain from (17) for all sufficiently large values of r that

$$\begin{aligned} M_{g_1 \cdot g_2}(r) &< M_{f_1}\left[\left(\left(\sigma_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\rho_{f_1}(g_i)}\right)^{\delta}\right] \\ i.e.,\ M_{g_1 \cdot g_2}(r) &< M_{f_1}\left[\left(\sigma_{f_1}(g_i) + \varepsilon\right) r^{\rho_{f_1}(g_i)}\right]. \end{aligned}$$

That is, we have

$$\begin{aligned} \frac{M_{f_1}^{-1} M_{g_1 \cdot g_2}(r)}{r^{\rho_{f_1}(g_i)}} &< \left(\sigma_{f_1}(g_i) + \varepsilon\right) \\ i.e.,\ \frac{M_{f_1}^{-1} M_{g_1 \cdot g_2}(r)}{r^{\rho_{f_1}(g_1 \cdot g_2)}} &< \left(\sigma_{f_1}(g_i) + \varepsilon\right) \\ i.e.,\ \sigma_{f_1}(g_1 \cdot g_2) &\le \sigma_{f_1}(g_i). \end{aligned} \tag{19}$$

In order to establish the equality of (19), let us restrict the functions f_1 and g_i with the property $2^{\rho_{f_1}(g_i)} \le 1$ $(i = 1, 2)$. Now let h, h_1, h_2 and k be any four entire functions such that $h = \frac{h_2}{h_1}$ and $\rho_k(h_1) < \rho_k(h_2)$. So $T_h(r) = T_{\frac{h_2}{h_1}}(r) \le T_{h_2}(r) + T_{h_1}(r) + O(1)$. Now, in view of Lemma 4, and as in the line of the procedure of the above proof, it follows that $\sigma_k(h) = \sigma_k\left(\frac{h_2}{h_1}\right) \le 2^{\rho_k(h_2)} \sigma_k(h_2)$.

Further without loss of any generality, let $g = g_1 \cdot g_2$ and $\rho_{f_1}(g_1) < \rho_{f_1}(g_2) = \rho_{f_1}(g)$. Then $\sigma_{f_1}(g) \le \sigma_{f_1}(g_2)$. Also let $g_2 = \frac{g}{g_1}$ and in this case we obtain from above arguments that $\sigma_{f_1}(g_2) \le \sigma_{f1}(g)$. Hence $\sigma_{f_1}(g) = \sigma_{f_1}(g_2) \Rightarrow \sigma_{f_1}(g_1 \cdot g_2) = \sigma_{f_1}(g_2)$. Thus, $\sigma_{f_1}(g_1 \cdot g_2) = \sigma_{f_1}(g_i) \mid i = 1, 2$ where $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k = i = 1, 2\}$ and $\rho_{f_1}(g_1) \ne \rho_{f_1}(g_2)$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g are all entire functions and also suppose that $\rho_{f_1}(g_2) < \rho_{f_1}(g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\sigma_{f_1}(g_1) = \sigma_{f_1}(g)$ as $\rho_{f_1}(g) > \rho_{f_1}(g_2)$ and $2^{\rho_{f_1}(g_1)} \le 1$.

Case II. In view of Lemma 3, f_1 is transcendental. Now let $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$ where k, $i = 1, 2$ with $g_k \ne g_i$. Therefore from (1) and (4) it follows for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r_n) \le M_{g_1}(r_n) \cdot M_{g_2}(r_n). \tag{20}$$

That is, we have

$$\begin{aligned} & M_{g_1 \cdot g_2}(r_n) \\ \le\ & M_{f_1}\left[\left(\sigma_{f_1}(g_k) + \frac{\varepsilon}{2}\right) r_n^{\rho_{f_1}(g_k)}\right] \cdot M_{f_1}\left[\left(\overline{\sigma}_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r_n^{\rho_{f_1}(g_i)}\right]. \end{aligned} \tag{21}$$

