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SOME REMARKS FOR KÜNNETH FORMULA ON BOUNDED COHOMOLOGY

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Abstract. Künneth formula is to compute (co)-homology of $\mathbf{A}\otimes\mathbf{B}$ for known (co)-homology of the complexes $\mathbf{A}$ and $\mathbf{B}$. In the ordinary case, this is done by using elementary homological methods in an abelian category. However, when we consider the bounded cochain complex with values in $\mathbb{R}$ and its structure as a real Banach space, the techniques of homological algebra for constructing Künneth type formulas on it are not effective. The most notable facts are the image of a morphism of Banach spaces is not necessarily closed, and also the closed summand of a Banach space need not be a topological direct summand. The main goal of this paper is to construct the abstract theory of Künneth type formula on bounded cohomology with real coefficients in the suitable category of Banach spaces with some restricted conditions.

1. Introduction

As our main concern is to construct Künneth formula on bounded cohomology when we consider a Banach space structure on bounded cochains, we first review the standard definition of bounded cohomology with real coefficients $\mathbb{R}$.

For a discrete group G and a positive integer $n \geq 1$, let $G^n = \underbrace{G\times G\times\cdots\times G}_{n}$ and let $B^n(G)$ be the space of all bounded functions $f\colon G^n \to \mathbb{R}$, that is, $B^n(G) = \{\, f\colon G^n \to \mathbb{R} \mid \|f\| < \infty\}$, where

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$\|f\| = \sup\{\, |f(g_1, \cdots, g_n)| \mid (g_1, \cdots, g_n) \in G^n \,\}$. The boundary operator $d_n \colon B^n(G) \to B^{n+1}(G)$ for $n \geq 1$ is defined by the formula

$$\begin{aligned} &d_n(f)(g_1, \cdots, g_{n+1}) = f(g_2, \cdots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1} f(g_1, \cdots, g_n). \end{aligned}$$

Then it is easy to check that the sequence

$$0 \to \mathbb{R} \xrightarrow{d_0=0} B^1(G) \xrightarrow{d_1} B^2(G) \xrightarrow{d_2} B^3(G) \xrightarrow{d_3} \cdots \tag{1.1}$$

forms a complex.

Definition 1.1. The space $\{B^*(G), d_*\}$ in (1.1) is called *the bounded cochain complex* and its n-th cohomology is called *the n-th bounded cohomology of G with trivial coefficients* $\mathbb{R}$. We denote it by $\widehat{H}^n(G)$.

Similarly, for a space X, we consider $S_n(X)$ be the set of n-dimensional singular simplices in X for every $n \geq 0$.

Let $B^n(X)$ be the space of bounded functions $S_n(X) \to \mathbb{R}$ with values in $\mathbb{R}$, that is,

$$B^n(X) = \{f \colon S_n(X) \to \mathbb{R} \mid \|f\| = \sup_{\sigma \in S_n(X)} |f(\sigma)| < \infty\}.$$

Define the boundary operators $d_n \colon B^n(X) \to B^{n+1}(X)$ by $d_n f(\sigma) = \sum_{i=0}^{n+1} (-1)^i f(\partial_i \sigma)$ and $\partial_i \sigma$ is the i-th face of the singular simplex σ. Then it is easy to check the sequence

$$0 \to \mathbb{R} \xrightarrow{d_0} B^1(X) \xrightarrow{d_1} B^2(X) \xrightarrow{d_2} \cdots \tag{1.2}$$

is a complex. The cohomology of the sequence (1.2) is called the *bounded cohomology of X*, and denoted by $\widehat{H}^*(X)$.

Let Γ denote a discrete group or a space. Notice that the space of bounded cochains $B^*(\Gamma)$ has a structure of a real vector space and so does $\widehat{H}^*(\Gamma)$. Moreover, there is a natural norm $\|\cdot\|$ on the space $B^*(\Gamma)$ and this natural norm turns it into a real Banach space. This norm induces a *seminorm* on $\widehat{H}^n(\Gamma)$ given by $\|[g]\| = inf\|f\|$, where $[g] \in \widehat{H}^n(\Gamma)$ and the infimum is taken over all bounded cochains $f \in [g] = g + \operatorname{Im} d_{n-1}$ lying in the bounded cohomology class corresponding to $[g]$. Thus this vector space $\widehat{H}^*(\Gamma)$ carries the structure of *a seminormed space* over $\mathbb{R}$, not necessarily *a normed space*.

Recall that the ranges of d_n in (1.1) and (1.2) may not be closed in $B^{n+1}(\Gamma)$. Thus the space of coboundaries $d_n(B^n(\Gamma))$ may not be a closed subspace of $B^{n+1}(\Gamma)$. As it is well known, the space of coboundaries $d_n\,(B^n(\Gamma))$ is closed in $B^*(\Gamma)$ if and only if the seminorm on $\widehat{H}^*(\Gamma)$ is a norm. Hence, the space of coboundaries is closed in $B^*(\Gamma)$ if and only if $\widehat{H}^*(\Gamma)$ is a Banach space.

In [3], it is proved that Im d_1 is closed and so $\widehat{H}^2(\Gamma)$ is always a Banach space.

From this point of view, Mitsumatsu [6] defined the *reduced bounded cohomology* as follows:

Definition 1.2. Let $\overline{d_{n-1}(B^{n-1}(\Gamma))}$ denote the closure of $d_{n-1}(B^{n-1}(\Gamma))$, where d_* is in (1.1). *The reduced bounded cohomology of* Γ *with trivial coefficients* $\mathbb{R}$, denoted by $\overline{H_b^*(\Gamma)}$, is defined as

$$\overline{H_b^*(\Gamma)} = \ker(d_n)/\overline{d_{n-1}(B^{n-1}(\Gamma))}.$$

Notice that $\overline{H_b^*(\Gamma)}$ is a Banach space and

$$\overline{H_b^n(\Gamma)} = \widehat{H}^n(\Gamma)/\{f \in \widehat{H}^n(\Gamma) \mid \|f\| = 0\}.$$

Remark 1.1. As shown in [4], for a short exact sequence

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

of the complexes of Banach spaces, there is an exact sequence

$$\cdots \to \widehat{H}^{n-1}(\mathbf{C}) \to \widehat{H}^n(\mathbf{A}) \to \widehat{H}^n(\mathbf{B}) \to \widehat{H}^n(\mathbf{C}) \to \cdots \tag{1.1.1}$$

of bounded cohomology spaces and a semiexact sequence

$$\cdots \to \overline{H_b^{n-1}}(\mathbf{C}) \to \overline{H_b^n}(\mathbf{A}) \to \overline{H_b^n}(\mathbf{B}) \to \overline{H_b^n}(\mathbf{C}) \to \cdots \tag{1.1.2}$$

of reduced bounded cohomology.

Notice that if all boundary operators of the complexes **A**, **B**, and **C** have the closed ranges, then the sequences (1.1.1) and (1.1.2) are the same.

We recall some basic important properties of bounded cohomology.

Theorem 1.1. 1. *For* Γ *a discrete group or a space,* $\widehat{H}^1(\Gamma) = 0$.
2. *Let* X *be a connected countable cellular space. Then* $\widehat{H}^n(X)$ *and* $\widehat{H}^n(\pi_1 X)$ *are (isometrically) isomorphic for every* $n \geq 0$.
3. *Let* X *be simply connected space. Then* $\widehat{H}^*(X) = 0$ *for every* $n > 0$.
4. *If* N *is an amenable normal subgroup of* G*, then* $\widehat{H}^n(G)$ *and* $\widehat{H}^n(G/N)$ *are (isometrically) isomorphic for every* $n \geq 0$.
5. *If* G *is amenable, then* $\widehat{H}^n(G) = 0$ *for every* $n > 0$.

6. *Let G be a free group of rank greater than 1. Then $\widehat{H}^2(G)$ is infinite dimensional as a real vector space.*

The proof of the Theorem 1.1 and other properties of the theory of bounded cohomology can be found in [2] and [3].

As we see in Theorem 1.1, amenable groups play an important role in the theory of bounded cohomology.

