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# $\Omega$-INTERVAL-VALUED FUZZY SUBSEMIGROUPS IN A SEMIGROUP

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**Abstract.** By using a set $\Omega$, we introduce the concept of $\Omega$-fuzzy subsemigroups and study some of it's properties. Also, we show that the homomorphic images and preimages of $\Omega$-fuzzy subsemigroups become $\Omega$-fuzzy subsemigroups.

## 1. Introduction

In 1975, Zadeh [12] suggested the notion of interval-valued fuzzy sets as generalization of fuzzy sets introduced by himself [11]. After that time, Biswas [1] applied it to group theory, and Gorzalczany [4] introduced a method of inference in approximate reasoning by using interval-valued fuzzy sets. Moreover, Mondal and Samanta [10] introduced the concept of interval-valued fuzzy topology and investigate some of it's properties. Recently, Hur et al. [6] studies interval-valued fuzzy relations in the sense of a lattice theory. Also, Choi et al. [3] introduced the concept of interval-valued smooth topological spaces and investigated some of it's properties. On the other hand, Cheong and Hur [2], and Lee et al. [9] studied interval-valued fuzzy ideals/bi-ideals in a semigroup. Kang [7], Kang and Hur [8] applied the notion of interval-valued fuzzy sets to algebra. In this paper, by using a set $\Omega$, we introduce the concept of $\Omega$-fuzzy subsemigroups and study some of it's properties. Also, we state how the homomorphic images and preimages of $\Omega$-fuzzy subsemigroups become $\Omega$-fuzzy subsemigroups.

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## 2. Preliminaries

In this section, we list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where $M^L$ and $M^U$ are the lower and the upper end points respectively. Especially, we denoted $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\boldsymbol{a} = [a, a]$ for every $a \in (0, 1)$, We also note that

(i) $(\forall M, N \in D(I))\ (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))\ (M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of $M$, denoted by $M^C$, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$ (See[10]).

**Definition 2.1** [10,12]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set*(in short, *IVFS*) in $X$, denoted by $A = [A^L, A^U]$, if $A^L, A^L \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$ [resp $A^U(x)$] is called the *lower* [resp *upper*] *end point of* $x$ *to* $A$. For any $[a, b] \in D(I)$, the interval-valued fuzzy set $A$ in $X$ defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVFS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in $X$, respectively.

We will denote the set of all IVFSs in $X$ as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

**Definition 2.2** [10]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.

**Result 2.A** [10, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(a) $\widetilde{0} \subset$ A $\subset \widetilde{1}$.

(b) $A \cup B = B \cup A$ , $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$ , $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$ , $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$ , $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha$ , $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

**Definition 2.3** [10]. Let $f : X \to Y$ be a mapping, let $A \in D(I)^X$ and let $B \in D(I)^Y$. Then

(i) the *image of* $A$ *under* $f$, denoted by $f(A)$, is an IVFS in $Y$ defined as follows: For each $y \in Y$,

$$f(A)^L(y) = \begin{cases} \bigvee_{y=f(x)} A^L(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(A)^U(y) = \begin{cases} \bigvee_{y=f(x)} A^U(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) the *preimage of* $B$ *under* $f$, denoted by $f^{-1}(B)$, is an IVFS in $Y$ defined as follows: For each $y \in Y$,

$$f^{-1}(B)^L(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B)^U(y) = (B^U \circ f)(x) = B^U(f(x)).$$

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

**Result 2.B** [10, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then

(a) $f^{-1}(B^c) = [f^{-1}(B)]^c$ , $\forall B \in D(I)^Y$.
(b) $[f(A)]^c \subset f(A^c)$ , $\forall A \in D(I)^Y$.
(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.
(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.
(e) $f(f^{-1}(B)) \subset B$, $\forall B \in D(I)^Y$.
(f) $A \subset f(f^{-1}(A))$, $\forall A \in D(I)^Y$.
(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I)^Z$.
(h) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.
(h) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

**Definition 2.4** [2]. Let $S$ be a semigroup and let $A \in D(I)^S$. Then $A$ is called an *interval-valued fuzzy subsemigroup*(in short, *IVSG*) in $S$, if it satisfies the conditions : $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$, $\forall x, y \in S$.

We will denote the set of all IVSGs in $S$ as IVSG($S$).

## 3. Ω-interval-valued fuzzy subsemigroups

In what follows let $S$ and $\Omega$ denote a semigroup and a nonempty set, respectively, unless otherwise specified.

