

## AREA OF TRIANGLES ASSOCIATED WITH A STRICTLY LOCALLY CONVEX CURVE

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**Abstract.** It is well known that the area  $U$  of the triangle formed by three tangents to a parabola  $X$  is half of the area  $T$  of the triangle formed by joining their points of contact. Recently, it was proved that this property is a characteristic one of parabolas. That is, among strictly locally convex  $C^{(3)}$  curves in the plane  $\mathbb{R}^2$  parabolas are the only ones satisfying the above area property. In this article, we study strictly locally convex curves in the plane  $\mathbb{R}^2$ . As a result, generalizing the above mentioned characterization theorem for parabolas we present some conditions which are necessary and sufficient for a strictly locally convex  $C^{(3)}$  curve in the plane to be an open part of a parabola.

### 1. Introduction

A regular plane curve  $X : I \rightarrow \mathbb{R}^2$  defined on an open interval is called *convex* if, for all  $t \in I$ , the trace  $X(I)$  lies entirely on one side of the closed half-plane determined by the tangent line at  $X(t)$  ([2]). A regular plane curve  $X : I \rightarrow \mathbb{R}^2$  is called *locally convex* if, for each  $t \in I$  there exists an open subinterval  $J \subset I$  containing  $t$  such that the curve  $X|_J$  restricted to  $J$  is a convex curve.

Hereafter, we will say that a locally convex curve  $X$  in the plane  $\mathbb{R}^2$  is *strictly locally convex* if the curve is smooth (that is, of class  $C^{(3)}$ ) and is of positive curvature  $\kappa$  with respect to the unit normal  $N$  pointing to the convex side. Hence, in this case we have  $\kappa(s) = \langle X''(s), N(X(s)) \rangle > 0$ , where  $X(s)$  is an arc-length parametrization of  $X$ . When  $f : I \rightarrow \mathbb{R}$  is a smooth function defined on an open interval, we will also say that  $f$  is *strictly convex* if the graph of  $f$  has positive curvature  $\kappa$  with respect to

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the upward unit normal  $N$ . This condition is equivalent to the positivity of  $f''(x)$  on  $I$ .

We consider a strictly locally convex  $C^{(3)}$  curve  $X$  in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. For a fixed point  $P = A \in X$  and a sufficiently small number  $h > 0$ , we consider the line  $m$  passing through  $P + hN(P)$  which is parallel to the tangent  $\ell$  to  $X$  at  $P$  and the points  $A_1$  and  $A_2$  where the line  $m$  intersects the curve  $X$ .

Let us denote by  $\ell_1$  and  $\ell_2$  the tangent lines to  $X$  at the points  $A_1$  and  $A_2$ , respectively and by  $B, B_1$  and  $B_2$  the points of intersection  $\ell_1 \cap \ell_2$ ,  $\ell \cap \ell_1$  and  $\ell \cap \ell_2$ , respectively. We let  $L_P(h)$ ,  $\ell_P(h)$  and  $H_P(h)$  denote the lengths  $|A_1A_2|$  and  $|B_1B_2|$  of the corresponding segments and the height of the triangle  $\triangle BA_1A_2$  from the vertex  $B$  to the edge  $A_1A_2$ , respectively. We also consider  $T_P(h)$ ,  $U_P(h)$ ,  $V_P(h)$  and  $W_P(h)$  defined by the areas  $|\triangle AA_1A_2|$ ,  $|\triangle BB_1B_2|$ ,  $|\triangle BA_1A_2|$  of corresponding triangles and the area  $|\square A_1A_2B_2B_1|$  of trapezoid  $\square A_1A_2B_2B_1$ , respectively. Then, obviously we have

$$T_P(h) = \frac{1}{2}hL_P(h),$$

$$U_P(h) = \frac{1}{2}\ell_P(h)\{H_P(h) - h\},$$

$$V_P(h) = \frac{1}{2}H_P(h)L_P(h)$$

and

$$W_P(h) = \frac{1}{2}h\{L_P(h) + \ell_P(h)\}.$$

If we denote by  $S_P(h)$  the area of the region bounded by the curve  $X$  and chord  $A_1A_2$ , then we have ([8])

$$S'_P(h) = L_P(h).$$

It is well known that parabolas satisfy the following properties.