Definition 1.3. A group G is called *amenable* if it has a left invariant mean on $B(G)$, that is, there exists a linear functional $m\colon B(G) \to \mathbb{R}$ such that

(1) $m(1_G) = 1$, where 1_G denotes the constant function with value 1,

(2) $m(f) \geq 0$ whenever $f \geq 0$ for $f \in B(G)$,

(3) $m(g \cdot f) = m(f)$,

where, for $g \in G$, $g \cdot f \in B(G)$ is defined by $(g \cdot f)(x) = f(gx)$ for all $x \in G$.

Finite groups, abelian groups, solvable groups, a homomorphic image of an amenable group, and a finite direct product of amenable groups are known to be amenable. On the other hand, it is known that a group containing a free group of rank 2 is not amenable [2].

Notice that, by Theorem 1.1-(3), bounded cohomology of a space X and its fundamental group coincide, and so it is not necessary to treat bounded cohomology of groups and spaces separately.

Using the facts in Theorem 1.1, we examine two forms of Universal Coefficient Theorem on bounded cohomology. For this we define bounded cohomology with coefficients in an abelian group A, where $A = \mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$.

By the same method as for $B^*(G)$, let $B^n(G, A)$ be the space of all bounded functions $f\colon G^n \to A$, that is,

$$B^n(G, A) = \{\, f\colon G^n \to A \mid \|f\| < \infty \,\}.$$

Then, with the same boundary operator as $B^*(G)$ in (1.1), the space $\{B^*(G, A), d_*\}$ forms a complex.

Notice that $B^*(G, \mathbb{R}/\mathbb{Z}) = C^*(G, \mathbb{R}/\mathbb{Z})$ and so

$$\widehat{H}^*(G, \mathbb{R}/\mathbb{Z}) = H^*(G, \mathbb{R}/\mathbb{Z}).$$

For a space X, similarly to the case of groups we can define $B^*(X, A)$ the bounded cochain group of X with coefficients A, that is,

$$B^n(X, A) = \{f \in C^n(X, A) \mid \|f\| < \infty\},$$

where $\|f\| = \sup\{|f(\sigma)| \,|\, \sigma \in S_n(X)\}$. Then with the same boundary operators as in $B^*(X)$, it is clear that the space $\{B^*(X,\ A)\}$ is a complex.

Definition 1.4. Let Γ be a space or a group. The n-th cohomology of the complex $\{B^*(\Gamma, \mathbb{Z})\}$ is called *the n-th bounded cohomology of* Γ *with coefficients* $\mathbb{Z}$ and is denoted by $\widehat{H}^n(\Gamma, \mathbb{Z})$.

The one form known as the dual Universal Coefficient Theorem relates homology and cohomology. In the ordinary homology and cohomology, it formulates as follows [9].

Theorem 1.2. *Let G be a group and A be an abelian group. Then, for every $n \geq 0$, there is a natural short exact sequence*

(1.2.1)

$$0 \to Ext\,(H_{n-1}(G, \mathbb{Z}), A) \to H^n(G, A) \to Hom_{\mathbb{Z}}\,(H_n(G, \mathbb{Z}), A) \to 0.$$

To see whether the dual Universal Coefficient Theorem holds in bounded cohomology, we define ℓ_1 homology as for the dual concept of bounded cohomology.

For simplicity, we consider only the case $A = \mathbb{Z}$ or $\mathbb{R}$. Recall that the n-th chain group $C_n(G,\ A)$ of a group G with coefficients A has A- basis consisting of n-tuples $[g_1|\cdots|g_n]$ of elements in G, so that

$$C_n(G, A) = \left\{ \sum_{i=1}^{k} a_i[g_{i_1}|\cdots|g_{i_n}],\ \text{where } a_i \in A \right\}.$$

The boundary operator $\partial_n : C_n(G,\ A) \to C_{n-1}(G,\ A)$ for $n \geq 2$ is given by the formula

$$\partial_n([g_{i_1}|\cdots|g_{i_n}]) = [g_{i_2}|\cdots|g_{i_n}] + \sum_{j=1}^{n-1}(-1)^j[g_{i_1}|\cdots|g_{i_j}g_{i_{j+1}}|\cdots|g_{i_n}] + (-1)^n[g_{i_1}|\cdots|g_{i_{n-1}}]$$

and $\partial_1 = \partial_0 = 0$. The homology of $\{C_*(G,\ A), \partial_*\}$ is called the homology of G with coefficients A and denoted by $H_*(G,\ A)$.

We define the norm on $C_*(G,\ A)$ by

$$\| \sum_{i=1}^{k} a_i[g_{i_1}|\cdots|g_{i_n}] \|_1 = \sum_{i=1}^{k} |a_i|.$$

Let

$$C_n^{\ell_1}(G,\ A) = \left\{ \sum_{i=1}^{\infty} a_i[g_{i_1}|\cdots|g_{i_n}] \,|\, \sum_{i=1}^{\infty} |a_i| < \infty \right\}.$$

It is easy to check that $C_*^{\ell_1}(G, A)$ with the same boundary operators ∂_* as in $C_*(G, A)$ is a complex.

Definition 1.5. The homology of the complex $C_*^{\ell_1}(G, A)$ is called *ℓ_1-homology of G with coefficients A* and is denoted by $H_*^{\ell_1}(G, A)$. In particular, $H_*^{\ell_1}(G, \mathbb{R})$ is called *the ℓ_1 homology of G with real coefficients* and denoted by $H_*^{\ell_1}(G)$.

It is easy to see that $C_*^{\ell}(G, \mathbb{Z})$ and $C_*(G, \mathbb{Z})$ are the same as the abelian groups. Thus $H_*^{\ell_1}(G, \mathbb{Z}) = H_*(G, \mathbb{Z})$. Meanwhile, for each $n \geq 0$,

$$(C_n(G, \mathbb{Z}))^* = \mathrm{Hom}_{\mathbb{Z}}(C_n(G, \mathbb{Z}), \mathbb{Z}) = C^n(G, \mathbb{Z})$$
$$\left(C_n^{\ell_1}(G, \mathbb{Z})\right)' = B^n(G, \mathbb{Z}),$$

where $(\cdot)^*$ and $(\cdot)'$ denote the algebraic and norm dual respectively.

Notice that, the spaces $C_*^{\ell_1}(G, \mathbb{R})$ turn into Banach spaces as the norm completion of $C_n(G, \mathbb{R})$. Also notice that, for each $n \geq 0$,

$$\left(C_n^{\ell_1}(G, \mathbb{R})\right)' = B^n(G).$$

In [5], it is proved that $\widehat{H}^{n+1}(G)$ is a Banach space if $C_*^{\ell_1}(G)$ satisfies n-uniform boundary condition.

Definition 1.6. A normed chain complex $C_*^{\ell_1}(G)$ is said to satisfy *n-uniform boundary condition* if there is a number $K > 0$ such that for any boundary $z \in \mathrm{Im}\ \partial_{n+1}$ there is a chain $c \in C_{n+1}^{\ell_1}(G)$ satisfying $\partial_{n+1} c = z$ and $\|c\|_1 \leq K\|z\|_1$.

Remark 1.2. The dual Universal Coefficient Theorem does not hold in bounded cohomology.

If it holds in bounded cohomology, from the sequence (1.2.1), there must be an exact sequence as follows:
(1.2.2)

$$0 \to \mathrm{Ext}\left(H_{n-1}^{\ell_1}(G, \mathbb{Z}), \mathbb{R}\right) \to \widehat{H}^n(G, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}\left(H_n^{\ell_1}(G, \mathbb{Z}), \mathbb{R}\right) \to 0.$$

Recall that $\widehat{H}^1(G, \mathbb{R}) = 0$ for every group G, and $H_*^{\ell_1}(G, \mathbb{Z}) = H_*(G, \mathbb{Z})$.

Let $G = \mathbb{Z} * \mathbb{Z}$ a free group of rank 2. As it is well known,

$$H_1(G, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \text{ and so } \mathrm{Hom}_{\mathbb{Z}}(H_1^{\ell_1}(G, \mathbb{Z}), \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}.$$

Then, it is clear that the sequence (1.2.2) can not be exact in general even in the degree $n = 1$. Also, as for $n = 2$, we know $H_2(\mathbb{Z} * \mathbb{Z}, \mathbb{Z}) = 0$ and so $\mathrm{Hom}_{\mathbb{Z}}\left(H_n^{\ell_1}(\mathbb{Z} * \mathbb{Z}, \mathbb{Z}), \mathbb{R}\right) = 0$. However, $\widehat{H}^2(\mathbb{Z} * \mathbb{Z}, \mathbb{R})$ is infinite dimensional from Theorem 1.1. Thus the sequence (1.2.2) can not be exact in degree 2.

