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YANG-MILLS CONNECTIONS IN THE BUNDLE OF AFFINE ORTHONORMAL FRAMES

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Abstract. We get a necessary and sufficient condition for a generalized affine connection in the affine orthonormal frame bundle over a smooth manifold (M, g) to be a Yang-Mills connection.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold, $A(M)$ $(M, A(n;R) := GL(n;R) \times R^n)$ the bundle of affine frames, and $O(M, g)$ the bundle of orthonormal frames over (M, g). Let $AO(M, g)$ be a principal fiber bundle over the manifold (M, g) with group $AO(n;R) := O(n;R) \times R^n$, which is a subbundle of $A(M)$. In this paper, we call the bundle $AO(M, g)$ *the affine orthonormal frame bundle* over (M, g). Let $\widetilde{\gamma}$: $AO(M, g) \to O(M, g) = AO(M, g)/R^n$ be the natural projection, and $\widetilde{\omega}$ an arbitrarily given connection in $AO(M, g)$. Let ω (resp. φ) be the connection form (resp. 1-form) on $O(M, g)$ such that $\widetilde{\gamma}^\star \widetilde{\omega} = \omega + \varphi$. Then we get the following results:

(1) Assume the linear connection ω $(\widetilde{\gamma}^\star \widetilde{\omega} = \omega + \varphi)$ becomes a Yang-Mills connection in $O(M, g)$. Then we obtain a necessary and sufficient condition for the generalized affine connection form $\widetilde{\omega}$ in $AO(M, g)$ to become a Yang-Mills connection (cf. Theorem 4.1).

(2) Assume the 1-form φ $(\widetilde{\gamma}^\star \widetilde{\omega} = \omega + \varphi)$ on $O(M, g)$ is the canonical 1-form on $O(M, g)$, i.e., $\varphi(X) := u^{-1}(\pi_\star(X))$ for $X \in T_u(O(M, g))$ $(u \in (O(M, g))$. And, assume the linear connection ω becomes a Yang-Mills connection. Then we obtain a necessary and sufficient condition for the affine connection form $\widetilde{\omega}$ in $AO(M, g)$ to become a Yang-Mills connection (cf. Theorem 4.2).

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(3) As an application of (1) and (2), let $G = M$ be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G, and g the canonical Riemannian metric which is defined by the Killing form of the Lie algebra $\mathfrak{g}$ of G. Then, the linear connection form in the orthonormal frame bundle by the Levi-Civita connection for g becomes a Yang-Mills connection in $O(M, g)$, but the corresponding affine connection in $AO(M, g)$ does not become a Yang-Mills connection (cf. Theorem 4.4).

2. Preliminaries

In general, when we regard R^n as an affine space, we denote it by A^n. The group $A(n; R)$ of all affine transformations of A^n is represented by the group of all matrices of the form

$$\widetilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix},$$

where $a = (a^i_j)_{i,j} \in GL(n; R)$ and $\xi = (\xi^i)$ $(\xi \in R^n)$ is a column vector. The element $\widetilde{a}$ maps a point η of A^n into $a\eta + \xi$. We have the following exact sequence:

$$0 \to R^n \to A(n; R) \to GL(n; R) \to 1.$$

Let (M, g) be an n-dimensional Riemannian manifold, $A(M)$ $(M, A(n; R) =: GL(n; R) \times R^n)$ the bundle of affine frames over M, and $O(M, g)$ the bundle of orthonormal frames over (M, g). Let $AO(M, g)$ be the principal fibre bundle over (M, g) with group $AO(n; R) = O(n; R) \times R^n$ which is a subbundle of $A(M)$. In this paper, we call the bundle $AO(M, g)$ *the affine orthonormal frame bundle* over (M, g). Let $\widetilde{\gamma}$: $O(M, g) \to AO(M, g)$ be the natural injection together with the group homomorphism $\gamma : O(n; R) \hookrightarrow AO(n; R)$, and $\widetilde{\omega}$ an arbitrarily given *Ehresmann connection* (*form*) in $AO(M, g)$, i.e.