Since $\rho_{f_1}(g_k) < \rho_{f_1}(g_i)$, so for a sequence of values of r tending to infinity

$$M_{f_1}\left[\left(\overline{\sigma}_{f_1}(g_i)+\frac{\varepsilon}{2}\right) r_n^{\rho_{f_1}(g_i)}\right] > M_{f_1}\left[\left(\sigma_{f_1}(g_k)+\frac{\varepsilon}{2}\right) r_n^{\rho_{f_1}(g_k)}\right]$$

holds. Therefore, from (21), we have

$$M_{g_1\cdot g_2}(r_n) < M_{f_1}\left[\left(\overline{\sigma}_{f_1}(g_i)+\frac{\varepsilon}{2}\right) r_n^{\rho_{f_1}(g_i)}\right]^2. \tag{22}$$

Now using the similar technique for a sequence of values of r tending to infinity as explored in the proof of Case I, the second part of Theorem 3 I(ii) follows from (22).

Therefore the first part of theorem follows Case I and case II.

Case III. By Lemma 3, g_1 is transcendental. Suppose that $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ $(k,\ i = 1,\ 2)$ with $f_i \neq f_k$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Therefore in view of (2) and (3), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{f_1\cdot f_2}(r_n) \le M_{f_1}(r_n)\cdot M_{f_2}(r_n). \tag{23}$$

That is, we have

$$\begin{aligned} & M_{f_1\cdot f_2}(r_n) \\ \le\ & M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]\cdot M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\sigma}_{f_k}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]. \end{aligned}$$

Now $M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right] > M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\sigma}_{f_k}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]$ because for all sufficiently large values of n and $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$, $\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}} > \left(\frac{r_n}{\left(\overline{\sigma}_{f_k}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}$ hold. Therefore from above arguments, it follows for a sequence of values of r tending to infinity that

$$M_{f_1\cdot f_2}(r_n) < M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]^2. \tag{24}$$

Now we observe that

$$\delta_1 := \frac{\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}}{\sigma_{f_i}(g_1)-\varepsilon} > 1$$

$$\Rightarrow \log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}} > \log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}$$

$$\Rightarrow \frac{\log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}}{\log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}} = \delta\,(\text{say}) > 1$$

$$(25) \quad \Rightarrow \log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}} = \delta\log\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}.$$

Since g_1 has the Property (A), in view of Lemma 2, the first part of Theorem H and (25) we obtain from (24) for a sequence $\{r_n\}$ of values of r tending to infinity that

$$\begin{aligned} M_{f_1\cdot f_2}(r_n) &< M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\frac{\varepsilon}{2}\right)}\right)^{\frac{\delta}{\rho_{f_i}(g_1)}}\right] \\ &< M_{g_1}\left[\left(\frac{r_n}{\left(\sigma_{f_i}(g_1)-\varepsilon\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]. \end{aligned}$$

That is, we have

$$\begin{aligned} \left(\sigma_{f_i}(g_1)-\varepsilon\right) r_n^{\rho_{f_i}(g_1)} &< M_{f_1\cdot f_2}^{-1} M_{g_1}(r_n) \\ i.e.,\ \left(\sigma_{f_i}(g_1)-\varepsilon\right) &< \frac{M_{f_1\cdot f_2}^{-1} M_{g_1}(r_n)}{r_n^{\rho_{f_1\cdot f_2}(g_1)}}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from above arguments that

$$(26) \qquad \sigma_{f_1\cdot f_2}(g_1) \geq \sigma_{f_i}(g_1).$$

In order to establish the equality of (26), let us restrict the functions f_i and g_1 with the property $2^{\rho_{f_i}(g_1)} \geq 1$ $(i = 1, 2)$. Now let h, h_1, h_2 and k be any four entire functions such that $h = \frac{h_1}{h_2}$ and $\rho_{h_1}(k) < \rho_{h_2}(k)$. So $T_h(r) = T_{\frac{h_1}{h_2}}(r) \leq T_{h_1}(r) + T_{h_2}(r) + O(1)$. Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that $\frac{\sigma_{h_1}(k)}{2^{\rho_{h_1}(k)}} \leq \sigma_h(k) = \sigma_{\frac{h_1}{h_2}}(k)$.