Even more, if G is amenable and so $\widehat{H}^n(G, \mathbb{R}) = 0$ for every $n \geq 1$, the sequence (1.2.2) is hardly exact in every degree $n \geq 1$.

We consider another version of Universal Coefficient Theorem for a topological space X.

Theorem 1.3. *Let A be an abelian group and X be a space of finite type. Then, there is a natural short exact sequence*

(1.3.1)

$$0 \to H^n(X,\, \mathbb{Z}) \otimes_{\mathbb{Z}} A \to H^n(X,\, A) \to \mathrm{Tor}_1^{\mathbb{Z}}\left(H^{n+1}(X,\, \mathbb{Z}),\, A\right) \to 0.$$

In the sequence (1.3.1), we let $A = \mathbb{R}$. Recall that $\widehat{H}^*(X)$ and $\widehat{H}^*(\pi_1 X)$ are isomorphic from Theorem 1.1. Hence, for a group G such that $H_*(G,\, \mathbb{Z})$ is finitely generated, we may write the corresponding formula to (1.3) for bounded cohomology as follows:

(1.3.2)

$$0 \to \widehat{H}^n(G,\, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{H}^n(G,\, \mathbb{R}) \to \mathrm{Tor}_1^{\mathbb{Z}}\left(\widehat{H}^{n+1}(G,\, \mathbb{Z}),\, \mathbb{R}\right) \to 0.$$

To check the exactness of the sequence (1.3.2) with a group $G = \mathbb{Z}$, we compute $\widehat{H}^n(\mathbb{Z},\, \mathbb{Z})$.

We consider the exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 1.$$

Let Γ denote either a group or a space. Then there is an induced exact sequence of abelian groups [7]

(1.3)

$$\cdots \to \widehat{H}^n(\Gamma,\, \mathbb{R}) \to \widehat{H}^n(\Gamma,\, \mathbb{R}/\mathbb{Z})$$
$$\to \widehat{H}^{n+1}(\Gamma,\, \mathbb{Z}) \to \widehat{H}^{n+1}(\Gamma,\, \mathbb{R}) \to \widehat{H}^{n+1}(\Gamma,\, \mathbb{R}/\mathbb{Z}) \to \widehat{H}^{n+2}(\Gamma,\, \mathbb{Z}) \to \cdots.$$

Notice that $\widehat{H}^*(\Gamma,\, \mathbb{R}/\mathbb{Z}) = H^*(\Gamma,\, \mathbb{R}/\mathbb{Z})$.

Let $\Gamma = \mathbb{Z}$. It is clear that $\widehat{H}^0(\mathbb{Z},\, \mathbb{Z}) = \mathbb{Z}$ and $\widehat{H}^0(\mathbb{Z},\, \mathbb{R}) = \mathbb{R}$. Since there are no nontrivial bounded homomorphisms $\mathbb{Z} \to \mathbb{Z}$ or $\mathbb{Z} \to \mathbb{R}$,

$$\widehat{H}^1(\mathbb{Z},\, \mathbb{Z}) = \widehat{H}^1(\mathbb{Z},\, \mathbb{R}) = 0.$$

Also, since $\mathbb{Z}$ is a free group of rank 1,

$$H^n(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \begin{cases} \mathbb{R}/\mathbb{Z}, & n = 0, 1 \\ 0, & n \geq 2. \end{cases}$$

Recall that $\mathbb{Z}$ is amenable. So, by Theorem 1.1,

$$\widehat{H}^n(\mathbb{Z},\mathbb{R}) = 0, \text{ for all } n \geq 1 .$$

Then for $n \geq 1$,

$$\widehat{H}^{n+1}(\mathbb{Z},\,\mathbb{Z}) = H^n(\mathbb{Z},\,\mathbb{R}/\mathbb{Z}).$$

This proves the following:

Theorem 1.4.

$$\widehat{H}^n(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{R}/\mathbb{Z}, & n = 2 \\ 0, & n = 1 \text{ or } n > 2. \end{cases}$$

Remark 1.3. The Universal Coefficient Theorem in (1.3.2) does not hold in bounded cohomology.

Let $G = \mathbb{Z}$. The sequence (1.3.2) is of the form

$$0 \to \widehat{H}^2(\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{H}^2(\mathbb{Z},\mathbb{R}) \to \mathrm{Tor}_1^{\mathbb{Z}}(\widehat{H}^3(\mathbb{Z},\mathbb{Z}),\mathbb{R}) \to 0.$$

Recall that $\widehat{H}^2(\mathbb{Z},\mathbb{R}) = \widehat{H}^3(\mathbb{Z},\mathbb{Z}) = 0$. However,

$$\widehat{H}^2(\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \neq 0.$$

This shows the sequence above is not exact.

We observe that there are some differences between bounded cohomology with *real coefficients* and the one with *integral coefficients*.

Remark 1.4. (1) Bounded cohomology of an amenable group with *real coefficients* is zero in every degree ≥ 1 by Theorem 1.1.(3). However, as we saw in $\widehat{H}^2(\mathbb{Z},\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$, this is not true for *integral coefficients*.
(2) From Theorem 1.1.(1), bounded cohomology of a space X with *real coefficients* is isomorphic with bounded cohomology of $\pi_1 X$. However, this is not true for *integral coefficients*.

For a space or group Γ, we recall the exact sequence (1.3)

$$\cdots \to \widehat{H}^n(\Gamma,\mathbb{R}) \to \widehat{H}^n(\Gamma,\mathbb{R}/\mathbb{Z}) \to \widehat{H}^{n+1}(\Gamma,\mathbb{Z}) \to \widehat{H}^{n+1}(\Gamma,\mathbb{R}) \to \cdots$$

Suppose Γ is an amenable group or a space such that $\pi_1\Gamma$ is amenable. Then $\widehat{H}^n(\Gamma,\mathbb{R}) = 0$ for every $n \geq 1$ and so, for $n > 1$

$$\widehat{H}^n(\Gamma,\mathbb{Z}) \cong H^{n-1}(\Gamma,\mathbb{R}/\mathbb{Z}).$$

In the ordinary cohomology, the cohomology groups of a space X and of its fundamental group $\pi_1 X$ are not isomorphic in general. We consider the space $\Gamma = S^2$, a 2-sphere. Recall that S^2 is simply connected and so $\pi_1 S^2$ is trivial. Hence $\widehat{H}^3(\pi_1 S^2, \mathbb{Z}) = H^2(\pi_1 S^2, \mathbb{R}/\mathbb{Z}) = 0$. However,

$$\widehat{H}^3(S^2, \mathbb{Z}) = H^2(S^2, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}.$$

Thus bounded cohomology of a space with integral coefficients does not depend on its fundamental group.
(3) The fact $\widehat{H}^3(S^2, \mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ from (2) above indicates bounded cohomology of a simply connected space with integral coefficients is not necessarily zero, contrary to Theorem 1.1.(2).

The Universal Coefficient Theorem gives us a glimpse of the Künneth Theorem. One of applications of Künneth Theorem is that, for spaces or groups Γ_1 and Γ_2, whether there is an isomorphism

(1.4)

$$H^n(\Gamma_1 \times \Gamma_2, A \otimes_{\mathbb{Z}} B) \cong \left(\bigoplus_{p+q=n} H^p(\Gamma_1, A) \otimes_{\mathbb{Z}} H^q(\Gamma_2, B) \right) \bigoplus \left(\bigoplus_{p+q=n+1} \operatorname{Tor}_1(H^p(\Gamma_1, A), H^q(\Gamma_2, B)) \right),$$

where A and B be abelian groups. We check the sequence (1.4) in case of bounded cohomoloogy. Let $\Gamma_1 = \Gamma_2 = \mathbb{Z}$, and $A = \mathbb{Z}$ and $B = \mathbb{R}$. Recall that $\mathbb{Z}$ is amenable and so $\widehat{H}^n(\mathbb{Z}, \mathbb{R}) = 0$ for every $n \geq 1$. Then the Tor term will be zero. Also, since $\mathbb{Z} \times \mathbb{Z}$ is amenable,

$$\widehat{H}^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}) = \widehat{H}^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{R}) = 0.$$

However,

$$\bigoplus_{p+q=2} \widehat{H}^p(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{H}^q(\mathbb{Z}, \mathbb{R}) = \widehat{H}^2(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{H}^0(\mathbb{Z}, \mathbb{R}) = \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Hence the groups $\bigoplus_{p+q=2} \widehat{H}^p(\mathbb{Z}, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{H}^q(\mathbb{Z}, \mathbb{R})$ and $\widehat{H}^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R})$ are not isomorphic.

