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THE JONES POLYNOMIAL OF KNOTS WITH SYMMETRIC UNION PRESENTATIONS

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Abstract. A symmetric union is a diagram of a knot, obtained from diagrams of a knot in the 3-space and its mirror image, which are symmetric with respect to an axis in the 2-plane, by connecting them with 2-tangles with twists along the axis and 2-tangles with no twists. This paper presents an invariant of knots with symmetric union presentations, which is called the minimal twisting number, and the minimal twisting number of 10_{42} is shown to be two. This paper also presents a sufficient condition for non-amphicheirality of a knot with a certain symmetric union presentation.

1. Introduction

A *symmetric union*, which is a generalized operation of the connected sum of a knot in the 3-space and its mirror image, was first introduced by Kinoshita and Terasaka [6]. They showed that the Alexander polynomial depends only on the parity of the number of half-twists of a trivial tangle on the symmetry axis and that the determinant is independent of the number of half-twists. In recent years, Lamm [7] generalized their results and also considered the relationship between a symmetric union and a ribbon knot. (See [3] for the definition.) It is easily seen that every knot with a symmetric union presentation is a ribbon knot. On the other hand, the converse question is still open. Lamm showed that every ribbon knot with minimal crossing number ≤ 10 has a symmetric union presentation, except 10_{87} [7]. The knot 10_{87} was also shown to be a symmetric union later in [2] and it is known that all two-bridge ribbon knots can be represented as symmetric unions. In fact, Lamm [8] has shown that certain three infinite families of two-bridge ribbon knots can be presented as a symmetric union. Recently, Lisca [11] has shown that the three families contain all two-bridge ribbon knots.

Let $\overline{V}_L(t) = V_L(t)/V_{O^n}(t)$ for an oriented link L of n components, where $V_L(t)$ and $V_{O^n}(t)$ are the Jones polynomial of L and the n-component trivial link O^n respectively. (See Section 2 for the definition.) We set i$=\sqrt{-1}$. In this

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paper we show the following formula for the Jones polynomial of knots with symmetric union presentations and its topological properties.

Theorem 1.1. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then*

$$V_{\overline{K}}(t) = (-1)^m t^{-m} V_{D_K \cup D_K^*(\infty,0)}(t) + (1 - (-1)^m t^{-m}) \overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t).$$

Corollary 1.2. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$ and s be a positive integer. If $m \equiv 0 \mod s$, then*

$$V_{\overline{K}}(-\exp(\frac{2\pi \mathrm{i}}{s})) = V_K(-\exp(\frac{2\pi \mathrm{i}}{s}))V_{K^*}(-\exp(\frac{2\pi \mathrm{i}}{s})).$$

A knot is called *amphicheiral* if it is isotopic to its mirror image. By Theorem 1.1, we have the following.

Theorem 1.3. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then*

$$t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) = (t^m + (-1)^m) V_K(t) V_K(t^{-1}).$$

In particular, if $\overline{K}$ is amphicheiral, then $V_{\overline{K}}(t) = V_K(t)V_K(t^{-1})$.

Now we restrict to the special values of the Jones polynomial. First we consider the values of the first derivative of the Jones polynomial at -1. We denote a (Laurent) polynomial $f(t)$ evaluated at r by $[f(t)]_{t=r}$.

Theorem 1.4. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then*

$$[\frac{d}{dt}V_{\overline{K}}(t)]_{t=-1} = m\{V_K(-1)^2 - \overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1)\}.$$

Corollary 1.5. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=-1} \equiv 0 \mod 8|m|$.*

Next we consider the values of the first derivative of the Jones polynomial at i.

Theorem 1.6. *Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then we have $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} = a + b\mathrm{i}$ where $a, b \in 4\mathbb{Z}$. In particular, if m is even $(1 \leq i \leq k)$ or $\overline{K}$ is amphicheiral, then $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} \in 4\mathbb{Z}$.*

Remark 1.7. Theorem 1.1, Corollary 1.2, Theorem 1.4 and Theorem 1.6 can be generalized to the case of $D \cup D^*(\infty, m_1, \ldots, m_k)$. However we do not give the detail since the goal of this paper is to give an obstruction for a knot to have a symmetric union presentation $D_K \cup D_K^*(\infty, m)$.

In this paper, all knots and links are oriented unless otherwise stated. In Section 2, we give the definitions of the Jones polynomial and a symmetric union. In Section 3, we shall prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. In Section 4, we shall prove Theorem 1.4 and Corollary 1.5 and observe a topological property of the Jones polynomial with respect to a symmetric union in special cases. In Section 5, we shall prove Theorem 1.6 and show a calculation in the case of the example in Section 4. In Section 6, we introduce the *minimal twisting number* of a knot with a symmetric union presentation. It is the smallest number of trivial tangles (with twists) appearing on the axis of a symmetric union presentation of a knot, the minimum taken over all symmetric union presentations for the knot. We shall show that the minimal twisting number of a knot 10_{42} is equal to two. In Section 7, we consider the amphicheirality of symmetric unions.