**Proposition 1.1.** Suppose that  $X$  denotes an open part of a parabola. For an arbitrary point  $P \in X$  and a sufficiently small number  $h > 0$ , it satisfies

$$S_P(h) = \frac{4}{3}T_P(h), \quad (1.1)$$

$$U_P(h) = \frac{1}{2}T_P(h), \quad (1.2)$$

$$U_P(h) = \frac{1}{4}V_P(h) \quad (1.3)$$

and

$$U_P(h) = \frac{1}{3}W_P(h). \quad (1.4)$$

**Proof.** For a proof of (1.1), see [13]. For an arbitrary point  $P = A \in X$  and a sufficiently small number  $h > 0$  the height function  $H_P(h)$  is given by

$$H_P(h) = 2h,$$

and hence we have

$$L_P(h) = 2\ell_P(h).$$

These complete the proofs of (1.2), (1.3) and (1.4). For another proof of (1.2), see [1].  $\square$

In fact, Archimedes showed that parabolas satisfy (1.1) ([13]). Recently, in [8] the first author of the present paper and Y. H. Kim proved that (1.1) is a characteristic property of parabolas and established some characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([13]). For the higher dimensional analogues of some results in [8], see [6] and [7].

The property (1.2) of parabolas have been studied by several authors ([1, 10, 11]). Very recently, the first author of the present paper et al. showed that (1.2) is also a characteristic property of parabolas ([4]). More precisely, we have

**Proposition 1.2.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then  $X$  is an open part of a parabola if and only if it satisfies one of the following conditions.

- (1) For all  $P \in X$  and sufficiently small  $h > 0$ ,

$$S_P(h) = \lambda(P)T_P(h), \quad (1.5)$$

where  $\lambda(P)$  is a function of  $P$ .

- (2) For all  $P \in X$  and sufficiently small  $h > 0$ ,

$$U_P(h) = \alpha(P)T_P(h), \quad (1.6)$$

where  $\alpha(P)$  is a function of  $P$ .

- (3) For all  $P \in X$  and sufficiently small  $h > 0$ ,

$$S_P(h) = \mu(P)W_P(h), \quad (1.7)$$

where  $\mu(P)$  is a function of  $P$ .

- (4) For all  $P \in X$  and sufficiently small  $h > 0$ ,

$$S_P(h) = \eta(P)V_P(h), \quad (1.8)$$

where  $\eta(P)$  is a function of  $P$ .

**Proof.** For a proof of (1), see Theorem 3 of [8]. For proofs of remaining characterizations of parabolas, we refer to [4].

In Proposition 1.2, obviously we have  $\lambda(P) = 4/3$ ,  $\alpha(P) = 1/2$ ,  $\mu(P) = 8/9$  and  $\eta(P) = 2/3$ .

In this paper, we study whether the remaining properties of parabolas in Proposition 1.1 characterize parabolas. As a result, first of all in Section 3 we prove

**Theorem 1.3.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then  $X$  is an open part of a parabola if and only if it satisfies one of the following conditions.

- (1) There exists a function  $\beta(P)$  of  $P \in X$  such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies

$$U_P(h) = \beta(P)V_P(h). \quad (1.9)$$

- (2) There exists a function  $\gamma(P)$  of  $P \in X$  such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies

$$U_P(h) = \gamma(P)W_P(h). \quad (1.10)$$

In Theorem 1.3, obviously we have  $\beta(P) = 1/4$  and  $\gamma(P) = 1/3$ .

Combining (1.6) and (1.9), it follows from (1.2) and (1.3) that for an arbitrary point  $P$  and a sufficiently small number  $h > 0$ , parabolas satisfy

$$U_P(h) = \alpha(P)T_P(h) + \beta(P)V_P(h), \quad (1.11)$$

whenever  $\alpha(P)$  and  $\beta(P)$  are functions of  $P$  with

$$\alpha(P) + 2\beta(P) = \frac{1}{2}. \quad (1.12)$$

Similarly, if we combine (1.6) and (1.10), it follows from (1.2) and (1.4) that for an arbitrary point  $P$  and a sufficiently small number  $h > 0$ , parabolas satisfy

$$U_P(h) = \alpha(P)T_P(h) + \gamma(P)W_P(h), \quad (1.13)$$

whenever  $\alpha(P)$  and  $\gamma(P)$  are functions of  $P$  with

$$2\alpha(P) + 3\gamma(P) = 1. \quad (1.14)$$

Next in section 3, we generalize both of Theorem 1.3 and (2) of Proposition 1.2 (Theorem 1.3 of [4]) as follows.

**Theorem 1.4.** Let  $X$  denote a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then  $X$  is an open part of a parabola if and only if it satisfies one of the following conditions.