In case of degree 2 in bounded cohomology with real coefficients, the sequence (1.4) is more interesting. As the $\widehat{H}^1(G, \mathbb{R}) = 0$ for every group G, the sequence (1.4) is to see if there is an isomorphism

$$\widehat{H}^2(\Gamma_1 \times \Gamma_2) \cong \widehat{H}^2(\Gamma_1) \otimes \widehat{H}^2(\Gamma_2).$$

As a special case, this is proved for compactly generated locally compact second countable groups in [7].

Recall that Künneth Theorem for cohomology requires the conditions of finite type on the homology level for the spaces involved. So, to discuss Künneth type formula on bounded cohomology without mentioning ℓ_1-homology, we recall an algebraic Künneth Theorem for the ordinary cohomology [9]:

Theorem 1.5. *Let R be a principal ideal domain. Let $\boldsymbol{A}$ and $\boldsymbol{C}$ be cochain cochain complexes. If the spaces of cocycles and coboundaries of $\boldsymbol{A}$ are flat R-modules, then there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H^p(\boldsymbol{A}) \otimes_R H^q(\boldsymbol{C}) \to H^n(\boldsymbol{A} \otimes_R \boldsymbol{C}) \to \bigoplus_{p+q=n+1} Tor_1^R(H^p(\boldsymbol{A}), H^q(\boldsymbol{C}) \to 0. \tag{1.5.1}$$

Since bounded cohomology with integral coefficients loses several nice properties that hold in real coefficients as we saw in Remark 1.4, we consider bounded cohomology with real coefficients and its real Banach space structure. If we consider only the real vector space structure on bounded cohomology, it is in an abelian category and so we get the similar result as in Theorem 1.5. However, in this case, the norms on bounded cohomology will not play their role properly. Hence, considering its structure of Banach space would be preferable. On the other hand, as mentioned it earlier, the standard method of homological algebra for an abelian category does not work directly on bounded cochains as Banach spaces. The most crucial is that the ranges of boundary operators of a Banach complex are not necessarily closed and so not Banach spaces. So, contrary to the usual case, there are no short exact sequences of Banach spaces induced from the boundary operators. To make up for these defect, in Section 2, we review some proper conditions for the category of Banach spaces following [1]. In Section 3, we investigate Künneth formula on Banach complex and so on Bounded cohomology.

2. The category of Banach spaces

In this section, we summarize some basic facts for the category of Banach spaces for our own use by avoiding categorical terms as much as possible. The details including proofs can be found in [1].

For the sake of bounded cohomology with real coefficients, we fix the ground field by $\mathbb{R}$. So a Banach space over $\mathbb{R}$ will be called a Banach space for short. To avoid some set-theoretical difficulties, we suppose all Banach spaces are small, that is, belong to some given universe. Also, we treat the zero vector space as a Banach space.

Let $\mathcal{B}an$ be the category of Banach spaces and bounded linear maps.

Remark 2.1. $\mathcal{B}an$ is an additive category possessing only finite product and coproduct. However, we remark that there is no infinite product and coproducts in $\mathcal{B}an$ (See [1]).

Let I be a finite index set and $\{E_i\}_{i\in I}$ be a family of Banach spaces equipped with the norm $\|\cdot\|_{E_i}$. Then it is easy to check the linear spaces

$$(1)\, \mathbf{E}_\infty = \{x = (x_i)_{i\in I} \mid \|x\|_\infty < \infty\}, \text{ where } \|x\|_\infty = \sup_{i\in I} \|x_i\|_{E_i},$$

and

$$(2)\, \mathbf{E}_1 = \{x = (x_i)_{i\in I} \mid \|x\|_1 < \infty\}, \quad \text{where} \quad \|x\|_1 = \sum_{i\in I} \|x_i\|_{E_i},$$

are Banach spaces. Also $\mathbf{E}_\infty$ and $\mathbf{E}_1$ are the product and coproduct of $\{E_i\}_{i\in I}$, respectively. In particular, as I is finite, $\mathbf{E}_\infty$ and $\mathbf{E}_1$ are isomorphic as Banach spaces.

Proposition 2.1. *(1) For a morphism $f : U \to V$ in $\mathcal{B}an$ the kernel of f exists and is identified with the space*

$$\{u \in U \mid f(u) = 0\}.$$

Also, the cokernel of f exists and is identified with the space

$$V/\overline{\{f(u) \mid u \in U\}},$$

where $\overline{(\cdot)}$ denotes the norm closure.
(2) All finite (co)-limits exist.

The proof can be found in the section IV.2.1. in [1].

Recall that an additive category such that every morphism has a kernel and cokernel is called preabelian and so $\mathcal{B}an$ is preabelian.

For a morphism $f : U \to V$ in $\mathcal{B}an$, we will denote $\{u \in U| \mid f(u) = 0\}$ by Ker f and $V/\overline{\{f(u) \mid u \in U\}}$ by Cok f. Then Ker f is equipped with a subspace topology of U and Cok f with a quotient topology.

Remark 2.2. Let $f : U \to V$ be a morphism in $\mathcal{B}an$.
(1) As categorical term, since image is defined as the kernel of a cokernel, the image of f is defined by $\overline{f(U)}$. We denote it by Im f. Notice that Im f is closed and so a Banach space.
(2) As categorical term, since coimage is defined as the cokernel of a

kernel, the coimage of f is defined by $U/(\mathrm{Ker}\ f)$. We denote it by Coim f. Notice that Coim f is closed and so a Banach space.

For a morphism $f : U \to V$ in $\mathcal{B}an$, notice that the following sequences

$$0 \to \mathrm{Ker}\ f \to U \to \mathrm{Im}\ f = \overline{f(U)} \to 0 \quad \text{and}$$
$$0 \to \mathrm{Ker}\ f \to U \to \text{range of } f = f(U) \to 0$$

are not exact in $\mathcal{B}an$.

To work with short exact sequences, we introduce the exact structure on $\mathcal{B}an$ following Definition IV. 2.3.1 in [1].

Definition 2.1. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{B}an$. is called *a kernel-cokernel pair* if $f = \ker g$ and $g = \mathrm{cok}\ f$. In this case, f is called *an admissible mono and g an admissible epi.*

Remark 2.3. Let $f\colon X \to Y$ be a morphism in $\mathcal{B}an$. Then f is a kernel if and only if f is topologically injective, that is, f is injective and has a closed range. Also, by Open mapping Theorem, f is a cokernel if and only if f is surjective.

Definition 2.2. *An exact structure on $\mathcal{B}an$* is a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{B}an$ that satisfies the following axioms:

Axiom 1: $\mathcal{E}$ is closed under isomorphisms.

Axiom 2: For each object X in $\mathcal{C}$, the identity $\mathrm{id}_X : X \to X$ is an admissible monomorphism and an admissible epimorphism.

Axiom 3: The compositions of admissible monomorphisms is an admissible monomorphism.

Axiom 3$^{\mathrm{op}}$: The compositions of admissible epimorphisms is an admissible epimorphism.

Axiom 4: If $f : X \to Y$ is an admissible monomorphism and $t : X \to T$ is a morphism, then the pushout of f and t exists and its coprojection from T is also an admissible monomorphism.

Axiom 4$^{\mathrm{op}}$: If $g : Y \to Z$ is an admissible epimorphism and $s : S \to Z$ is a morphism, then the pullback of g and s exists and its projection to S is also an admissible epimorphism.

We call the pair $(\mathcal{B}an, \mathcal{E})$ *an exact category.*

In [1], it is shown that an exact structure on $\mathcal{B}an$ is not unique and the following:

Proposition 2.2. *Let $\mathcal{E}_{\max}$ be the class of all kernel-cokernel pais in $\mathcal{B}an$. Then $\mathcal{E}_{\max}$ is the largest exact structure on $\mathcal{B}an$.*

From now on, we consider the exact structure $\mathcal{E}_{\max}$ only on $\mathcal{B}an$ and so denote it by $\mathcal{E}$ in short.