Further without loss of any generality, let $f = f_1 \cdot f_2$ and $\rho_{f_1}(g_1) = \rho_f(g_1) < \rho_{f_2}(g_1)$. Then $\sigma_f(g_1) \geq \sigma_{f_1}(g_1)$. Also let $f_1 = \frac{f}{f_2}$ and in this case we obtain from above arguments that $\sigma_{f_1}(g_1) \geq \frac{\sigma_f(g_1)}{2^{\rho_f(g_1)}}$. Hence $\sigma_f(g_1) = \sigma_{f_1}(g_1)$ implies that $\sigma_{f_1 \cdot f_2}(g_1) = \sigma_{f_1}(g_1)$. Thus, $\sigma_{f_1 \cdot f_2}(g_1) = \sigma_{f_i}(g_1)$ $(i = 1, 2)$, where $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1)\}$ $(k = 1, 2)$, $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$ and $2^{\rho_{f_i}(g_1)} \geq 1$ $(i = 1, 2)$.

Next one may suppose that $f = \frac{f_2}{f_1}$ with f_1, f_2, f are all entire and $\rho_{f_2}(g_1) < \rho_{f_1}(g_1)$. We have $f_2 = f \cdot f_1$. Therefore $\sigma_{f_2}(g_1) = \sigma_f(g_1)$ as $\rho_{f_1}(g_1) > \rho_f(g_1)$ and $2^{\rho_{f_1}(g_1)} \geq 1$ $(i = 1, 2)$.

Case IV. By Lemma 3, g_1 is transcendental. Suppose $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ $(k, i = 1, 2)$ where $f_i \neq f_k$ $(i \neq k)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2.

Therefore in view of (2) we obtain for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) \leq M_{f_1}(r) \cdot M_{f_2}(r) . \tag{27}$$

That is, we have

$$\begin{aligned} & M_{f_1 \cdot f_2}(r) \\ \leq\ & M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right] \cdot M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_k}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]. \end{aligned}$$

Therefore $M_{g_1}\left[\left(\frac{r}{(\overline{\sigma}_{f_i}(g_1) - \frac{\varepsilon}{2})}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right] > M_{g_1}\left[\left(\frac{r}{(\overline{\sigma}_{f_k}(g_1) - \frac{\varepsilon}{2})}\right)^{\frac{1}{\rho_{f_k}(g_1)}}\right]$ as $\rho_{f_i}(g_1) < \rho_{f_k}(g_1)$ and from above arguments it follows all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) < M_{g_1}\left[\left(\frac{r}{\left(\overline{\sigma}_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\rho_{f_i}(g_1)}}\right]^2 . \tag{28}$$

Therefore, using the similar technique as in the proof of Case III, for all sufficiently large values of r, Theorem 3 II (ii) follows from (28).

Thus the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem J (ii) and the above cases.

Proof of Theorem 4. Case I. By Lemma 3, f_1 is transcendental. Suppose that $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ $(k, i = 1, 2)$ with $g_k \neq g_i$ $(k \neq i)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Now for any arbitrary $\varepsilon > 0$, from (9), (12) and (20), we obtain for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r_n) \le M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_k) + \frac{\varepsilon}{2}\right) r_n^{\lambda_{f_1}(g_k)}\right] \cdot M_{f_1}\left[\left(\tau_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r_n^{\lambda_{f_1}(g_i)}\right].$$

As $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, we get from above arguments for a sequence $\{r_n\}$ of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r_n) < M_{f_1}\left[\left(\tau_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r_n^{\lambda_{f_1}(g_i)}\right]^2. \tag{29}$$

Now using the similar technique as explored in the proof of Case II of Theorem 3, we have from (29) and the second part of Theorem H that

$$\tau_{f_1}(g_1 \cdot g_2) \le \tau_{f_1}(g_i). \tag{30}$$

In order to establish the equality of (30), let us restrict the functions f_1 and g_i with the property $2^{\lambda_{f_1}(g_i)} \le 1$ $(i = 1, 2)$. Now let h, h_1, h_2 and k be any four entire functions such that $h = \frac{h_2}{h_1}$ and $\lambda_k(h_1) < \lambda_k(h_2)$. So $T_h(r) = T_{\frac{h_2}{h_1}}(r) \le T_{h_2}(r) + T_{h_1}(r) + O(1)$. Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that $\tau_k(h) = \tau_k\left(\frac{h_2}{h_1}\right) \le \tau^{\lambda_k(h_2)}\sigma_k(h_2)$.