- (1) There exist functions  $\alpha(P)$  and  $\beta(P)$  of  $P \in X$  such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies (1.11), that is,

$$U_P(h) = \alpha(P)T_P(h) + \beta(P)V_P(h).$$

- (2) There exist functions  $\alpha(P)$  and  $\gamma(P)$  of  $P \in X$  such that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies (1.13), that is,

$$U_P(h) = \alpha(P)T_P(h) + \gamma(P)W_P(h).$$

Finally, in Section 4, we establish the following characterization theorem for parabolas which extends both of Theorems 1.3 and 1.4.

**Theorem 1.5.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then, the following are equivalent.

- (1) There exist functions  $\alpha(P)$ ,  $\beta(P)$  and  $\gamma(P)$  of  $P \in X$  with  $(\alpha(P), \beta(P), \gamma(P)) \neq (0, 1, -1)$  for any  $P \in X$  such that the curve  $X$  satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$U_P(h) = \alpha(P)T_P(h) + \beta(P)V_P(h) + \gamma(P)W_P(h). \quad (1.15)$$

- (2)  $X$  is an open part of a parabola.

In Theorem 1.5, by considering the limit of both sides of (1.15) as  $h$  tends to 0 (cf. Section 2), we see that the coefficient functions satisfy

$$2\alpha(P) + 4\beta(P) + 3\gamma(P) = 1. \quad (1.16)$$

Note that every strictly locally convex curve  $X$  in the plane  $\mathbb{R}^2$  satisfies (1.15) with  $(\alpha, \beta, \gamma) = (0, 1, -1)$  because we always have

$$U_P(h) = V_P(h) - W_P(h).$$

Some characterizations of parabolas or conic sections by properties of tangent lines were given in [3] and [9]. In [5], using curvature function  $\kappa$  and support function  $h$  of a plane curve, the first author and Y. H. Kim gave a characterization of ellipses and hyperbolas centered at the origin.

B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([12]). In their papers, a parabola means the graph of a quadratic polynomial in one variable.

Throughout this article, all curves are of class  $C^{(3)}$  and connected, unless otherwise mentioned.

## 2. Some lemmas

In this section, we prove some lemmas. In order to prove Theorems 1.3, 1.4 and 1.5 in Section 1, first of all we need the following.

**Lemma 2.1.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. Then for the curvature function  $\kappa(P)$  of  $X$  at  $P$  with respect to the unit normal  $N$  we have

$$\lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \quad (2.1)$$

$$\lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} \ell_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}, \quad (2.2)$$

$$\lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}, \quad (2.3)$$

$$\lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} U_P(h) = \frac{\sqrt{2}}{2\sqrt{\kappa(P)}}, \quad (2.4)$$

$$\lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} V_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}} \quad (2.5)$$

and

$$\lim_{h \rightarrow 0} \frac{1}{h\sqrt{h}} W_P(h) = \frac{3\sqrt{2}}{2\sqrt{\kappa(P)}}. \quad (2.6)$$

**Proof.** It follows from [8] that (2.1) holds. For proofs of (2.2), (2.5) and (2.6), see [4]. For proofs of (2.3) and (2.4), we refer to [10].  $\square$

Now, the derivative  $L'_P(h)$  of  $L_P(h)$  with respect to  $h$  is given by the following lemma.

**Lemma 2.2.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  with the unit normal  $N$  pointing to the convex side. Then we have

$$hL'_P(h) = L_P(h) - \ell_P(h), \quad (2.7)$$

where  $L'_P(h)$  means the derivative of  $L_P(h)$  with respect to  $h$ .

**Proof.** For a proof, see Lemma 2.3 in [4].  $\square$

In Section 1,  $L_P(h)$ ,  $\ell_P(h)$  and  $H_P(h)$  were defined by the lengths  $|A_1A_2|$  and  $|B_1B_2|$  of the corresponding segments and the height of the

triangle  $\triangle BA_1A_2$  from the vertex  $B$  to the edge  $A_1A_2$ , respectively. Hence, we get

$$L_P(h) : \ell_P(h) = H_P(h) : H_P(h) - h, \tag{2.8}$$

which yields

$$H_P(h) = \frac{hL_P(h)}{L_P(h) - \ell_P(h)}. \tag{2.9}$$

Together with Lemma 2.2, this shows that

$$H_P(h) = \frac{L_P(h)}{L'_P(h)}. \tag{2.10}$$

Finally, we prove a lemma which is useful in the proofs of main theorems.