Proposition 2.3. *Let $f\colon X \to Y$ be a morphism in $\mathcal{B}an$. Then*

1. *f is a kernel if and only if a sequence $X \xrightarrow{f} Y \to Y/f(X)$ is in $(\mathcal{B}an, \mathcal{E})$,*
2. *f is a cokernel if and only if a sequence Ker $f \to X \xrightarrow{f} Y$ is in $(\mathcal{B}an, \mathcal{E})$.*

Proof. From Remark 2.3, f is a kernel if and only if X and $f(X)$ are topologically injective. Hence X and $f(X)$ is isomorphic as Banach spaces and so $f(X) = \overline{f(X)} = \operatorname{Im} f$. Then $\operatorname{Cok} f = Y/(\operatorname{Im} f) = Y/\overline{f(X)} = Y/f(X)$. Also, from Remark 2.3, f is a cokernel if and only if f is surjective. Hence $Y/(\operatorname{Ker} f)$ and Y are isomorphic as Banach spaces. □

Now we consider the tensor product on Banach spaces. For Banach spaces U and V, we consider their algebraic tensor product $U \otimes_{\mathbb{R}} V$, i.e., the tensor product of vector spaces over $\mathbb{R}$. Then we define the **projective tensor norm** $\|\cdot\|_\pi$ on $U \otimes_{\mathbb{R}} V$ as follows:

$$\text{for } \omega \in U \otimes_{\mathbb{R}} V, \|\omega\|_\pi = \inf\left\{\sum \|\alpha_i\|_U \|\beta_i\|_V, \text{ where } \omega = \sum \alpha_i \otimes \beta_i\right\},$$

where the infimum is taken over all representations of ω as a finite sum of elementary tensors. Notice that $\|\cdot\|_\pi$ defines a seminorm on $U \otimes_{\mathbb{R}} V$.

Definition 2.3. *The projective tensor product of Banach spaces U and V* is defined as the completion of $U \otimes_{\mathbb{R}} V$ with respect to $\|\cdot\|_\pi$. We denote it by $U \widehat{\otimes} V$.

Remark 2.4. We state some elementary properties of projective tensor product. Let $\mathcal{B}an(U, V)$ be the space of all bounded linear maps.

1. It is symmetric and associative monoidal structure on $\mathcal{B}an$.
2. $U \widehat{\otimes} \mathbb{R} = \mathbb{R} \widehat{\otimes} U = U$.
3. Let $f \in \mathcal{B}an(U, U_1)$ and $g \in \mathcal{B}an(V, V_1)$. Then the linear map $f \otimes g\colon U \otimes V \to U_1 \otimes V_1$ extends to bounded linear map $f \widehat{\otimes} g\colon U \widehat{\otimes} V \to U_1 \widehat{\otimes} V_1$. Furthermore, if $f_1 \in \mathcal{B}an(U_1, U_2)$ and $g_1 \in \mathcal{B}an(V_1, V_2)$, then $(f \widehat{\otimes} g) \circ (f_1 \widehat{\otimes} g_1) = (f \circ f_1) \widehat{\otimes} (g \circ g_1)$.
4. $\mathcal{B}an(U \widehat{\otimes} V, W) = \mathcal{B}an\left(U, \mathcal{B}an(V, W)\right)$.
5. It preserves colimits and so cokernels.
6. For a closed subspace A of X, if $f\colon X \to X/A$ is a quotient map, then
$$f \widehat{\otimes} 1_Y\colon X \widehat{\otimes} Y \to X/A \widehat{\otimes} Y$$

is also a quotient map. Also notice that the kernel of $f\widehat{\otimes}1_Y$ is the norm closure of $A\otimes Y$ in $X\widehat{\otimes}Y$.

Definition 2.4. A Banach space V is called *flat* if the functor

$$V\widehat{\otimes}\bullet : (\mathcal{B}an,\mathcal{E}) \to (\mathcal{B}an,\mathcal{E})$$

is exact.

3. Künneth type formula

In this section we formulate the Künneth type formula in the exact category $(\mathcal{B}an,\mathcal{E})$ first, then in the bounded cohomology.

Proposition 3.1. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a morphism in $\mathcal{B}an$ such that $g\circ f = 0$ and f has a closed range. Then the sequences*

$$\textit{Ker } f \to X \to \textit{Im } f \quad \text{and} \quad \textit{Im } f \to \textit{Ker } g \to \textit{Ker } g/\textit{Im } f$$

are in $(\mathcal{B}an,\mathcal{E})$.

Proof. Since f has a closed range, $f(X) = \overline{f(X)} = \text{Im } f$. Then $X \xrightarrow{f} f(X) = \text{Im } f$ is surjective and also $X/\text{Ker } f$ and $\text{Im } f$ are isomorphic as Banach spaces. Also notice that $\text{Im } f$ is a closed subspace of $\text{Ker } f$. Hence it follows from Proposition 2.3. □

Proposition 3.2. *Let the Banach complex $(\boldsymbol{U}, d_*)$ such that every boundary operator d_n has a closed range for $n \geq 0$. Let V be a flat Banach space. Then, for every $n \geq 0$*

$$V\widehat{\otimes}\, H^n(\boldsymbol{U}) = H^n(V\widehat{\otimes}\, \boldsymbol{U}).$$

Proof. Consider the sequence

$$\{V\widehat{\otimes}\mathbf{U}, 1_V\widehat{\otimes}d_*\} : 0 \xrightarrow{0} V\widehat{\otimes}U^0 \xrightarrow{1_V\widehat{\otimes}d_0} V\widehat{\otimes}U^1 \xrightarrow{1_V\widehat{\otimes}d_1} V\widehat{\otimes}U^2 \xrightarrow{1_V\widehat{\otimes}d_2} \cdots .$$

Since $(1_V\widehat{\otimes}d_{k+1})\circ(1_V\widehat{\otimes}d_k) = 1_V\widehat{\otimes}(d_{k+1}\circ d_k) = 0$, it is a Banach complex.

Since every range of d_n is closed for $n \geq 0$, from Proposition 3.1 the sequences

$$\text{Ker } d_n \xrightarrow{i_n} U^n \to \text{Im } d_n \quad \text{and} \quad \text{Im } d_{n-1} \xrightarrow{j_n} \text{Ker } d_n \to H^n(\mathbf{U})$$

are in $(\mathcal{B}an,\mathcal{E})$. Then, since V is flat, by definition the following sequences

$$V\widehat{\otimes}\text{Ker } d_n \xrightarrow{1_V\widehat{\otimes}i_n} V\widehat{\otimes}U^n \to V\widehat{\otimes}\text{Im } d_n \tag{3.2.1}$$

$$V\widehat{\otimes}\text{Im } d_{n-1} \xrightarrow{1_V\widehat{\otimes}j_n} V\widehat{\otimes}\text{Ker } d_n \to V\widehat{\otimes}H^n(\mathbf{U}) \tag{3.2.2}$$

are in $(\mathcal{B}an, \mathcal{E})$ and so are kernel-cokernel pairs in $\mathcal{B}an$. Notice that $V\widehat{\otimes}\text{Ker } d_n$ and $V\widehat{\otimes}\text{Im } d_{n-1}$ are closed subspaces in $V\widehat{\otimes}U^n$ and $V\widehat{\otimes}\text{Ker } d_n$, respectively. Hence the inclusion morphisms $1_V\widehat{\otimes}i_n$ and $1_V\widehat{\otimes}j_n$ are closed. From the sequence (3.2.2), we get

$$\begin{aligned} V\widehat{\otimes}H^n(\mathbf{U}) &= \text{Cok } \big(1_V\widehat{\otimes}j_n\big) \\ &= \big(V\widehat{\otimes}\text{Ker } d_n\big) \,/\, \big(\text{Im } 1_V\widehat{\otimes}j_n\big) \\ &= \big(V\widehat{\otimes}\text{Ker } d_n\big) \,/\, \big(V\widehat{\otimes}\text{Im } d_{n-1}\big) . \end{aligned}$$