$$(2.1) \quad \begin{aligned} &\widetilde{\omega}(X^\star) = X \quad (X \in \mathfrak{ao}(n; R) = \mathfrak{o}(n; R) + R^n \text{ (semidirect sum)}), \\ &{R_g}^\star \widetilde{\omega} = Ad(g^{-1})\widetilde{\omega} \quad (g \in A(n; R)), \end{aligned}$$

where $\mathfrak{ao}(n; R)$ (resp. $\mathfrak{o}(n; R)$) is the Lie algebra of $AO(n; R)$ (resp. $O(n; R)$), $X^\star$ is the fundamental vector field corresponding to $X \in \mathfrak{ao}(n; R)$ which is defined on $AO(M, g)$, and ${R_g}^\star \widetilde{\omega}$ is the pull back of $\widetilde{\omega}$ by the action R_g on $AO(M, g)$. Let ω (resp. φ) be the Ehresmann connection form (resp. the tensorial 1-form) of type $(\mathrm{Ad}(AO(n; R)), R^n)$ on $O(M, g)$ such that $\widetilde{\gamma}^\star \widetilde{\omega} = \omega + \varphi$ (cf. [1]).

In this paper, we obtain the following results:

(1) Assume ω becomes a Yang-Mills connection in $O(M,g)$. Then we obtain a necessary and sufficient condition for the connection form $\widetilde{\omega}$ in $AO(M,g)$ to become a Yang-Mills connection (cf. Theorem 4.1).

(2) Assume the 1-form φ ($\widetilde{\gamma}^{\star}\widetilde{\omega} = \omega + \varphi$) on $O(M,g)$ is the canonical 1-form on $O(M,g)$, i.e., $\varphi(X) := u^{-1}(\pi_{\star}(X))$ for $X \in T_u(O(M,g))$ ($u \in (O(M,g))$. And, assume ω becomes a Yang-Mills connection. Then we obtain a necessary and sufficient condition for the connection form $\widetilde{\omega}$ in $AO(M,g)$ to become a Yang-Mills connection (cf. Theorem 4.2).

(3) Let $G = M$ be a compact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G, and g_o the canonical Riemannian metric which is defined by the Killing form of the Lie algebra $\mathfrak{g}$ of G. Then, the linear connection form A in the orthonormal frame bundle by the Levi-Civita connection of g_o becomes a Yang-Mills connection in $O(M,g)$, but the corresponding affine connection $\widetilde{A}$ ($\widetilde{A} = A + \theta$) in $AO(M,g)$ does not become a Yang-Mills connection, where $\theta(X) := u^{-1}(\pi_{\star}(X))$ for $X \in T_u(O(M,g))$ ($u \in (O(M,g))$ (cf. Theorem 4.4).

Traditionally, the words 'linear connection' and 'affine connection' have been used interchangeably ([1, Theorem 3.3, p.129]). But, strictly speaking, a linear connection is a connection in $L(M)$ ($\supset O(M,g)$), and an affine connection is a connection in $A(M)$ ($\supset AO(M,g)$).

By virtue of Theorem 4.4, we show the fact that there exists a Yang-Mills linear connection in $O(M,g)$ of which the corresponding affine connection in $AO(M,g)$ does not become a Yang-Mills connection.

3. Yang-Mills connections in principal fibre bundles

Let $P(M,G)$ be a principal fiber bundle with semisimple Lie group G over an n-dimensional closed (compact and connected) Riemannian manifold (M,g), and $\mathfrak{g}$ the Lie algebra of the structure group G, and $\{U,V,W,\ldots\}$ an open covering of M generated by local triviality of P. Let the mappings $\phi_{UV} : U \cap V \to G$ corresponding to the open covering $\{U,V,W,\ldots\}$ of M be transition functions. If the family $A = \{A_U, A_V, A_W, \ldots\}$ of $\mathfrak{g}$-valued 1-forms which are defined on open subsets $U,V,W,\ldots$ of M satisfies the following cocycle condition

$$(3.1)\quad (A_V)_x = L_{\phi_{VU}(x)_\star}(d(\phi_{UV}))_x + Ad(\phi_{VU}(x))(A_U)_x \quad (x \in U \cap V),$$