Further without loss of generality, let $g = g_1 \cdot g_2$ and $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2) = \lambda_{f_1}(g)$. Then $\tau_{f_1}(g) \le \tau_{f_1}(g_2)$. Also let $g_2 = \frac{g}{g_1}$ and in this case we obtain from above arguments that $\tau_{f_1}(g_2) \le 2^{\lambda_{f_1}(g)}\tau_{f1}(g) \le \sigma_{f1}(g)$. Hence $\tau_{f_1}(g) = \tau_{f_1}(g_2) \Rightarrow \tau_{f_1}(g_1 \cdot g_2) = \tau_{f_1}(g_2)$. Thus, $\tau_{f_1}(g_1 \cdot g_2) = \sigma_{f_1}(g_i) \mid i = 1, 2$ where $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k)\}$ $(k, i = 1, 2)$, $\lambda_{f_1}(g_1) \ne \lambda_{f_1}(g_2)$ and $2^{\lambda_{f_1}(g_i)} \le 1$ $(i = 1, 2)$.

Next we may suppose that $g = \frac{g_1}{g_2}$ with g_1, g_2, g all entire functions and also suppose that $\lambda_{f_1}(g_2) < \lambda_{f_1}(g_1)$. We have $g_1 = g \cdot g_2$. Therefore $\lambda_{f_1}(g_1) = \lambda_{f_1}(g)$ as $\lambda_{f_1}(g) > \lambda_{f_1}(g_2)$ and $2^{\lambda_{f_1}(g_1)} \le 1$.

Case II. In view of Lemma 3, f_1 is transcendental. Now let $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$ $(k, i = 1, 2)$ with $g_k \ne g_i$ $(k \ne i)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Therefore from (16) and (9) it follows for all sufficiently large values of r that

$$\begin{aligned} & M_{g_1 \cdot g_2}(r) \\ \le\ & M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_k) + \frac{\varepsilon}{2}\right) r^{\lambda_{f_1}(g_k)}\right] \cdot M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\lambda_{f_1}(g_i)}\right]. \end{aligned} \tag{31}$$

Since $\lambda_{f_1}(g_k) < \lambda_{f_1}(g_i)$, so for all sufficiently large values of r,

$$M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\lambda_{f_1}(g_i)}\right] > M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_k) + \frac{\varepsilon}{2}\right) r^{\lambda_{f_1}(g_k)}\right]$$

holds and therefore from (31) we get for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}\left[\left(\overline{\tau}_{f_1}(g_i) + \frac{\varepsilon}{2}\right) r^{\lambda_{f_1}(g_i)}\right]^2 . \tag{32}$$

Now using the similar technique of Case I of Theorem 3, Theorem 4 I (i) follows from (32).

Therefore combining Case I and Case II, the first part of the theorem follows.

Case III. By Lemma 3, g_1 is transcendental. Suppose that $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ $(k,\, i = 1,\, 2)$ with $f_i \neq f_k$ $(i \neq k)$.

Therefore, in view of (10), we obtain from (27) for all sufficiently large values of r that

$$\begin{aligned} & M_{f_1 \cdot f_2}(r) \\ \leq\ & M_{g_1}\left[\left(\frac{r}{\left(\tau_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right] \cdot M_{g_1}\left[\left(\frac{r}{\left(\tau_{f_k}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]. \end{aligned}$$

As $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$, we find from above arguments that, for all sufficiently large values of r,

$$M_{f_1 \cdot f_2}(r) < M_{g_1}\left[\left(\frac{r_n}{\left(\tau_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]^2 . \tag{33}$$