Also, from the sequence (3.2.1), we get

$$\begin{aligned} V\widehat{\otimes}\text{Ker } d_n &= \ker[V\widehat{\otimes}U^n \to V\widehat{\otimes}\text{Im}d_n] \\ &= \ker[V\widehat{\otimes}U^n \xrightarrow{1_V\widehat{\otimes}d_n} V\widehat{\otimes}U^{n+1}]. \end{aligned}$$

Notice that the morphism $1_V\widehat{\otimes}d_{n-1} : V\widehat{\otimes}U^{n-1} \to V\widehat{\otimes}U^n$ is the composition

$$V\widehat{\otimes}U^{n-1} \to V\widehat{\otimes}\text{Im } d_{n-1} \to V\widehat{\otimes}\text{Ker } d_n \to V\widehat{\otimes}U^n.$$

The first morphism is surjective with closed range from the sequence (3.2.1) for boundary operator d_{n-1} . The second and third ones are inclusion from the sequences (3.2.2) and (3.2.1). Hence Im $(1_V\otimes d_{n-1}) = V\widehat{\otimes}\text{Im } d_{n-1}$ and so $1_V\widehat{\otimes}d_{n-1}$ has a closed range. This shows that

$$\begin{aligned} V\widehat{\otimes}H^n(\mathbf{U}) &= V\widehat{\otimes}\text{Ker } d_n/V\widehat{\otimes}\text{Im } d_{n-1} \\ &= \text{Ker } (1_V\widehat{\otimes}d_n)/\text{Im } (1_V\widehat{\otimes}d_{n-1}) \\ &= H^n(V\widehat{\otimes}\mathbf{U}). \end{aligned}$$

□

Recall that, as mentioned it earlier, there is no infinite (co)product in $\mathcal{B}an$ in general.

Definition 3.1. We say a Banach cocomplex $\{\mathbf{U} = \{U^*\}, d_*\}$ is *finite dimensional* if $U^n = 0$ for sufficiently large n. Similarly, we say $H^*(\mathbf{U})$ is *finite dimensional* if $H^n(\mathbf{U}) = 0$ for sufficiently large n.

Let $\{\mathbf{U}, d_*\}$ and $\{\mathbf{V}, \partial_*\}$ be finite dimensional Banach complexes. As usual, we can form the bicomplex $\{U^p\widehat{\otimes}V^q\}$ of Banach spaces with boundaries $1_U\widehat{\otimes}\partial_q$ and $d_p\widehat{\otimes}1_V$. For each n, we let

$$(\mathbf{U}\widehat{\otimes}\mathbf{V})^n = \oplus_{p+q=n}(U^p\widehat{\otimes}V^q)$$

and define $\Delta_n\colon (\mathbf{U}\widehat{\otimes}\mathbf{V})^n \to (\mathbf{U}\widehat{\otimes}\mathbf{V})^{n+1}$ by the formula

$$\Delta_n(u_p\widehat{\otimes}v_q) = du_p\widehat{\otimes}v_q + (-1)^p u_p\widehat{\otimes}\partial(v_q)$$

for an elementary summand $u_p\widehat{\otimes}v_q \in U^p\widehat{\otimes}V^q$.

Definition 3.2. Let $\{\mathbf{U}, d_*\}$ and $\{\mathbf{V}, \partial_*\}$ be the finite dimensional Banach complexes. Their total complex $\mathbf{U}\widehat{\otimes}\mathbf{V}$ is defined by $\{(\mathbf{U}\widehat{\otimes}\mathbf{V})^*,\ \Delta_*\}$ and is called *the tensor product of* $\{\boldsymbol{U}, d_*\}$ *and* $\{\boldsymbol{V}, \partial_*\}$.

Notice that, as $\{\mathbf{U}, d_*\}$ and $\{\mathbf{V}, \partial_*\}$ are finite dimensional, so is the total complex $\mathbf{V}\widehat{\otimes}\mathbf{U}$.

Proposition 3.3. *Let $\{\boldsymbol{A}, \partial_*\}$ be a finite dimensional Banach complex having zero differentiations $\partial_* = 0$ and every A^p is flat. Let $\{\boldsymbol{U}, d_*\}$ be a finite dimensional Banach complex such that the images of d_* are closed. Then, for each $n \geq 0$,*

$$\bigoplus_{p+q=n} A^p\widehat{\otimes}H^q(\boldsymbol{U}) = H^n(\boldsymbol{A}\widehat{\otimes}\boldsymbol{U}).$$

Proof. Notice that the sequences

$$\text{Ker } d_q \to U^q \to \text{Im } d_q \quad \text{and} \quad \text{Im } d_{q-1} \to \text{Ker } d_q \to H^q(\mathbf{U})$$

are in $(\mathcal{B}an, \mathcal{E})$. Since A^p is flat, the sequences

$$A^p\widehat{\otimes}\text{Ker } d_q \to A^p\widehat{\otimes}U^q \to A^p\widehat{\otimes}\text{Im } d_q \tag{3.3.1}$$

$$A^p\widehat{\otimes}\text{Im } d_{q-1} \to A^p\widehat{\otimes}\text{Ker} d_q \to A^p\widehat{\otimes}H^q(\mathbf{U}) \tag{3.3.2}$$

are in $(\mathcal{B}an, \mathcal{E})$ and so form kernel-cokernel pairs.

Since the boundary operators of $\mathbf{A}$ is zero, they have closed ranges and the boundary operator of

$$\left(\mathbf{A}\widehat{\otimes}\mathbf{U}\right)^n \to \left(\mathbf{A}\widehat{\otimes}\mathbf{U}\right)^{n+1}$$

is $(\pm)1_{\mathbf{A}}\widehat{\otimes}d_*$. Then from the sequences (3.3.1) and (3.3.2) we get

$$\text{Ker } (1_{A^p}\widehat{\otimes}d_q) = A^p\widehat{\otimes}\text{Ker } d_q \quad \text{and} \quad \text{Im}(1_{A^p}\widehat{\otimes}d_q) = A^p\widehat{\otimes}\text{Im } d_q.$$

This shows that $\text{Im}(1_{A^p}\widehat{\otimes}d_q)$ is closed. Hence

$$\begin{aligned} A^p\widehat{\otimes}H^q(\mathbf{U}) &= A^p\widehat{\otimes}\text{Ker } d_q/A^p\widehat{\otimes}\text{Im } d_{q-1} \\ &= \text{Ker } (1_{A^p}\widehat{\otimes}d_q)/\text{Im } (1_{A^p}\widehat{\otimes}d_{q-1}) \\ &= H^q(A^p\widehat{\otimes}\mathbf{U}). \end{aligned}$$

Taking the direct sum over $p + q = n$, we have

$$\bigoplus_{p+q=n} A^p\widehat{\otimes}H^q(\mathbf{U}) = H^n(\mathbf{A}\widehat{\otimes}\mathbf{U}).$$

□

As we consider Banach space structure on bounded cohomology, we first prove Künneth theorem on the cohomology of a Banach complex following the proof of the Künneth Theorem in [9].

Theorem 3.4. *Let* $\{\boldsymbol{U}, d_*\}$ *and* $\{\mathbf{V}, \partial_*\}$ *be a finite dimensional Banach complexes satisfying the following conditions:*

1. *boundary operators* d_* *and* ∂_* *have closed ranges,*
2. *every Im* d_* *and* $H^*(U)$ *are flat.*

Then there is an isomorphism

$$\bigoplus_{p+q=n} H^p(\boldsymbol{U})\widehat{\otimes}H^q(\mathbf{V}) \cong H^n(\boldsymbol{U}\widehat{\otimes}\mathbf{V})$$

as Banach spaces.

Proof. First, recall that $(\mathbf{U}\widehat{\otimes}\mathbf{V})^n = \bigoplus_{p+q=n} U^p\widehat{\otimes}V^q$. Since $\mathbf{U}$ and $\mathbf{V}$ are finite dimensional Banach complexes, their total complex $\mathbf{U}\widehat{\otimes}\mathbf{V}$ with boundary operators Δ_* as in Definition 3.2 is also a finite dimensional Banach complex.