then A is said to be a *connection* (*form*) ([2, Definition 3.1.1, p.74]) in $P(M,G)$. Let $\{\sigma_U, \sigma_V, \sigma_W, \ldots\}$ be the family of local cross sections of the open neighborhoods $U,V,W,\ldots$ into P. Let $\mathfrak{A}_P$ be the space of all connections in P, and $\mathfrak{C}_P$ the space of all Ehresmann connections in P

which are defined as in (2.1). Then, $\mathfrak{A}_P$ and $\mathfrak{C}_P$ are 1-1 correspondent as follows ([2, Theorem 3.1.4, p.76]):

if we put $\sigma_U{}^\star\omega =: A_U$ for a given $\omega \in \mathfrak{C}_P$, then the family $A := \{A_U, A_V, A_W, \ldots\}$ of $\mathfrak{g}-$valued 1-forms defined on the open neighborhoods $U, V, W, \ldots$ satisfies the cocycle condition (3.1). On the other hand, we put $\omega^U(Z) := A_U(Y) + X$ for $Z = (\sigma_{U\star})_x(Y) + (X^\star)_{\sigma_U(x)}$ $(Z \in T_{\sigma_U(x)}(P),\ Y \in T_x(M),\ X \in \mathfrak{g})$, and if $Z \in T_v(P)$ $(g \in G,\ \sigma_U(x)g = v \in P)$, then we put $\omega^U(Z) := \mathrm{Ad}(g^{-1})\ \omega^U(R_{g^{-1}}{}_\star Z)$. Here $\omega^U, \omega^V, \omega^W, \ldots$ coincide on the overlapping neighborhoods of P, and the family $\omega := \{\omega^U, \omega^V, \omega^W, \ldots\}$ satisfy the conditions as in (2.1).

The curvature form $F(A)$ ([1, 2]) of a connection A $(\in \mathfrak{A}_P)$ in the principal fibre bundle $P(M, g)$ is given by

$$F(A) = dA + A \wedge A. \tag{3.2}$$

We fix an $\mathrm{Ad}(G)$-invariant inner product $<\ ,\ >$ on $\mathfrak{g}$. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F(A)\|^2\ v_g \quad (A \in \mathfrak{A}_P)$$

which is defined on the space $\mathfrak{A}_P$, where v_g is the volume element of (M, g) and $\|F(A)\|^2 =< F(A), F(A) >$. Let $\{X_i\}_{i=1}^n$ be a (locally defined) orthonormal frame on (M, g). Then a necessary and sufficient condition ([2, Theorem 4.1.3, p.107]) for a connection A in the principal fibre bundle P to become a Yang-Mills connection is the fact that

$$\begin{aligned}(\delta_A F(A))(X_i) = -\sum_j \{&(\nabla_{X_j} F(A))(X_j, X_i) \\ &+ [A(X_j), F(A)(X_j, X_i)]\}\end{aligned} \tag{3.3}$$

vanishes, where δ_A is the formal adjoint operator of the covariant exterior differentiation d_A, and ∇ is the Levi-Civita connection on (M, g).

4. Yang-Mills connections in the affine orthonormal frame bundle $AO(M, g)$ over (M, g)

4.1. A generalized affine (Ehresmann) connection and an affine (Ehresmann) connection

Let (M, g) be an n-dimensional compact connected smooth manifold. A (Ehresmann) connection in the bundle $L(M)$ of all linear frames on M is called a *linear connection* of M. A *generalized affine* (*Ehresmann*) *connection* is a (Ehresmann) connection in the affine frame bundle $A(M)$

over M. Moreover, a generalized affine (Ehresmann) connection $\widetilde{\omega}$ in $A(M)$ is called an *affine (Ehresmann) connection* in $A(M)$, if the R^n-valued 1-form φ ($\widetilde{\gamma}^\star(\widetilde{\omega}) = \omega + \varphi$) on $L(M)$ is the canonical 1-form θ ($\theta(X) = u^{-1}\pi_\star(X)$ ($X \in T_u(L(M))$)), where $\gamma : GL(n;R) \hookrightarrow A(n;R)$ and $\widetilde{\gamma} : L(M) \to A(M)$.