**Lemma 2.3.** Suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$ . Then, the function  $L_P(h)$  satisfies

$$\lim_{h \rightarrow 0} \frac{hL'_P(h)}{L_P(h)} = \frac{1}{2}. \tag{2.11}$$

**Proof.** It follows from (2.7) that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{hL'_P(h)}{L_P(h)} &= \lim_{h \rightarrow 0} \frac{L_P(h) - \ell_P(h)}{L_P(h)} \\ &= \lim_{h \rightarrow 0} \left\{ 1 - \frac{\ell_P(h)}{L_P(h)} \right\}. \end{aligned} \tag{2.12}$$

Together with (2.12), (2.1) and (2.2) complete the proof of Lemma 2.3.  $\square$

### 3. Theorems 1.3 and 1.4

In this section, first of all, we prove Theorem 1.3.

It follows from Proposition 1.1 that any open parts of parabolas satisfy (1) and (2) in Theorem 1.3 with  $\beta(P) = 1/4$  and  $\gamma(P) = 1/3$ , respectively.

Conversely, suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$U_P(h) = \beta(P)V_P(h),$$

where  $\beta = \beta(P)$  is a function of  $P \in X$ . Then, by letting  $h \rightarrow 0$ , Lemma 2.1 shows that  $\beta = 1/4$ .

We fix an arbitrary point  $P \in X$ . Note that we have

$$\begin{aligned} U_P(h) &= \frac{1}{2}\{H_P(h) - h\}l_P(h) \\ &= \frac{1}{2}\left\{\frac{L_P(h)^2}{L'_P(h)} - 2hL_P(h) + h^2L'_P(h)\right\}, \end{aligned} \quad (3.1)$$

where the second equality follows from (2.7) and (2.10). Since the area  $V_P(h)$  is given by

$$V_P(h) = \frac{1}{2}H_P(h)L_P(h) = \frac{1}{2}\frac{L_P(h)^2}{L'_P(h)}, \quad (3.2)$$

we see that the assumption (1.9) implies

$$\frac{3}{4}L_P(h)^2 - 2hL_P(h)L'_P(h) + h^2L'_P(h)^2 = 0. \quad (3.3)$$

Hence we obtain

$$\{3L_P(h) - 2hL'_P(h)\}\{L_P(h) - 2hL'_P(h)\} = 0. \quad (3.4)$$

It follows from Lemma 2.3 and (3.4) that the curve  $X$  satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$L_P(h) = 2hL'_P(h). \quad (3.5)$$

With the aid of Lemma 2.1, integrating (3.5) shows that

$$L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \quad (3.6)$$

Since  $S'_P(h) = L_P(h)$  with  $S_P(0) = 0$ , from (3.6) we get

$$S_P(h) = \frac{4\sqrt{2}}{3\sqrt{\kappa(P)}}h\sqrt{h}. \quad (3.7)$$

Together with (3.6), (3.7) shows that for all  $P \in X$  and sufficiently small  $h > 0$  the curve  $X$  satisfies

$$S_P(h) = \frac{4}{3}T_P(h).$$

Thus, Proposition 1.2 implies that the curve  $X$  is an open part of a parabola.

Now, suppose that  $X$  denotes a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$U_P(h) = \gamma(P)W_P(h),$$



where  $\gamma = \gamma(P)$  is a function of  $P \in X$ . Then, by letting  $h \rightarrow 0$ , from Lemma 2.1 we see that  $\gamma = 1/3$ . Note that the area  $W_P(h)$  is given by

$$\begin{aligned} W_P(h) &= \frac{1}{2}h\{L_P(h) + \ell_P(h)\} \\ &= \frac{1}{2}h\{2L_P(h) - hL'_P(h)\}, \end{aligned} \tag{3.8}$$

where the second equality follows from Lemma 2.2.

Using (3.1) and (3.8), just as in the above proof we may show that the curve  $X$  is an open part of a parabola. This completes the proof of Theorem 1.3.

Finally, we prove Theorem 1.4.

It follows from Proposition 1.1 that any open parts of parabolas satisfy (1) and (2) in Theorem 1.4 with  $(\alpha, \beta) = (1/2, 0)$  and  $(\alpha, \gamma) = (1/2, 0)$ , respectively.

Conversely, suppose that  $X$  is a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies for all  $P \in X$  and sufficiently small  $h > 0$

$$U_P(h) = \alpha(P)T_P(h) + \beta(P)V_P(h),$$

where  $\alpha = \alpha(P)$  and  $\beta = \beta(P)$  are some functions of  $P \in X$ . Then, by letting  $h \rightarrow 0$ , Lemma 2.1 shows that (1.12) holds, that is,

$$\alpha(P) + 2\beta(P) = \frac{1}{2}.$$

Now, we fix an arbitrary point  $P \in X$ .