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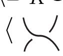
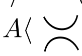
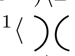
2. Definitions

Definition 2.1. Let K be a knot in the 3-space. We denote a diagram of K by D_K. The *bracket polynomial* of a diagram of a knot K, $\langle D_K \rangle$ can be defined as a polynomial which satisfies the following identities.

i) $\langle \bigcirc \rangle = 1$,

ii) $\langle D_K \cup \bigcirc \rangle = -(A^2 + A^{-2})\langle D_K \rangle$,

iii) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$.

We defines $V_{D_K}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ by $V_{D_K}(t) = \{(-A^3)^{-\omega(D_K)}\langle D_K \rangle\}_{t^{1/2}=A^{-2}}$ for any diagram D_K for K, where ω is the *writhe* of the diagram. (The writhe is the number of positive crossings of D_K minus the number of negative crossings of D_K.) It is shown that $V_{D_K}(t)$ is an invariant of the link [5], [9]. Then we denote $V_{D_K}(t)$ by $V_K(t)$ and call it the *Jones polynomial* of K.

Here we define a symmetric union in [7] as follows. We denote the tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1.

Definition 2.2. Let D be an unoriented knot diagram and D^* the diagram D reflected at an axis in the plane. If in the symmetric placement of D and D^* we replace the tangles $T_i = 0, (i = 0, \ldots, k)$ on the symmetry axis by $T_0 = \infty$ and $T_i = m_i \in \mathbb{Z}$ for $i = 1, \ldots, k$, we call the result a *symmetric union* of D and D^* and write $D \cup D^*(\infty, m_1, \ldots, m_k)$. The *partial knot* $\hat{K}$ of the symmetric union is the knot given by the diagram D. See Figure 1 for an illustration of the definition.

If a knot K has a diagram $D \cup D^*(\infty, m_1, \ldots, m_k)$, then the diagram is called a *symmetric union presentation* for K and we say that the knot K is a *symmetric union.*

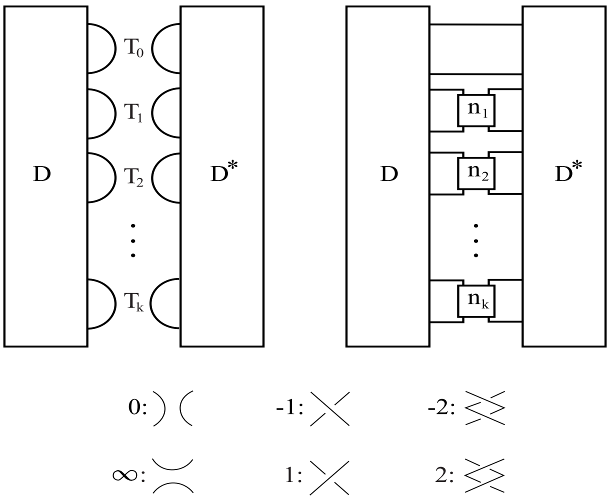


FIGURE 1

3. A formula of the Jones polynomial

Proof of Theorem 1.1. By using a skein relation of Kauffman bracket polynomial, we have

$$\begin{aligned}
&\langle D_K \cup D_K^*(\infty, m)\rangle \\
&= A^{m/|m|} D_K \cup D_K^*(\infty, m-1) + F_1\langle D_K \cup D_K^*(\infty,\infty)\rangle \\
&= (A^{m/|m|})^{|m|}\langle D_K \cup D_K^*(\infty, 0)\rangle + (F_1 + \cdots + F_{|m|})\langle D_K \cup D_K^*(\infty,\infty)\rangle \\
&= A^m\langle D_K \cup D_K^*(\infty, 0)\rangle + (\sum_{j=1}^{|m|} F_j)\langle D_K \cup D_K^*(\infty,\infty)\rangle,
\end{aligned}$$

where each F_j is the polynomial obtained by applying the skein relation and a finite number of *type I Reidemeister moves* [9] to the bracket polynomial. In fact, a single type I Reidemeister move changes the bracket polynomial by a factor of $-A^{\pm 3}$ ([9], Lemma 3.2).

Now we calculate a formula of $\sum_{j=1}^{|m|} F_j$ by considering the unknot instead of K as follows. We assume that D_K is a diagram as in Figure 2 so that we have a symmetric union of the unknot. We denote the diagram by $D_\circ$.