4.2. Generalized affine Yang-Mills connections in the affine orthonormal frame bundle $AO(M,g)$ over (M,g)

We get the following exact sequence:

$$0 \hookrightarrow R^n \overset{\alpha}{\hookrightarrow} AO(n;R) \overset{\beta}{\longrightarrow} O(n;R) = AO(n;R)/R^n \longrightarrow 1.$$

By the virtue of the principal fibre bundle homomorphism

$$\widetilde{\beta} : AO(M,g) \to O(M,g) = AO(M,g)/R^n$$

associated with the group homomorphism

$$\beta : AO(n;R) \to O(n;R) = AO(n;R)/R^n,$$

we obtain the fact that the set of all affine (Ehresmann) connections in $AO(M,g)$ ($\subset A(M)$) and the set of all linear connections in $O(M,g)$ ($\subset L(M)$) are 1-1 correspondent (cf. [1, Theorem 3.3, p.129]).

Let $\widetilde{\omega}$ be a generalized affine (Ehresmann) connection in $AO(M,g)$, and $(X_1, X_2, \ldots, X_n)$ an (locally defined) orthonormal frame on (M,g). Let σ_U be the cross section of $O(M,g)$ ($\subset AO(M,g)$) over U ($\subset M$) which assigns to each $x \in U$ the linear frame $((X_1)_x, (X_2)_x, \ldots, (X_n)_x)$.

Let $\widetilde{\gamma} : O(M,g) \to AO(M,g)$ be the bundle homomorphism associated with the group homomorphism $\gamma : O(n;R) \hookrightarrow AO(n;R)$. Since $\widetilde{\gamma}^\star\widetilde{\omega} = \omega + \varphi$ on $O(M,g)$,

$$\sigma_U{}^\star(\widetilde{\gamma}^\star\widetilde{\omega}) = \sigma_U{}^\star\omega + \sigma_U{}^\star\varphi \equiv \widetilde{\omega}$$

on U ($\subset M$). Here and from now on, we put

$$\sigma_U{}^\star\widetilde{\omega} =: \widetilde{A}_U,\ \sigma_U{}^\star\omega =: A_U,\ \sigma_U{}^\star\varphi =: \varphi_U =: \varphi \tag{4.1}$$

on U ($\subset M$). Then in (4.1), the $\mathfrak{ao}(n;R)$-valued 1-form $\widetilde{A}$ and the $\mathfrak{o}(n;R)$-valued 1-form A on M satisfy the cocycle condition (3.1). So, $\widetilde{A}$ (resp. A) is a connection (form) in $AO(M,g)$ (resp. $O(M,g)$). Similarly, we call $\widetilde{A}$ (resp. A) a *generalized affine connection* (*form*) in $AO(M,g)$ (resp. the *linear connection* (*form*) in $O(M,g)$ which is related to the connection $\widetilde{A}$). From the above facts, we get

$$\widetilde{A}_U = A_U + \varphi_U \quad (\widetilde{A} = A + \varphi,\ \text{simply}). \tag{4.2}$$

From (3.2) and (4.2), we obtain ([1, Proposition 3.4, p.130])

$$F(\widetilde{A}) = F(A) + d\varphi + A \wedge \varphi. \tag{4.3}$$

Let $(\ ,\ )$ be the inner product on $\mathfrak{o}(n;R)$ defined by $(X,Y) := -trace(XY)$ $(X,Y \in \mathfrak{o}(n;R)$, and let $\{e_i\}_i$ be the natural basis of R^n.

Now, we fix an inner product $<\ ,\ >$ on $\mathfrak{ao}(n;R)$ such that

$$\{Z_i\}_{i=1}^{(n^2-n)/2} \cup \{e_j\}_{j=1}^{n}$$

is an orthonormal basis of $\mathfrak{ao}(n;R)$ with respect to $<\ ,\ >$, where $\{Z_i\}_{i=1}^{(n^2-n)/2}$ is an orthonormal basis on $(\mathfrak{o}(n;R), (\ ,\ ))$. Then, the inner product $<\ ,\ >$ on $\mathfrak{ao}(n;R)$ is $\mathrm{Ad}(O(n;R))$-invariant. So, $<\ ,\ >$ is $\mathrm{Ad}(\phi(x))$-invariant, where $x \in M$ and ϕ are transition functions appeared by local triviality of the principal fibre bundle $O(M,g)$.