Further using the similar technique as explored in the proof of case II in Theorem 3, we have from (33) and Theorem F that

$$\tau_{f_1 \cdot f_2}(g_1) \geq \tau_{f_i}(g_1) . \tag{34}$$

In order to establish the equality of (34), let us restrict the functions f_i and g_1 with the property $2^{\lambda_{f_i}(g_1)} \geq 1$ $(i = 1,\, 2)$. Now let h, h_1, h_2 and k be any four entire functions such that $h = \frac{h_1}{h_2}$ and $\lambda_{h_1}(k) < \lambda_{h_2}(k)$. So $T_h(r) = T_{\frac{h_1}{h_2}}(r) \leq T_{h_1}(r) + T_{h_2}(r) + O(1)$. Now in view of Lemma 4 and as in the line of procedure of the above proof, it follows that $\frac{\tau_{h_1}(k)}{2^{\lambda_{h_1}(k)}} \leq \tau_h(k) = \tau_{\frac{h_1}{h_2}}(k)$.

Further without loss of generality, let $f = f_1 \cdot f_2$ and $\lambda_{f_1}(g_1) = \lambda_f(g_1) < \lambda_{f_2}(g_1)$. Then $\tau_f(g_1) \geq \tau_{f_1}(g_1)$. Also let $f_1 = \frac{f}{f_2}$ and in this case we obtain from above arguments that $\tau_{f_1}(g_1) \geq \frac{\tau_f(g_1)}{2^{\lambda_f(g_1)}}$. Hence $\tau_f(g_1) = \tau_{f_1}(g_1)$ implies that $\tau_{f_1 \cdot f_2}(g_1) = \tau_{f_1}(g_1)$. Thus, $\tau_{f_1 \cdot f_2}(g_1) =$

$\tau_{f_i}(g_1)$ $(i = 1, 2)$ where $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1)\}$ $(k = 1, 2)$, $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$ and $2^{\lambda_{f_i}(g_1)} \geq 1$ $(i = 1, 2)$.

Next one may suppose that $f = \frac{f_2}{f_1}$ with f_1, f_2, f are all entire and $\lambda_{f_2}(g_1) < \lambda_{f_1}(g_1)$. We have $f_2 = f \cdot f_1$. Therefore $\tau_{f_2}(g_1) = \tau_f(g_1)$ as $\lambda_{f_1}(g_1) > \lambda_f(g_1)$ and $2^{\lambda_{f_1}(g_1)} \geq 1$ $(i = 1, 2)$.

Case IV. By Lemma 3, g_1 is transcendental. Suppose $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ $(k, i = 1, 2)$ with $f_i \neq f_k$ $(i \neq k)$.

Therefore, in view of (23), (10) and (11), we obtain that, for a sequence $\{r_n\}$ of values of r tending to infinity,

$$M_{f_1 \cdot f_2}(r_n) \leq M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\tau}_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right] \cdot M_{g_1}\left[\left(\frac{r_n}{\left(\tau_{f_k}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right].$$

Therefore $M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\tau}_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right] > M_{g_1}\left[\left(\frac{r_n}{\left(\tau_{f_k}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_k}(g_1)}}\right]$ as $\lambda_{f_i}(g_1) < \lambda_{f_k}(g_1)$ and from above arguments it follows that, for a sequence $\{r_n\}$ of values of r tending to infinity,

$$M_{f_1 \cdot f_2}(r_n) < M_{g_1}\left[\left(\frac{r_n}{\left(\overline{\tau}_{f_i}(g_1) - \frac{\varepsilon}{2}\right)}\right)^{\frac{1}{\lambda_{f_i}(g_1)}}\right]^2. \tag{35}$$

Therefore using the similar technique, for all sufficiently large values of r , as in the proof of Case III , the second part of Theorem 4 II (ii) follows from (35).

Thus the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the theorem is omitted as it can be carried out in view of Theorem J (ii) and the above cases.