**Definition 3.1**. A mapping $A_\Omega : S \times \Omega \to D(I)$ is called an *Ω-interval-valued fuzzy set*(in short, *Ω − IVFS*) in $S$, denoted by $A_\Omega = [A_\Omega^L, A_\Omega^U]$, where $A_\Omega^L, A_\Omega^U \in I^{S\times\Omega}$ are called the *degree of lower membership* and *degree of upper membership* of *the element* $(x, \alpha) \in A_\Omega \subset S \times \Omega$, respectively.

We will denote the set of all Ω-IVFSs in $S$ as $D(I)^{S\times\Omega}$.

**Definition 3.2**. Let $A_\Omega \in D(I)^{S\times\Omega}$. Then $A_\Omega$ is called an *Ω-interval-valued fuzzy subsemigroup* (in short, *Ω − IVSG*)of $S$, if it satisfies the followings : For each $\alpha \in \Omega$ and each $x,\ y \in S$,

(i) $A_\Omega^L(xy, \alpha) \geq A_\Omega^L(x, \alpha) \wedge A_\Omega^L(y, \alpha)$,
(ii) $A_\Omega^U(xy, \alpha) \geq A_\Omega^U(x, \alpha) \wedge A_\Omega^U(y, \alpha)$.

We will denote the set of all Ω-IVSGs of $S$ as Ω-IVSG(S).

**Example 3.3**. Consider $S = \{a, b\}$ with the following Cayley table:

| | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $b$ | $a$ |

Let $\Omega = \{1, 2\}$ and let $A_\Omega$ be an Ω-IVFS in $S$ defined as follows: $A_\Omega(a, 1) = A_\Omega(a, 2) = [1, 1]$, $A_\Omega(b, 1) = [0.1, 0.8]$ and $A_\Omega(b, 2) = [0.3, 0.5]$. Then it is easy to see that $A_\Omega \in$ Ω-IVSG(S).

Let $S^\Omega = \{u | u : \Omega \to S\}$. For any $u, v \in S^\Omega$, we define $(uv)(\alpha) = u(\alpha)v(\alpha)$ for each $\alpha \in \Omega$. Then $S^\Omega$ is a semigroup (see[5]).

**Example 3.4**. Let $A \in IVSG(\mathrm{S})$ and let $A_\Omega$ be an Ω-IVFS in $S^\Omega$ defined as follows : $A^L_\Omega(u, \alpha) = A^L(u(\alpha))$ and $A^U_\Omega(u, \alpha) = A^U(u(\alpha))$ for each $u \in S^\Omega$ and $\alpha \in \Omega$. Then $A \in \Omega - IVSG(S^\Omega)$.

**Proposition 3.5**. Let $A_\Omega \in \Omega - IVSG(S)$. For each $\omega \in \Omega$, let $A^\omega_\Omega$ be the IVFS in $S$ defined as follows : For each $x \in S$,

$$(A^\omega_\Omega)^L(x) = A^L_\Omega(x, \omega) \text{ and } (A^\omega_\Omega)^U(x) = A^U_\Omega(x, \omega).$$

Then $A^\omega_\Omega$ is an IVSG of $S$.

*Proof.* Let $x, y \in S$. Then

$$\begin{aligned}(A^\omega_\Omega)^L(xy) &= A^L_\Omega(xy, \omega)\\ &\geq A^L_\Omega(x, \omega) \wedge A^L_\Omega(y, \omega)\\ &= (A^\omega_\Omega)^L(x) \wedge (A^\omega_\Omega)^L(y),\end{aligned}$$

and

$$\begin{aligned}(A^\omega_\Omega)^U(xy) &= A^U_\Omega(xy, \omega)\\ &\geq A^U_\Omega(x, \omega) \wedge A^U_\Omega(y, \omega)\\ &= (A^\omega_\Omega)^U(x) \wedge (A^\omega_\Omega)^U(y).\end{aligned}$$

This completes the proof. □

**Proposition 3.6**. For each $\omega \in \Omega$, let $A^\omega_\Omega \in IVSG(S)$. Let $A_\Omega$ be the Ω-IVFS in $S$ defined as follows : For each $x \in S$,

$$A^L_\Omega(x, \omega) = (A^\omega_\Omega)^L(x) \text{ and } A^U_\Omega(x, \omega) = (A^\omega_\Omega)^U(x).$$

Then $A_\Omega$ is an Ω-IVSG of $S$.