For convenience' sake, for a (locally defined) orthonormal frame $\{X_i\}_{i=1}^{n}$ on (M,g), we put

$$\begin{aligned}
&(\delta_{\widetilde{A}}F(\widetilde{A}))(X_i) =: (\delta_{\widetilde{A}}\widetilde{F})_i, \qquad (\delta_A F(A))(X_i) =: (\delta_A F)_i,\\
&(\nabla_{X_k} d\varphi)(X_i,X_j) =: \nabla_k d\varphi_{ij},\\
&(\nabla_{X_k} A \wedge \varphi)(X_i,X_j) =: \nabla_k (A\wedge\varphi)_{ij},\\
&(F(A))(X_k,X_i) =: F_{ki}, \qquad \varphi(X_j) =: \varphi_j, \quad A(X_k) =: A_k,\\
&(A\wedge\varphi)(X_i,X_j) =: (A\wedge\varphi)_{ij},\ (d\varphi)(X_i,Xj) =: d\varphi_{ij}.
\end{aligned} \tag{4.4}$$

From (3.2), (3.3), (4.3), (4.4), and the properties of the inner product $<\ ,\ >$ on $\mathfrak{ao}(n;R)$, we obtain

$$\begin{aligned}
(\delta_{\widetilde{A}}\widetilde{F})_i = (\delta_A F)_i - \sum_k \{&\nabla_k d\varphi_{ki} + \nabla_k (A\wedge\varphi)_{ki}\\
&- F_{ki}\varphi_k + [A_k, d\varphi_{ki}] + [A_k, (A\wedge\varphi)_{ki}]\}.
\end{aligned} \tag{4.5}$$

By the help of (4.5), we get

Theorem 4.1. *Let $\widetilde{A}$ be a generalized affine connection $(form)$ in $AO(M,g)$, and A a linear connection in $O(M,g)$ such that $\widetilde{A} = A + \varphi$. Assume the linear connection A in $O(M,g)$ is a Yang-Mills connection. Then, a necessary and sufficient condition for the connection $(form)$ $\widetilde{A}$ to become a Yang-Mills connection is*

$$\begin{aligned}
\sum_k \{&\nabla_k d\varphi_{ki} + \nabla_k (A\wedge\varphi)_{ki} - F_{ki}\varphi_k\\
&+ [A_k, d\varphi_{ki}] + [A_k, (A\wedge\varphi)_{ki}]\} = 0.
\end{aligned}$$

4.3. Affine Yang-Mills connections in the affine orthonormal frame bundle $AO(M,g)$

Assume $\widetilde{A}$ is an affine connection in $AO(M,g)$. Then we get

$$\widetilde{A}_U = A_U + \varphi_U = A_U + \sigma_U{}^{\star}\theta =: A_U + \theta_U \quad (\widetilde{A} = A + \theta,\ \text{briefly}),$$

where θ is the canonical 1-form on $O(M,g)$, i.e.

$$\theta(Z) := u^{-1}\pi_{\star}(Z) \quad (Z \in T_u(O(M,g)).$$

Since $\theta_i := \theta(X_i) := (\sigma_U{}^{\star})\theta(X_i) = e_i$, we have

$$X_k(\varphi_i) = X_k(\theta_i) = 0. \tag{4.6}$$

Let $\{Y^i\}_i$ be the (locally defined) dual frame of the (locally defined) orthonormal frame $\{X_i\}_i$ on (M,g). Then we put

$$Y^t(\nabla_{X_i}X_k) =: \Gamma_{ik}{}^t, \quad [X_i, X_j] =: \sum_k C_{ij}{}^k X_k. \tag{4.7}$$

Since ∇ is the Levi-Civita connection on (M,g), we have from (4.7)

$$\Gamma_{ij}{}^k - \Gamma_{ji}{}^k = C_{ij}{}^k, \quad \Gamma_{ij}{}^k = -\Gamma_{ji}{}^k. \tag{4.8}$$