Proof of Theorem 5. Case I. Suppose that $\rho_{f_1}(g_1) = \rho_{f_1}(g_2)$ $(0 < \rho_{f_1}(g_1), \rho_{f_1}(g_2) < \infty)$. Now in view of Theorem C it is easy to see that $\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_1) = \rho_{f_1}(g_2)$. If possible let

$$\rho_{f_1}(g_1 \pm g_2) < \rho_{f_1}(g_1) = \rho_{f_1}(g_2). \tag{36}$$

Let $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$. Then in view of Theorem K and (36) we obtain that $\sigma_{f_1}(g_1) = \sigma_{f_1}(g_1 \pm g_2 \mp g_2) = \sigma_{f_1}(g_2)$ which is a contradiction. Hence $\rho_{f_1}(g_1 \pm g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2)$. Similarly with the help of the first part of Theorem 1, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_1}(g_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ $(0 < \rho_{f_1}(g_1), \rho_{f_2}(g_1) < \infty)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2. Therefore in view of the first part of Theorem G, it follows that $\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ and if possible let

$$\rho_{f_1 \pm f_2}(g_1) > \rho_{f_1}(g_1) = \rho_{f_2}(g_1). \tag{37}$$

Let us consider that $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$. Then. in view of the Theorem 1 II (i) and (37) we obtain that $\sigma_{f_1}(g_1) = \sigma_{f_1 \pm f_2 \mp f_2}(g_1) = \sigma_{f_2}(g_1)$ which is a contradiction. Hence $\rho_{f_1 \pm f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$. Also in view of Theorem 1 II (ii) one can derive the same conclusion for the condition $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1)$ and therefore the second part of the theorem is established.

Proof of Theorem 6. Case I. Let $\lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$ $(0 < \lambda_{f_1}(g_1), \lambda_{f_1}(g_2) < \infty)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Now, in view of Theorem G(ii), it is easy to see that $\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. If possible let

$$\lambda_{f_1}(g_1 \pm g_2) < \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2). \tag{38}$$

Let $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$. Then in view of Theorem 2 I (i) and (38) we obtain that $\tau_{f_1}(g_1) = \tau_{f_1}(g_1 \pm g_2 \mp g_2) = \tau_{f_1}(g_2)$ which is a contradiction. Hence $\lambda_{f_1}(g_1 \pm g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. Similarly with the help of Theorem 2 I (ii) , one can establish the same conclusion under the hypothesis $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$. This prove the first part of the theorem.

Case II. Let us consider that $\lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$ $(0 < \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) < \infty)$. Therefore in view of Theorem E it follows that

$$\lambda_{f_1 \pm f_2}(g_1) \geq \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$$

and if possible let

$$\lambda_{f_1 \pm f_2}(g_1) > \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1). \tag{39}$$

Suppose $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$. Then in view of Theorem 2 II (i) and (39) we obtain that $\tau_{f_1}(g_1) = \tau_{f_1 \pm f_2 \mp f_2}(g_1) = \tau_{f_2}(g_1)$ which is a contradiction. Hence $\lambda_{f_1 \pm f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$. Analogously with the help of Theorem 2 II (ii), the same conclusion can also be derived under the condition $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_2}(g_1)$ and therefore the second part of the theorem is established.

Proof of Theorem 7. Case I. Suppose that $\rho_{f_1}(g_1) = \rho_{f_1}(g_2)$ $(0 < \rho_{f_1}(g_1), \rho_{f_1}(g_2) < \infty)$. Now in view of Theorem D it is easy to see that $\rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_1) = \rho_{f_1}(g_2)$. If possible let

$$\rho_{f_1}(g_1 \cdot g_2) < \rho_{f_1}(g_1) = \rho_{f_1}(g_2). \tag{40}$$

Let $\sigma_{f_1}(g_1) \neq \sigma_{f_1}(g_2)$. Now in view of Theorem 3 I (i) and (40) we obtain that $\sigma_{f_1}(g_1) = \sigma_{f_1}\left(\frac{g_1 \cdot g_2}{g_2}\right) = \sigma_{f_1}(g_2)$ which is a contradiction. Hence $\rho_{f_1}(g_1 \cdot g_2) = \rho_{f_1}(g_1) = \rho_{f_1}(g_2)$. Similarly with the help of Theorem 3 I (ii), one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_1}(g_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ $(0 < \rho_{f_1}(g_1), \rho_{f_2}(g_1) < \infty)$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2. Therefore in view of the first part of Theorem H, it follows that $\rho_{f_1 \cdot f_2}(g_1) \geq \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$ and if possible let