Since  $T_P(h) = \frac{1}{2}hL_P(h)$ , together with the assumption (1.11), (3.1) and (3.2) imply that

$$h^2L'_P(h)^2 + (2\beta - \frac{5}{2})hL_P(h)L'_P(h) + (1 - \beta)L_P(h)^2 = 0, \tag{3.9}$$

where we use (1.12). It follows from (3.9) that for all sufficiently small  $h > 0$ ,  $x = x(h) = hL'_P(h)/L_P(h)$  satisfies the following quadratic equation

$$x^2 + (2\beta - \frac{5}{2})x + (1 - \beta) = 0.$$

This shows that  $x(h) = hL'_P(h)/L_P(h)$  is constant. Hence, due to Lemma 2.3 we see that  $x = 1/2$ , that is,

$$hL'_P(h) = \frac{1}{2}L_P(h). \tag{3.10}$$

By integrating (3.10) and using Lemma 2.1, we get

$$L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}\sqrt{h}. \quad (3.11)$$

In the same manner just as in the proof of Theorem 1.3, we obtain

$$S_P(h) = \frac{4}{3}T_P(h).$$

Therefore, Proposition 1.2 shows that  $X$  is an open part of a parabola.

Now, suppose that a strictly locally convex  $C^{(3)}$  curve  $X$  in the plane  $\mathbb{R}^2$  satisfies (2) of Theorem 1.4. Then, by letting  $h \rightarrow 0$ , from Lemma 2.1 we see that  $(\alpha, \gamma)$  satisfies (1.14).

Using (3.1) and (3.8), just as in the above proof we may show that the curve  $X$  is an open part of a parabola. This completes the proof of Theorem 1.4.

#### 4. Theorem 1.5

In this section, we prove Theorem 1.5.

It follows from Proposition 1.1 that any open parts of parabolas satisfy (1) in Theorem 1.5 with  $(\alpha, \beta, \gamma) = (1/2, 0, 0)$ . Note that they also satisfy the equation (1.15) with either  $(\alpha, \beta, \gamma) = (0, 1/4, 0)$  or  $(\alpha, \beta, \gamma) = (0, 0, 1/3)$ .

Conversely, suppose that  $X$  is a strictly locally convex  $C^{(3)}$  curve in the plane  $\mathbb{R}^2$  which satisfies (1.15) for all  $P \in X$  and sufficiently small  $h > 0$ , that is, for some functions  $\alpha = \alpha(P)$ ,  $\beta = \beta(P)$  and  $\gamma = \gamma(P)$ , the curve  $X$  satisfies

$$U_P(h) = \alpha(P)T_P(h) + \beta(P)V_P(h) + \gamma(P)W_P(h).$$

Then, Lemma 2.1 shows that for all  $P \in X$  the coefficient functions  $\alpha, \beta$  and  $\gamma$  satisfy (1.16), that is,

$$2\alpha(P) + 4\beta(P) + 3\gamma(P) = 1.$$

Now, we fix an arbitrary point  $P \in X$ .

Using  $T_P(h) = \frac{1}{2}hL_P(h)$ , together with (3.1), (3.2) and (3.8), the assumption (1.15) implies that

$$(1 + \gamma)h^2L'_P(h)^2 - (\alpha + 2\gamma + 2)hL_P(h)L'_P(h) + (1 - \beta)L_P(h)^2 = 0. \quad (4.1)$$

It follows from (4.1) that  $x = x(h) = hL'_P(h)/L_P(h)$  satisfies a quadratic equation as follows.

$$(1 + \gamma)x^2 - (\alpha + 2\gamma + 2)x + (1 - \beta) = 0. \quad (4.2)$$

Note that by the assumption, we have  $(\alpha, \beta, \gamma) \neq (0, 1, -1)$ . Hence, (4.2) is a polynomial of degree 1 or 2 with nonvanishing leading term. Thus, we see that  $x(h) = hL'_P(h)/L_P(h)$  is a nonzero constant. With the help of Lemma 2.3, we get  $x = 1/2$ . In other words, for all sufficiently small  $h > 0$  the curve  $X$  satisfies

$$\frac{hL'_P(h)}{L_P(h)} = \frac{1}{2}. \quad (4.3)$$

Since  $P$  was an arbitrary point of  $X$ , together with (4.3) the same argument as in the proof of Theorem 1.3 completes the proof of Theorem 1.5.

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