Then the resultant symmetric union is a diagram of the unknot with r crossings where $r = |m|$ such that it can be transformed into a diagram of the unknot with no crossings by r type I Reidemeister moves. Thus we have

$$\begin{aligned}
&\langle D_\circ \cup D_\circ^*(\infty, m)\rangle = (-A^{-3m/|m|})^{|m|} = (-1)^{|m|}A^{-3m}, \\
&\langle D_\circ \cup D_\circ^*(\infty, 0)\rangle = 1,
\end{aligned}$$

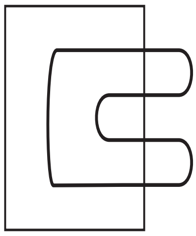


FIGURE 2

$$\langle D_\circ \cup D_\circ^*(\infty,\infty)\rangle = -A^{-2} - A^2.$$

Then we have

$$\sum_{j=1}^{|m|} F_j = \frac{\langle D_\circ \cup D_\circ^*(\infty,m)\rangle - A^m\langle D_\circ \cup D_\circ^*(\infty,0)\rangle}{\langle D_\circ \cup D_\circ^*(\infty,\infty)\rangle}$$
$$= \frac{(-1)^{|m|}A^{-3m} - A^m}{-A^{-2} - A^2} = \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2}.$$

Since $\omega(D_K \cup D_K^*(\infty,m)) = -m$, we obtain that

$$\begin{aligned} &V_{D_K\cup D_K^*(\infty,m)}(A)\\ &= (-A^3)^m\langle D_K \cup D_K^*(\infty,m)\rangle\\ &= (-A^3)^m\{A^m\langle D_K \cup D_K^*(\infty,0)\rangle + \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2}\langle D_K \cup D_K^*(\infty,\infty)\rangle\}\\ &= (-1)^m A^{4m}\langle D_K \cup D_K^*(\infty,0)\rangle + \frac{1-(-1)^m A^{4m}}{-A^{-2} - A^2}\langle D_K \cup D_K^*(\infty,\infty)\rangle. \end{aligned}$$

Using $t^{1/2} = A^{-2}$, we have

$$\begin{aligned} &V_{D_K\cup D_K^*(\infty,m)}(t)\\ &= (-1)^m t^{-m} V_{D_K\cup D_K^*(\infty,0)}(t) + (1-(-1)^m t^{-m})\overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t). \end{aligned}$$ □

Remark 3.1. By a result of Eisermann ([1], Theorem 1), we know that

$$\overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t)$$

in the statement of Theorem 1.1 is always a Laurent polynomial.

Proof of Corollary 1.2. By Theorem 1.1 and a property that the Jones polynomial is multiplicative under connected sum of knots ([9], p. 29), we have

$$\begin{aligned} V_{\overline{K}}(-\exp(\frac{2\pi \mathrm{i}}{s})) &= \overline{V}_{D_K\cup D_K^*(\infty,0)}(-\exp(\frac{2\pi \mathrm{i}}{s}))\\ &= V_K(-\exp(\frac{2\pi \mathrm{i}}{s}))\cdot V_{K^*}(-\exp(\frac{2\pi \mathrm{i}}{s})). \end{aligned}$$ □

Proof of Theorem 1.3. The first part of the theorem is obtained as follows. By Theorem 1.1, we have

$$
\begin{aligned}
& t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) \\
={}& t^m((-1)^m t^{-m} V_{D_K \cup D_K^*(\infty,0)}(t) + (1-(-1)^m t^{-m})\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)) \\
& + (-1)^m((-1)^m t^m V_{D_K \cup D_K^*(\infty,0)}(t^{-1}) + (1-(-1)^m t^m)\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t^{-1})) \\
={}& (-1)^m V_{D_K \cup D_K^*(\infty,0)}(t) + (t^m - (-1)^m)\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)) \\
& + t^m V_{D_K \cup D_K^*(\infty,0)}(t) + ((-1)^m - t^m)\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)) \\
={}& (t^m + (-1)^m) V_{D_K \cup D_K^*(\infty,0)}(t) \\
={}& (t^m + (-1)^m) V_{\overline{K}}(t) \cdot V_{\overline{K}}(t^{-1}).
\end{aligned}
$$

The latter part of the theorem follow immediately from the first part because $V_{\overline{K}}(t) = V_{\overline{K}}(t^{-1})$ if $\overline{K}$ is amphicheiral ([9], p. 29). □

4. Evaluation of the derivative at −1

Proof of Theorem 1.4. By Theorem 1.1, we know that

$$
\begin{aligned}
\frac{d}{dt} V_{\overline{K}}(t) ={}& \frac{d}{dt}((-1)^m t^{-m} V_{D_K \cup D_K^*(\infty,0)}(t)) \\
& + \frac{d}{dt}((1-(-1)^m t^{-m})\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)).
\end{aligned}
$$

Since

$$
\begin{aligned}
& \overline{V}_{D_K \cup D_K^*(\infty,0)}(t) = V_{D_K}(t) \cdot V_{D_K}(t^{-1}) \quad \text{and} \\
& [\frac{d}{dt} V_{D_K}(t)]_{t=-1} = -[\frac{d}{dt} V_{D_K}(t^{-1})]_{t=-1},
\end{aligned}
$$

we have $[\frac{d}{dt}(V_{D_K}(t) \cdot V_{D_K}(t^{-1}))]_{t=-1} = 0$.