As boundary operators d_* and ∂_* have closed ranges, the spaces Im d_*, Im ∂_*, $H^*(\mathbf{U})$, and $H^*(\mathbf{V})$ are all Banach spaces. For each $d_p\colon U^p \to U^{p+1}$, we let Ker $d_p = Z^p$ and Im $d_p = L^{p+1}$. We consider for each $p \geq 0$ the sequence

$$Z^p \xrightarrow{i_p} U^p \xrightarrow{d_p} L^{p+1} \tag{3.4.1}$$

$$L^p \xrightarrow{j_p} Z^p \xrightarrow{d_p} H^p(\mathbf{U}) \tag{3.4.2}$$

which are kernel-cokernel pairs in $\mathcal{B}an$. Since the spaces L^* and $H^*(\mathbf{U})$ are flat, by applying Corollary 2.5.3 (*ii*) in [1] to the sequence (3.4.2), the space Z^* is flat. Then, by applying it to the sequence (3.4.1), the space U^p is flat from the first sequence (3.4.1). Then, from Corollary 2.5.3 (*i*) in [1], for every V^q the sequence

$$Z^p\widehat{\otimes}V^q \xrightarrow{i_p\widehat{\otimes}1} U^p\widehat{\otimes}V^q \xrightarrow{d_p\widehat{\otimes}1} L^{p+1}\widehat{\otimes}V^q$$

is a kernel-cokernel pair in $\mathcal{B}an$ and so are in $(\mathcal{B}an, \mathcal{E})$. Notice that $\mathbf{Z} = \{Z^p\}$ and $\mathbf{L} = \{L^p\}$ are also finite dimensional Banach complexes with zero differentiations. So the spaces $\mathbf{Z}\widehat{\otimes}\mathbf{V}$ and $\mathbf{L}\widehat{\otimes}\mathbf{V}$ are finite dimensional Banach complexes with boundary operators $(\pm)1_{\mathbf{Z}}\widehat{\otimes}\partial_*$ and $(\pm)1_{\mathbf{L}}\widehat{\otimes}\partial_*$, respectively. Hence there is a short exact sequence

$$0 \to \mathbf{Z}\widehat{\otimes}\mathbf{V} \xrightarrow{i\widehat{\otimes}1} \mathbf{U}\widehat{\otimes}\mathbf{V} \xrightarrow{d\widehat{\otimes}1} \mathbf{L}_+\widehat{\otimes}\mathbf{V} \to 0$$

of Banach complexes, where $\mathbf{L}_+$ means that every L^* has one degree higher in the sequence (3.4.1). Notice that $H^n(\mathbf{L}_+\widehat{\otimes}\mathbf{V}) = H^{n+1}(\mathbf{L}\widehat{\otimes}\mathbf{V})$. By Remark 1.1, there is an exact sequence of Banach cohomology

$$\cdots \to H^n(\mathbf{L}\widehat{\otimes}\mathbf{V}) \xrightarrow{D_n} H^n(\mathbf{Z}\widehat{\otimes}\mathbf{V}) \xrightarrow{(i\widehat{\otimes}1)_n} H^n(\mathbf{U}\widehat{\otimes}\mathbf{V}) \xrightarrow{(d\widehat{\otimes}1)_n} H^{n+1}(\mathbf{L}\widehat{\otimes}\mathbf{V}) \xrightarrow{D_{n+1}} H^{n+1}(\mathbf{Z}\widehat{\otimes}\mathbf{V}) \to \cdots . \tag{3.4.3}$$

Since $\mathbf{Z}$ and $\mathbf{L}$ are flat with zero differentiations, from Proposition 3.3

$$H^n(\mathbf{Z}\widehat{\otimes}\mathbf{V}) = \big(\mathbf{Z}\widehat{\otimes}H^*(\mathbf{V})\big)^n = \oplus_{p+q=n} Z^p\widehat{\otimes}H^q(\mathbf{V})$$
$$H^n(\mathbf{L}\widehat{\otimes}\mathbf{V}) = \big(\mathbf{L}\widehat{\otimes}H^*(\mathbf{V})\big)^n = \oplus_{p+q=n} L^p\widehat{\otimes}H^q(\mathbf{V}).$$

We claim that $D_n\colon H^n(\mathbf{L}\widehat{\otimes}\mathbf{V}) \to H^n(\mathbf{Z}\widehat{\otimes}\mathbf{V})$ induced by the inclusion $\mathbf{L} = \mathrm{Im}\ d_* \xrightarrow{j_*} \mathbf{Z} = \mathrm{Ker}\ d_*$. Consider the following diagram

$$\begin{array}{ccccc}
(\mathbf{Z}\widehat{\otimes}\mathbf{V})^{n+1} & \xrightarrow{i\widehat{\otimes}1_{\mathbf{V}}} & (\mathbf{U}\widehat{\otimes}\mathbf{V})^{n+1} & \longrightarrow & (\mathbf{L}\widehat{\otimes}\mathbf{V})^{n+2} \\
{\scriptstyle (\pm)1_{\mathbf{Z}}\widehat{\otimes}\partial_*}\big\uparrow & & {\scriptstyle \Delta_n}\big\uparrow & & {\scriptstyle (\pm)1_{\mathbf{L}}\widehat{\otimes}\partial_*}\big\uparrow \\
(\mathbf{Z}\widehat{\otimes}\mathbf{V})^{n} & \longrightarrow & (\mathbf{U}\widehat{\otimes}\mathbf{V})^{n} & \longrightarrow & (\mathbf{L}\widehat{\otimes}\mathbf{V})^{n+1}.
\end{array}$$

Let $p+q=n$. An elementary factor of a typical cocycle in $(\mathbf{L}\widehat{\otimes}\mathbf{V})^{n+1}$ has the form $x\otimes y$, where $x \in L^{p+1}$ and $y \in V^{q-1}$. Notice that there is an element $u \in U^p$ such that $d_p(u) = x$. Then

$$\begin{aligned}
D(x\otimes y) &= (i\otimes 1_{\mathbf{V}})^{-1}\Delta(u\otimes y) \\
&= (i\otimes 1_{\mathbf{V}})^{-1}(d_p u\otimes y + (-1)^{p+1}u\otimes \partial_{q-1}y) \\
&= (i\otimes 1_{\mathbf{V}})^{-1}(x\otimes y + (-1)^{p+1}(1\otimes \partial_{q-1})(u\otimes y)).
\end{aligned}$$

This shows that $x\otimes y + (-1)^{p+1}(1\otimes\partial_{q-1})(u\otimes y)$ is in $(\mathbf{Z}\widehat{\otimes}\mathbf{V})^{n+1}$. Since the boundary operator of the complex $(\mathbf{Z}\widehat{\otimes}\mathbf{V})$ is $(\pm)1_{\mathbf{Z}}\otimes\partial_*$, the element $(-1)^{p+1}(1\widehat{\otimes}\partial_{q-1})(u\otimes y)$ is in $(\mathbf{Z}\widehat{\otimes}\mathbf{V})^n$ and so $x\otimes y+(-1)^{p+1}(1\widehat{\otimes}\partial_{q-1})(u\otimes y)$ and $x\otimes y$ are the same cohomology class. The claim is proved. Notice that we can rewrite the sequence (3.4.3) as

$$\cdots \to (\mathbf{L}\widehat{\otimes}H^*(\mathbf{V}))^n \xrightarrow{D_n} (\mathbf{Z}\widehat{\otimes}H^*(\mathbf{V}))^n \xrightarrow{(i\widehat{\otimes}1_V)_n} H^n(\mathbf{U}\widehat{\otimes}\mathbf{V}) \xrightarrow{(d\widehat{\otimes}1)_n} (\mathbf{L}\widehat{\otimes}H^*(\mathbf{V}))^{n+1} \xrightarrow{D_{n+1}} H^{n+1}(\mathbf{Z}\widehat{\otimes}\mathbf{V}) \to \cdots . \tag{3.4.4}$$

Finally, recall the sequence (3.4.2)

$$L^p \to Z^p \to H^p(\mathbf{U}).$$

Since Z^p and $H^p(\mathbf{U})$ are flat for every $p \geq 0$, by Corollary 2.5.3 (i) in [1], for every $H^q(\mathbf{V})$ the sequence

$$L^p \widehat{\otimes} H^q(\mathbf{V}) \to Z^p \widehat{\otimes} H^q(\mathbf{V}) \to H^p(\mathbf{U}) \widehat{\otimes} H^q(\mathbf{V})$$

is in $\mathcal{E}$ and so a kernel-cokernel pair in $\mathcal{B}an$. For every $n \geq 0$, there is a sequence