Then, by the help of (3.2), (4.4), (4.6) and (4.7), we get

$$\begin{aligned}
\nabla_k d\theta_{ki} &= -\frac{1}{2}\{X_k(\sum_l C_{ki}{}^l)e_l - \sum_{l,t}(\Gamma_{kk}{}^t C_{ti}{}^l + \Gamma_{ki}{}^t C_{kt}{}^l)e_l\},\\
\nabla_k(A\wedge\theta)_{ki} &= \frac{1}{2}[X_k(A_k e_i - A_i e_k)\\
&\qquad + \sum_l\{\Gamma_{ki}{}^l(A_l e_k - A_k e_l) + \Gamma_{kk}{}^l(A_i e_l - A_l e_i)\}],\\
F_{ki}\theta_k &= \frac{1}{2}\{X_k(A_i) - X_i(A_k) + A_kA_i - A_iA_k - \sum_l C_{ki}{}^l A_l\}e_k,\\
[A_k, d\theta_{ki}] &= -\frac{1}{2}\sum_l A_k C_{ki}{}^l e_l,\\
[A_k, (A\wedge\theta)_{ki}] &= \frac{1}{2}A_k(A_k e_i - A_i e_k).
\end{aligned} \tag{4.9}$$

From (4.5) and (4.9), we get

$$
\begin{aligned}
(\delta_{\widetilde{A}}\widetilde{F})_i = (\delta_A F)_i - \frac{1}{2}\sum_k \{ & X_k(A_k e_i - A_i e_k - \sum_l C_{ki}{}^l e_l) \\
& + A_k(A_k e_i - 2A_i e_k) + \sum_{l,t} \Gamma_{ki}{}^t C_{kt}{}^l e_l \\
& + \sum_t \Gamma_{kk}{}^t (A_i e_t - A_t e_i + \sum_l C_{ti}{}^l e_l) \\
& + (X_i(A_k) - X_k(A_i) + A_i A_k) e_k \\
& + 2\sum_t (\Gamma_{ti}{}^k + \Gamma_{ik}{}^t + \Gamma_{kt}{}^i) A_k e_t \}.
\end{aligned} \tag{4.10}
$$

By virtue of (4.10), we obtain

Theorem 4.2. *Let $\widetilde{A}$ be an affine connection (form) in $AO(M,g)$, and A a linear connection in $O(M,g)$ such that $\widetilde{A} = A + \theta$. Assume the linear connection A in $O(M,g)$ is a Yang-Mills connection. Then, a necessary and sufficient condition for the connection (form) $\widetilde{A}$ to become a Yang-Mills connection is*

$$
\begin{aligned}
\sum_k \{ & X_k(A_k e_i - A_i e_k - \sum_l C_{ki}{}^l e_l) + A_k(A_k e_i - 2A_i e_k) \\
& + \sum_{l,t} \Gamma_{ki}{}^t C_{kt}{}^l e_l + \sum_t \Gamma_{kk}{}^t (A_i e_t - A_t e_i + \sum_l C_{ti}{}^l e_l) \\
& + (X_i(A_k) - X_k(A_i) + A_i A_k) e_k \\
& + 2\sum_t (\Gamma_{ti}{}^k + \Gamma_{ik}{}^t + \Gamma_{kt}{}^i) A_k e_t \} = 0.
\end{aligned}
$$

4.4. The connection form in $O(M,g)$ defined by the Levi-Civita connection for the metric g of (M,g)

Let $M = G$ be an $n-$dimensional closed (connected and compact) semisimple Lie group. Let g be the canonical bi-invariant Riemannian metric induced by the Killing form of the Lie algebra $\mathfrak{g}$ of the group M. Let $\{X_i\}_i$ be an orthonormal basis on $(\mathfrak{g}, (\ ,\ ) := g_e)$, where e is the identity element of the group M. Let ∇ be the Levi-Civita connection on $(M = G, g)$, and $\{Y^k\}_k$ be the dual frame of the orthonormal frame $\{X_i\}_i$ on (M,g). Now, we put $Y^k(\nabla_{X_j} X_i) =: \Gamma_{ji}{}^k$, $\sum_k \Gamma_{kj}{}^i Y^k =: a^i_j$, and $(a^i_j)_{i,j} =: A$. Then, A is a linear connection (form) in $O(M,g)$. Let $\widetilde{A}$ be an affine connection in $AO(M,g)$ such that $\widetilde{A} = A + \theta$ on $O(M,g)$, where θ is the canonical form on $O(M,g)$.