*Proof.* Let $x, y \in S$. Then

$$\begin{aligned} A^L_\Omega(xy, \omega) &= (A^\omega_\Omega)^L(xy) \\ &\geq (A^\omega_\Omega)^L(x) \wedge (A^\omega_\Omega)^L(y) \\ &= A^L_\Omega(x, \omega) \wedge A^L_\Omega(y, \omega), \end{aligned}$$

and

$$\begin{aligned} A^U_\Omega(xy, \omega) &= (A^\omega_\Omega)^U(xy) \\ &\geq (A^\omega_\Omega)^U(x) \wedge (A^\omega_\Omega)^U(y) \\ &= A^U_\Omega(x, \omega) \wedge A^U_\Omega(y, \omega). \end{aligned}$$

Hence $A_\Omega$ is an $\Omega$-IVSG of $S$. □

**Proposition 3.7**. Let $\Phi \in IVSG(S^\Omega)$ and let $A_\Phi$ be the $\Omega$-IVFS in $S^\Omega$ defined as follows: For each $x \in S$ and $\alpha \in \Omega$,

$$A^L_\Phi(x, \alpha) = \bigvee_{\substack{u \in S^\Omega \\ u(\alpha)=x}} \Phi^L(u),$$

and

$$A^U_\Phi(x, \alpha) = \bigvee_{\substack{u \in S^\Omega \\ u(\alpha)=x}} \Phi^U(u).$$

Then $A_\Phi$ is an $\Omega$-IVSG of $S$.

*Proof.* Let $x, y \in S$ and let $\alpha \in \Omega$. Then

$$\begin{aligned} A^L_\Phi(xy, \alpha) &= \bigvee_{\substack{u \in S^\Omega \\ u(\alpha)=xy}} \Phi^L(u) \\ &\geq \bigvee_{\substack{u, v \in S^\Omega \\ u(\alpha)=x, v(\alpha)=y}} [\Phi^L(u) \wedge \Phi^L(v)] \\ &\geq (\bigvee_{\substack{u \in S^\Omega \\ u(\alpha)=x}} \Phi^L(u)) \wedge (\bigvee_{\substack{v \in S_\Omega \\ v(\alpha)=y}} \Phi^L(v)) \\ &\geq A^L_\Phi(x, \alpha) \wedge A^L_\Phi(y, \alpha). \end{aligned}$$

Similarly, we have that

$$A^U_\Phi(xy, \alpha) \geq A^U_\Phi(x, \alpha) \wedge A^U_\Phi(y, \alpha).$$

Hence $A_\Phi$ is an $\Omega$-IVSG of $S$. □

**Example 3.8**. Let $S = \{a, b\}$ be a semigroup in Example 3.3 and let $\Omega = \{1, 2\}$. Then $S^\Omega = \{e, u, v, w\}$, where $e(1) = e(2) = v(1) = w(2) = a$ and $u(1) = v(2) = w(1) = b$, is a semigroup (in fact, a commutative group) under the following Cayley table :

| | $e$ | $u$ | $v$ | $w$ |
|---|---|---|---|---|
| $e$ | $e$ | $u$ | $v$ | $w$ |
| $u$ | $u$ | $e$ | $w$ | $v$ |
| $v$ | $v$ | $w$ | $e$ | $u$ |
| $w$ | $w$ | $v$ | $u$ | $e$ |

Let $\Phi$ be an IVFS in $S^\Omega$ defined as follows:

$$\Phi(e) = [0.1, 0.9], \Phi(u) = \Phi(v) = [0.2, 0.7], \Phi(w) = [0.3, 0.8].$$

Then $\Phi$ is an IVSG of $S^\Omega$. Thus we can obtain an $\Omega$-IVSG $A_\Phi$ of $S$ as follows:

$$A^L_\Phi(a, 1) = \bigvee_{\substack{t \in S^\Omega \\ t(1)=a}} \Phi^L(t) = \Phi^L(e) \vee \Phi^L(v) = 0.2,$$

$$A^U_\Phi(a, 1) = \bigvee_{\substack{t \in S^\Omega \\ t(1)=a}} \Phi^U(t) = \Phi^U(e) \vee \Phi^U(v) = 0.9,$$

$$A^L_\Phi(a, 2) = \bigvee_{\substack{t \in S^\Omega \\ t(2)=a}} \Phi^L(t) = \Phi^L(e) \vee \Phi^L(w) = 0.3,$$