$$\rho_{f_1 \cdot f_2}(g_1) > \rho_{f_1}(g_1) = \rho_{f_2}(g_1). \tag{41}$$

Further suppose that $\sigma_{f_1}(g_1) \neq \sigma_{f_2}(g_1)$. Therefore in view of the first part of Theorem 3 II(i) and (37), we obtain that $\sigma_{f_1}(g_1) = \sigma_{\frac{f_1 \cdot f_2}{f_2}}(g_1) = \sigma_{f_2}(g_1)$ which is a contradiction. Hence $\rho_{f_1 \cdot f_2}(g_1) = \rho_{f_1}(g_1) = \rho_{f_2}(g_1)$. Likewise with the help of Theorem 3 II (ii), one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{f_1}(g_1) \neq \overline{\sigma}_{f_2}(g_1)$. This proves the second part of the theorem.

We omit the proof for quotient as it is an easy consequence of the above two cases.

Proof of Theorem 8. Case I. Let $\lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$ $(0 < \lambda_{f_1}(g_1), \lambda_{f_1}(g_2) < \infty)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Now in view of Theorem H (ii) it is easy to see that $\lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. If possible let

$$\lambda_{f_1}(g_1 \cdot g_2) < \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2). \tag{42}$$

Also let $\tau_{f_1}(g_1) \neq \tau_{f_1}(g_2)$. Then in view of Theorem 4 I (i) and (42), we obtain that $\tau_{f_1}(g_1) = \tau_{f_1}\left(\frac{g_1 \cdot g_2}{g_2}\right) = \tau_{f_1}(g_2)$ which is a contradiction. Hence $\lambda_{f_1}(g_1 \cdot g_2) = \lambda_{f_1}(g_1) = \lambda_{f_1}(g_2)$. Analogously with the help of Theorem 4 I (ii), the same conclusion can also be derived under the condition $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_1}(g_2)$. Hence the first part of the theorem is established.

Case II. Let us consider that $\lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$ $(0 < \lambda_{f_1}(g_1), \lambda_{f_2}(g_1) < \infty)$. Therefore in view of Theorem F it follows that $\lambda_{f_1 \cdot f_2}(g_1) \geq \lambda_{f_1}(g_1) =$

$\lambda_{f_2}(g_1)$ and if possible let

$$\lambda_{f_1 \cdot f_2}(g_1) > \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1). \tag{43}$$

Further let $\tau_{f_1}(g_1) \neq \tau_{f_2}(g_1)$. Then in view of the second part of Theorem 4 II (i) and (43) we obtain that $\tau_{f_1}(g_1) = \tau_{\frac{f_1 \cdot f_2}{f_2}}(g_1) = \tau_{f_2}(g_1)$ which is a contradiction. Hence $\lambda_{f_1 \cdot f_2}(g_1) = \lambda_{f_1}(g_1) = \lambda_{f_2}(g_1)$. Similarly by Theorem 4 II (ii), we get the same conclusion when $\overline{\tau}_{f_1}(g_1) \neq \overline{\tau}_{f_2}(g_1)$ and therefore the second part of the theorem follows.

We omit the proof for quotient as it is an easy consequence of the above two cases.

5. Concluding Remarks

In this paper, we investigate certain properties of *relative type* (*relative lower type*) and *relative weak type* of entire functions. Here we actually prove Theorem 1 to Theorem 4 under some different conditions stated in Theorem A to Theorem J, respectively. Moreover, the treatment of these notions may also be extended for meromorphic functions, in the field of slowly changing functions and also in case of entire or meromorphic functions of several complex variables. Further some natural questions may arise about the sum and product properties for *relative type* (*relative lower type*) and *relative weak type* of entire functions when the conditions of Theorem 5 to Theorem 8 are, respectively, provided. Answers of these last questions are left to the interested readers or the involved authors for future study in this research subject.

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