(3.4.5)

$$\begin{array}{ccc} 0 \to (\mathbf{L}\widehat{\otimes} H^*(\mathbf{V}))^n & \xrightarrow{D_n} & (\mathbf{Z}\widehat{\otimes} H^*(\mathbf{V}))^n \to H^*(\mathbf{U})\widehat{\otimes} H^*(\mathbf{V}))^n \to 0 \\ \| & & \| \\ H^n(\mathbf{L}\widehat{\otimes}\mathbf{V}) & \xrightarrow{D_n} & H^n(\mathbf{Z}\widehat{\otimes}\mathbf{V}) \end{array}$$

which is a kernel-cokernel pair in $\mathcal{B}an$. This shows that in the sequence (3.4.4) the kernel of every D_n is zero and so Im $(d\widehat{\otimes}1_V)_n = 0$. Then $i\widehat{\otimes}1_V$ in the sequence (3.4.4) is surjective, so that there is an isomorphism of Banach spaces

$$H^n(\mathbf{U}\widehat{\otimes}\mathbf{V}) \cong \left(\mathbf{Z}\widehat{\otimes} H^*(\mathbf{V})\right)^n / \text{Ker } (i\widehat{\otimes}1_V)_n. \tag{3.4.6}$$

Notice that from the exact sequence (3.4.4), since D_n is injective as an inclusion, we have

$$\text{Ker } (i\widehat{\otimes}1_V)_n = \text{Im } D_n = (\mathbf{L}\widehat{\otimes} H^q(\mathbf{V}))^n.$$

From the sequence (3.4.5), the space $(\mathbf{L}\widehat{\otimes} H^q(\mathbf{V}))^n$ and so Im D_n is closed. This shows that there is an isomorphism of Banach spaces

$$\begin{aligned} H^n(\mathbf{U}\widehat{\otimes}\mathbf{V}) &\cong \left(\mathbf{Z}\widehat{\otimes} H^*(\mathbf{V})\right)^n / \text{Ker } (i\widehat{\otimes}1_V)_n && \text{by (3.4.6)} \\ &\cong \left(\mathbf{Z}\widehat{\otimes} H^*(\mathbf{V})\right)^n / (\mathbf{L}\widehat{\otimes} H^*(\mathbf{V}))^n \\ &\cong \left(\oplus_{p+q=n} Z^p \widehat{\otimes} H^q(\mathbf{V})\right) / \left(\oplus_{p+q=n} L^p \widehat{\otimes} H^q(\mathbf{V})\right) \\ &\cong \oplus_{p+q=n} H^p(\mathbf{U})\widehat{\otimes} H^q(\mathbf{V}), && \text{by (3.4.5).} \end{aligned}$$

□

Now we formulate an abstract form of Künneth theorem on bounded cohomolgy.

For discrete groups G and K, if the complexes $\mathbf{U} = \{B^*(G), d_*\}$ and $\mathbf{V} = \{B^*(K), \partial_*\}$ satisfy the conditions in Theorem 3.4, then there is an isomorphism

$$\bigoplus_{p+q=n} \widehat{H}^p(G)\widehat{\otimes}\widehat{H}^q(K) \cong \widehat{H}^n(B^*(G)\widehat{\otimes} B^*(K))$$

as Banach spaces.

One of the problems is that even in the case that Γ is an one point space or a trivial group, bounded cochain $B^n(\Gamma)$ is never zero for every $n \geq 0$. So we can not assume bounded cochain complex $B^n(\Gamma)$ is finite dimensional.

Remark 3.1. In Theorem 1.5, the flat condition is given to cocycles and coboundaries. In Theorem 3.4, if we give flat condition to cocycles and coboundaries instead of coboundaries and cohomologies, we need a stronger condition such as for every $n \geq 0$ and for every V^q the sequence

$$\text{Im } d_n \widehat{\otimes} V^q \to \text{Ker } d_n \widehat{\otimes} V^q \to H^n(\mathbf{U}) \widehat{\otimes} V^q$$

is also a kernel-cokernel pair in $\mathcal{B}an$.

Corollary 3.5. *Let G and K be discrete groups. Let the complexes $\{B^*(G), d_*\}$ and $\{B^*(K), \partial_*\}$ satisfy the followings:*

1. *boundary operators d_* and ∂_* are closed.*
2. *$\widehat{H}^*(G)$ and $\widehat{H}^*(K)$ are finite dimensional, say $\widehat{H}^p(G) = 0$ for $p > p_0$ and $\widehat{H}^q(K) = 0$ for $q > q_0$,*
3. *every Ker d_n is finite dimensional as vector spaces over $\mathbb{R}$.*

Then for every $n \leq p_0 + q_0$ there is an isomorphism

$$\bigoplus_{p+q=n} \widehat{H}^p(G) \widehat{\otimes} \widehat{H}^q(K) \cong \widehat{H}^n(B^*(G) \widehat{\otimes} B^*(K))$$

as Banach spaces.

Proof. Set $\mathbf{U} = B^*(G)$ and $\mathbf{V} = B^*(K)$. Since Ker d_n is finite dimensional as vector spaces over $\mathbb{R}$, so are Im d_n and $\widehat{H}^n(G)$ for every $n \geq 0$. Then Im d_n and $\widehat{H}^n(G)$ are the finite coproducts of $\mathbb{R}$, that is, they are of the form $\coprod_{\text{finite}} \mathbb{R}$. Then it is clear that Im d_n and $\widehat{H}^n(G)$ are flat for every $n \geq 0$. Recall that $\widehat{H}^p(G) = 0$ for $p > p_0$ and $\widehat{H}^q(K) = 0$ for $q > q_0$. Recall that $\big(B^*(G) \widehat{\otimes} B^*(K)\big)^n = \oplus_{p+q=n} B^p(G) \widehat{\otimes} B^q(K)$. So $B^p(G)$ for $p > n+1$ and $B^q(K)$ for $q > n+1$ do not affect the cohomologies $\widehat{H}^p(G)$, $\widehat{H}^q(K)$, and $\widehat{H}^n(B^*(G) \widehat{\otimes} B^*(K))$ for $n \leq p_0 + q_0$. Hence we consider the complex

(3.5.1)

$$0 \to \mathbb{R} \to B^1(G) \xrightarrow{d_1} B^2(G) \to \cdots \to B^{n+1}(G) \xrightarrow{d_{n+1}} \text{Im } d_{n+1} \to 0$$

(3.5.2)

$$0 \to \mathbb{R} \to B^1(K) \xrightarrow{\partial_1} B^2(K) \to \cdots \to B^{n+1}(K) \xrightarrow{\partial_{n+1}} \text{Im } \partial_{n+1} \to 0.$$

Then, the sequences (3.5.1) and (3.5.2) are finite dimensional complex. Hence, from Theorem 3.4, for $n \leq p_0 + q_0$ we have the isomorphism

$$\bigoplus_{p+q=n} \widehat{H}^p(G)\widehat{\otimes}\widehat{H}^q(K) \cong \widehat{H}^n(B^*(G)\widehat{\otimes}B^*(K))$$

□

As an simplest case, we consider an amenable group A. Recall that $\widehat{H}^n(A) = 0$ for $n > 0$ and $\widehat{H}^0(A) = \mathbb{R}$. Notice that, in this case, every boundary operator is closed. Also, in [8] it is shown that $B^n(A) \cong \mathbb{R}$. Hence, for a group G such that its bounded cochains $\{B^*(G)\}$ having closed boundaries, we have

$$B^*(G)\widehat{\otimes}B^*(A) \cong B^*(G)\widehat{\otimes}\mathbb{R} = B^*(G).$$

Then $\widehat{H}^n(B^*(G)\widehat{\otimes}B^*(A)) = \widehat{H}^n(B^*(G)$ and also

$$\bigoplus_{p+q=n} \widehat{H}^p(G)\widehat{\otimes}\widehat{H}^q(A) \cong \widehat{H}^n(G)\widehat{\otimes}\widehat{H}^0(A) \cong \widehat{H}^n(G)\widehat{\otimes}\mathbb{R} = \widehat{H}^n(G).$$

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