$$A^U_\Phi(a, 2) = \bigvee_{\substack{t \in S^\Omega \\ t(2)=a}} \Phi^U(t) = \Phi^U(e) \vee \Phi^U(w) = 0.9,$$

$$A^L_\Phi(b, 1) = \bigvee_{\substack{t \in S^\Omega \\ t(1)=b}} \Phi^L(t) = \Phi^L(u) \vee \Phi^L(w) = 0.3,$$

$$A^U_\Phi(b, 1) = \bigvee_{\substack{t \in S^\Omega \\ t(1)=b}} \Phi^U(t) = \Phi^U(u) \vee \Phi^U(w) = 0.8,$$

$$A^L_\Phi(b, 2) = \bigvee_{\substack{t \in S^\Omega \\ t(2)=b}} \Phi^L(t) = \Phi^L(u) \vee \Phi^L(v) = 0.2,$$

$$A^U_\Phi(b, 2) = \bigvee_{\substack{t \in S^\Omega \\ t(2)=b}} \Phi^U(t) = \Phi^U(u) \vee \Phi^U(v) = 0.7.$$

**Proposition 3.9**. Let $A_\Omega$ be an $\Omega$-IVSG of $S$ and let $\Phi$ be an IVFS of $S^\Omega$ defined as follows: For each $u \in S^\Omega$,

$$\Phi(u) = [\bigwedge_{\alpha\in\Omega} A^L_\Omega(u(\alpha),\alpha), \bigwedge_{\alpha\in\Omega} A^U_\Omega(u(\alpha),\alpha)].$$

Then $\Phi$ is an IVSG of $S^\Omega$.

*Proof.* Let $u, v \in S^\Omega$. Then

$$\begin{aligned}\Phi^L(uv) &= \bigwedge_{\alpha\in\Omega} A^L_\Omega(uv(\alpha),\alpha)\\ &= \bigwedge_{\alpha\in\Omega} A^L_\Omega(u(\alpha)v(\alpha),\alpha)\\ &\geq \bigwedge_{\alpha\in\Omega} [A^L_\Omega(u(\alpha),\alpha) \wedge A^L_\Omega(v(\alpha),\alpha)]\\ &= (\bigwedge_{\alpha\in\Omega} A^L_\Omega(u(\alpha),\alpha)) \wedge (\bigwedge_{\alpha\in\Omega} A^L_\Omega(v(\alpha),\alpha))\\ &= \Phi^L(u) \wedge \Phi^L(v).\end{aligned}$$

By the similar arguments, we have that

$$\Phi^U(uv) \geq \Phi^U(u) \wedge \Phi^U(v).$$

Hence $\Phi$ is an IVSG of $S^\Omega$. □

**Example 3.10**. Let $A_\Omega$ be the $\Omega$-IVSG of $S$ in Example 3.3 and let $S^\Omega$ be the commutative group in Example 3.8. Then we can obtain an IVSG $\Phi$ of $\Omega$ as follows:

$$\begin{aligned}\Phi^L(e) &= \bigwedge_{\alpha\in\Omega} A^L_\Omega(e(\alpha),\alpha) = A^L_\Omega(e(1),1) \wedge A^L_\Omega(e(2),2)\\ &= A^L_\Omega(a,1) \wedge A^L_\Omega(a,2) = 1,\end{aligned}$$

$$\begin{aligned}\Phi^U(e) &= \bigwedge_{\alpha\in\Omega} A^U_\Omega(e(\alpha),\alpha) = A^U_\Omega(e(1),1) \wedge A^U_\Omega(e(2),2)\\ &= A^U_\Omega(a,1) \wedge A^U_\Omega(a,2) = 1,\end{aligned}$$

$$\begin{aligned}\Phi^L(u) &= \bigwedge_{\alpha\in\Omega} A^L_\Omega(u(\alpha),\alpha) = A^L_\Omega(u(1),1) \wedge A^L_\Omega(u(2),2)\\ &= A^L_\Omega(b,1) \wedge A^L_\Omega(b,2) = 0.1,\end{aligned}$$

$$\Phi^U(u) = \bigwedge_{\alpha\in\Omega} A^U_\Omega(u(\alpha),\alpha) = A^U_\Omega(u(1),1) \wedge A^U_\Omega(u(2),2)$$
$$= A^U_\Omega(b,1) \wedge A^U_\Omega(b,2) = 0.5,$$