From these facts, we get

$$A_k = A(X_k) = ({\Gamma_{kj}}^i)_{i,j} \tag{4.11}$$

which belongs to $\mathfrak{o}(n;R)$. We fix an inner product $(\ ,\ )$ on $\mathfrak{o}(n;R)$ such that $(X,Y) := g_e(X,Y)$ $(X,Y \in \mathfrak{o}(n;R))$. Then the inner product $(\ ,\ )$ is $\mathrm{Ad}(O(n;R))$-invariant. So, we obtain from (4.8) and (4.11)

$${\Gamma_{kj}}^i = -{\Gamma_{jk}}^i = -{\Gamma_{ki}}^j,\ 2{\Gamma_{kj}}^i = {C_{kj}}^i,\ {\Gamma_{kj}}^i = {\Gamma_{ji}}^k = {\Gamma_{ik}}^j. \tag{4.12}$$

Since ${C_{ij}}^k$ are constants, we have

$$X_k({C_{ij}}^k) = 2X_k({\Gamma_{ij}}^k) = 0. \tag{4.13}$$

From (3.2), (4.11) and (4.12), we get

$$\begin{aligned}(F(A))(X_k, X_i) &=: F_{ki} \\ &= \frac{1}{2}\Big(\sum_l ({\Gamma_{kl}}^s{\Gamma_{ij}}^l - {\Gamma_{il}}^s{\Gamma_{kj}}^l - 2{\Gamma_{ki}}^l{\Gamma_{lj}}^s)\Big)_{s,j}\end{aligned} \tag{4.14}$$

which belongs to $\mathfrak{o}(n;R)$. From (4.4), (4.11), (4.12) and (4.14), we get

$$\sum_k [A_k, F_{ki}]^s_j = \sum_{k,l,t}\Big(3{\Gamma_{kj}}^t{\Gamma_{ki}}^l{\Gamma_{lt}}^s + \frac{1}{2}{\Gamma_{kt}}^s{\Gamma_{kl}}^t{\Gamma_{ij}}^l + \frac{1}{2}{\Gamma_{kj}}^t{\Gamma_{il}}^s{\Gamma_{kt}}^l\Big) \tag{4.15}$$

which is the (s,j)th component of $\sum_k [A_k, F_{ki}]$ $(\in \mathfrak{o}(n;R))$. From (4.12) and (4.14), we obtain

$$\Big(\sum_k \nabla_k F_{ki}\Big)^s_j = \sum_{k,l,t}({\Gamma_{ki}}^t{\Gamma_{tl}}^s{\Gamma_{kj}}^l + {\Gamma_{ki}}^t{\Gamma_{kt}}^l{\Gamma_{lj}}^s) \tag{4.16}$$

which is the (s,j)th component of $\sum_k \nabla_k F_{ki}$ $(\in \mathfrak{o}(n;R))$. We get from (3.3), (4.4), (4.12), (4.15) and (4.16),

$$\begin{aligned}(\delta_A F)_i = -\sum_{k,l,t}\Big(&4{\Gamma_{kj}}^t{\Gamma_{ki}}^l{\Gamma_{lt}}^s + {\Gamma_{ki}}^t{\Gamma_{kt}}^l{\Gamma_{lj}}^s \\ &+ \frac{1}{2}{\Gamma_{ks}}^t{\Gamma_{kt}}^l{\Gamma_{li}}^j + \frac{1}{2}{\Gamma_{kj}}^t{\Gamma_{kt}}^l{\Gamma_{ls}}^i\Big).\end{aligned} \tag{4.17}$$

Since $[[X_j, X_s], X_k] + [[X_s, X_k], X_j] + [[X_k, X_j], X_s] = 0$, we have

$$\sum_l {C_{js}}^l{C_{lk}}^t = \sum_l ({C_{ks}}^l{C_{lj}}^t + {C_{jk}}^l{C_{ls}}^t). \tag{4.18}$$

From (4.12) and (4.18), we get

$$\sum_{k,l,t}{\Gamma_{ki}}^t{\Gamma_{kt}}^l{\Gamma_{lj}}^s = 2\sum_{k,l,t}{\Gamma_{jk}}^t{\Gamma_{ki}}^l{\Gamma_{lk}}^s. \tag{4.19}$$

Similarly, we have from (4.12) and (4.19)

$$
\begin{aligned}
&\frac{1}{2}\sum_{k,l,t}\Gamma_{ks}{}^{t}\Gamma_{kt}{}^{l}\Gamma_{li}{}^{j} = \sum_{k,l,t}\Gamma_{jk}{}^{t}\Gamma_{ki}{}^{l}\Gamma_{lt}{}^{s},\\
&\frac{1}{2}\sum_{k,l,t}\Gamma_{kj}{}^{t}\Gamma_{kt}{}^{l}\Gamma_{ls}{}^{i} = \sum_{k,l,t}\Gamma_{jk}{}^{t}\Gamma_{ki}{}^{l}\Gamma_{lt}{}^{s}.
\end{aligned} \tag{4.20}
$$

By virtue of (4.12), (4.17), (4.19) and (4.20), we get

$$
(\delta_A F)_i = 0. \tag{4.21}
$$

Thus we obtain the following

Lemma 4.3. *Let M be a closed semisimple Lie group, and g the canonical bi-invariant Riemannian metric which is induced from the Killing form of the Lie algebra $\mathfrak{g}$ of the Lie group M. Let A be the connection form in $O(M,g)$ which is defined by the Levi-Civita connection on (M,g). Then, A becomes a Yang-Mills connection.*

Moreover from (4.10), (4.11), (4.13) and (4.21), we obtain

$$
\begin{aligned}
(\delta_{\widetilde{A}}\widetilde{F})_i = -\frac{1}{2}\sum_{k}\{&A_k(A_k e_i - 2A_i e_k) + \sum_{l,t}\Gamma_{ki}{}^{t}C_{kt}{}^{l}e_l\\
&+ \sum_{t}\Gamma_{kk}{}^{t}(A_i e_t - A_t e_i + \sum_{l}C_{ti}{}^{l}e_l)\\
&+ A_i A_k e_k + 2\sum_{t}(\Gamma_{ti}{}^{k} + \Gamma_{ik}{}^{t} + \Gamma_{kt}{}^{i})A_k e_t\}.
\end{aligned} \tag{4.22}
$$

We get from (4.11), (4.12) and (4.22), we get

$$
\begin{aligned}
(\delta_{\widetilde{A}}\widetilde{F})_i = -\frac{1}{2}\sum_{k}\{&\sum_{t}A_k(\Gamma_{ki}{}^{t}e_t - 2\Gamma_{ik}{}^{t}e_t)\\
&+ 2\sum_{l,t}(\Gamma_{ki}{}^{t} + 3\Gamma_{ti}{}^{k})\Gamma_{kt}{}^{l}e_l\}.
\end{aligned} \tag{4.23}
$$

By the help of (4.11), (4.12) and (4.23), we have

$$
(\delta_{\widetilde{A}}\widetilde{F})_i = -\frac{1}{2}\sum_{k,l,t}\Gamma_{kt}{}^{i}\Gamma_{kt}{}^{l}e_l = -\frac{1}{8}\sum_{k,l,t}C_{kt}{}^{i}C_{kt}{}^{l}e_l. \tag{4.24}
$$

So, from (4.24) we get the following:

if the affine connection $\widetilde{A}$ ($\widetilde{A} = A + \theta$) in $AO(M,g)$ becomes a Yang-Mills connection, then the Lie algebra $\mathfrak{g}$ of M ($= G$) is abelian.
This contradicts the fact that $\mathfrak{g}$ is semisimple.

Combining this result with Lemma 4.3, we obtain

Theorem 4.4. *Let M be a closed semisimple Lie group, and g the canonical bi-invariant Riemannian metric which is induced from the Killing form of the Lie algebra $\mathfrak{g}$ of the group M. Let A be the connection form in $O(M, g)$ which is defined by the Levi-Civita connection on (M, g), and $\widetilde{A}$ the affine connection in $AO(M, g)$ such that $\widetilde{A} = A + \theta$. Then, A becomes a Yang-Mills connection in $O(M, g)$, but $\widetilde{A}$ does not become a Yang-Mills connection in $AO(M, g)$.*

References

[1] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I,* John Wiley and Sons, New York, 1963.

[2] I. Mogi and M. Itoh, *Differential Geometry and Gauge Theory (in Japanese),* Kyoritsu Publ., 1986.

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