$$\Phi^L(v) = \bigwedge_{\alpha\in\Omega} A^L_\Omega(v(\alpha),\alpha) = A^L_\Omega(v(1),1) \wedge A^L_\Omega(v(2),2)$$
$$= A^L_\Omega(a,1) \wedge A^L_\Omega(b,2) = 0.3,$$

$$\Phi^U(v) = \bigwedge_{\alpha\in\Omega} A^U_\Omega(v(\alpha),\alpha) = A^U_\Omega(v(1),1) \wedge A^U_\Omega(v(2),2)$$
$$= A^U_\Omega(a,1) \wedge A^U_\Omega(b,2) = 0.5,$$

$$\Phi^L(w) = \bigwedge_{\alpha\in\Omega} A^L_\Omega(w(\alpha),\alpha) = A^L_\Omega(w(1),1) \wedge A^L_\Omega(w(2),2)$$
$$= A^L_\Omega(b,1) \wedge A^L_\Omega(a,2) = 0.1,$$

$$\Phi^U(w) = \bigwedge_{\alpha\in\Omega} A^U_\Omega(w(\alpha),\alpha) = A^U_\Omega(w(1),1) \wedge A^U_\Omega(w(2),2)$$
$$= A^U_\Omega(b,1) \wedge A^U_\Omega(a,2) = 0.8.$$

## 4. Properties under a homomorphism

**Definition 4.1**. Let $\varphi : S \to T$ be a mapping, let $A_\Omega \in D(I)^{S\times\Omega}$ and let $B_\Omega \in D(I)^{T\times\Omega}$. Then

(i) the *preimage* of $B_\Omega$, denoted $\varphi^{-1}(B_\Omega)$, is an Ω-IVFS in $S$ defined as follows: For each $(x,\alpha) \in S \times \Omega$,

$$[\varphi^{-1}(B_\Omega)](x,\alpha) = [\varphi^{-1}(B^L_\Omega)(x,\alpha), \varphi^{-1}(B^U_\Omega)(x,\alpha)],$$

where $\varphi^{-1}(B^L_\Omega)(x,\alpha) = B^L_\Omega(\varphi(x),\alpha)$ and $\varphi^{-1}(B^U_\Omega)(x,\alpha) = B^U_\Omega(\varphi(x),\alpha)$.

(ii) the *image* of $A_\Omega$, denoted by $\varphi(A_\Omega)$, is an Ω-IVFS in $T$ defined as follows: For each $(y,\alpha) \in T \times \Omega$,

$$[\phi(A_\Omega)](y,\alpha) = [\varphi(A^L_\Omega)(y,\alpha), \varphi(A^U_\Omega)(y,\alpha)],$$

where

$$\varphi(A^L_\Omega)(y,\alpha) = \begin{cases} \bigvee_{y=\varphi(x)} A^L_\Omega(x,\alpha) & \text{if } \varphi^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi(A^U_\Omega)(y,\alpha) = \begin{cases} \bigvee_{y=\varphi(x)} A^U_\Omega(x,\alpha) & \text{if } \varphi^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 4.2**. Let $\varphi : S \to T$ be a homomorphism of semigroups. If $B_\Omega$ is an $\Omega$-IVSG of $T$, the preimage $\varphi^{-1}(B_\Omega)$ is an $\Omega$-IVSG of $S$.

*Proof.* Let $x, y \in S$ and let $\alpha \in \Omega$. Then

$$\begin{aligned}\varphi^{-1}(B^L_\Omega)(xy,\alpha) &= B^L_\Omega(\varphi(xy),\alpha) = B^L_\Omega(\varphi(x)\varphi(y),\alpha) \\ &\geq B^L_\Omega(\varphi(x),\alpha) \wedge B^L_\Omega(\varphi(y),\alpha) \\ &= \varphi^{-1}(B^L_\Omega)(x,\alpha) \wedge \varphi^{-1}(B^L_\Omega)(y,\alpha).\end{aligned}$$

Similarly, we have that

$$\varphi^{-1}(B^U_\Omega)(xy,\alpha) \geq \varphi^{-1}(B^U_\Omega)(x,\alpha) \wedge \varphi^{-1}(B^U_\Omega)(y,\alpha).$$

Hence $\varphi^{-1}(B_\Omega)$ is an $\Omega$-IVSG of $S$. □

**Proposition 4.3**. Let $\varphi : S \to T$ be a homomorphism of semigroups. If $A_\Omega$ is an $\Omega$-IVSG of $S$, the image $\varphi(A_\Omega)$ is an $\Omega$-IVSG of $T$.

*Proof.* We show that

$$\varphi^{-1}(y_1)\varphi^{-1}y_2 \subset \varphi^{-1}(y_1y_2) \qquad (*)$$

for any $y_1,\ y_2 \in T$.

Let $x \in \varphi^{-1}(y_1)\varphi^{-1}y_2$. Then $x = x_1x_2$ for some $x_1 \in \varphi^{-1}(y_1)$ and $x_2 \in \varphi^{-1}y_2$. Since $\varphi$ is a homomorphism, $\varphi(x) = \varphi(x_1x_2) = \varphi(x_1)\varphi(x_2) = y_1y_2$. Thus $x \in \varphi^{-1}(y_1y_2)$. So $(*)$ holds.

Now let $y_1, y_2 \in T$ and $\alpha \in \Omega$. Assume that $y_1y_2 \notin Im(\varphi)$. Then $\varphi(A_\Omega)(y_1y_2,\alpha) = [0,0]$. On the other hand, since $y_1y_2 \notin Im(\varphi)$, $\varphi^{-1}(y_1y_2) = \emptyset$. Then, by $(*)$, $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$. Thus $\varphi(A_\Omega)(y_1,\alpha) = [0,0]$ or $\varphi(A_\Omega)(y_2,\alpha) = [0,0]$. So it follows that

$$\varphi(A^L_\Omega)(y_1y_2,\alpha) = 0 = \varphi(A^L_\Omega)(y_1,\alpha) \wedge \varphi(A^L_\Omega)(y_2,\alpha),$$

$$\varphi(A^U_\Omega)(y_1y_2,\alpha) = 0 = \varphi(A^U_\Omega)(y_1,\alpha) \wedge \varphi(A^U_\Omega)(y_2,\alpha).$$

Suppose $\varphi^{-1}(y_1y_2) \neq \emptyset$. Then we can consider two cases as follows:

(i) $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$,

(ii) $\varphi^{-1}(y_1) \neq \emptyset$ and $\varphi^{-1}(y_2) \neq \emptyset$.

Case(i). We have $\varphi(A_\Omega)(y_1,\alpha) = [0,0]$ or $\varphi(A_\Omega)(y_2,\alpha) = [0,0]$. Thus

$$\varphi(A^L_\Omega)(y_1y_2,\alpha) \geq \varphi(A^L_\Omega)(y_1,\alpha) \wedge \varphi(A^L_\Omega)(y_2,\alpha) = 0,$$

$$\varphi(A^U_\Omega)(y_1y_2,\alpha) \geq \varphi(A^U_\Omega)(y_1,\alpha) \wedge \varphi(A^U_\Omega)(y_2,\alpha) = 0.$$

Case(ii). From (∗), we have that

$$\begin{aligned}\varphi(A^L_\Omega)(y_1y_2,\alpha) &= \bigvee_{x\in\varphi^{-1}(y_1y_2)} A^L_\Omega(x,\alpha)\\ &\geq \bigvee_{x\in\varphi^{-1}(y_1)\varphi^{-1}(y_2)} A^L_\Omega(x,\alpha)\\ &= \bigvee_{\substack{x_1\in\varphi^{-1}(y_1)\\ x_2\in\varphi^{-1}(y_2)}} A^L_\Omega(x_1x_2,\alpha)\\ &\geq \bigvee_{\substack{x_1\in\varphi^{-1}(y_1)\\ x_2\in\varphi^{-1}(y_2)}} [A^L_\Omega(x_1,\alpha)\wedge A^L_\Omega(x_2,\alpha)]\\ &= (\bigvee_{x_1\in\varphi^{-1}(y_1)} A^L_\Omega(x_1,\alpha))\wedge(\bigvee_{x_2\in\varphi^{-1}(y_2)} A^L_\Omega(x_2,\alpha))\\ &= \phi(A^L_\Omega)(y_1,\alpha)\wedge\phi(A^L_\Omega)(y_2,\alpha).\end{aligned}$$

By the Similar arguments, we have that

$$\varphi(A^U_\Omega)(y_1y_2,\alpha)\geq\phi(A^U_\Omega)(y_1,\alpha)\wedge\phi(A^U_\Omega)(y_2,\alpha).$$

This completes the proof. □

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