Then

$$
\begin{aligned}
[\frac{d}{dt}((-t^{-1})^m \overline{V}_{D_K \cup D_K^*(\infty,0)}(t))]_{t=-1} &= [\frac{d}{dt}(-t^{-1})^m]_{t=-1}(V_{D_K}(-1))^2 \\
&= m(V_{D_K}(-1))^2.
\end{aligned}
$$

On the one hand, we have

$$
\begin{aligned}
& [\frac{d}{dt}((1-(-t^{-1})^m)\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t))]_{t=-1} \\
={}& -m[\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)]_{t=-1} + [(1-(-t^{-1})^m)\frac{d}{dt}\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(t)]_{t=-1} \\
={}& -m\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1).
\end{aligned}
$$

Therefore we have

$$
[\frac{d}{dt} V_{\overline{K}}(t)]_{t=-1} = mV_{D_K}(-1)^2 - m\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1).
$$

□

Here we need the following theorem due to Eisermann to prove Corollary 1.5.

Theorem 4.1 ([1])**.** *If K is a ribbon link, then $\overline{V}_K(-1) \equiv 1 \mod 8$.*

Proof of Corollary 1.5. By Theorem 1.4, we have

$$\begin{aligned}[\frac{d}{dt}V_{\overline{K}}(t)]_{t=-1} &= mV_K(-1)^2 - m\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1) \\ &= mV_{D_K \cup D_K^*(\infty,0)}(-1) - m\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1).\end{aligned}$$

By Theorem 4.1, we know that $V_{D_K \cup D_K^*(\infty,0)}(-1)$ and $\overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1) \equiv 1 \mod 8$. Thus we have $m(V_{D_K \cup D_K^*(\infty,0)}(-1) - \overline{V}_{D_K \cup D_K^*(\infty,\infty)}(-1)) \equiv 0 \mod 8|m|$. □

Example 4.2. Let K_m ($m \in \mathbb{Z}$) be a symmetric union as described in Figure 3.

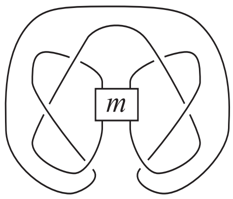


FIGURE 3

Then by Theorem 1.4, $[\frac{d}{dt}V_{K_m}(t)]_{t=-1} = m \cdot 9 - m = 8m$. Thus we know that K_m cannot be expressed as $D_K \cup D_K^*(\infty, n)$ if m is not divisible by $|n|$. In particular, if m is odd, then K_m cannot have a symmetric union presentation $D_K \cup D_K^*(\infty, n)$ where n is even.

5. Evaluation of the derivative at $\sqrt{-1}$

We say that a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ is *symmetric* if is satisfies that $f(t) = f(t^{-1})$. For example, the Jones polynomial of an amphicheiral knot is known to be symmetric ([9], p. 29).

Proposition 5.1. *Let K be a knot. Then we have*

$$[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} = a + b\mathrm{i},$$

where $a, b \in 2\mathbb{Z}$.

Proof. By results of Jones [4], $V_K(t)$ can be divided by $(t-1)(t^3-1)$. Let $V_K(t) = (t-1)(t^3-1)w(t)$. Then $V_K(t) = (t-1)^2(t^2+t+1)w(t)$ and $\frac{d}{dt}V_{\overline{K}}(t) =$

$2(t-1)(t^2+t+1)w(t)+(t-1)^2\dfrac{d}{dt}((t^2+t+1)w(t))$. So $[\dfrac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} = -2(\mathrm{i}+1)w(\mathrm{i})-2\mathrm{i}[\dfrac{d}{dt}((\mathrm{t}^2+\mathrm{t}+1)w(t))]_{t=\mathrm{i}}$. □

Proof of Theorem 1.6. By Theorem 1.1, we have

$$\begin{aligned}\frac{d}{dt}V_{\overline{K}}(t) &= \frac{d}{dt}((-1)^m t^{-m}))\overline{V}_{D_K\cup D_K^*(\infty,0)}(t) + (-1)^m t^{-m}\frac{d}{dt}\overline{V}_{D_K\cup D_K^*(\infty,0)}(t)\\ &\quad + \frac{d}{dt}(1-(-1)^m t^{-m})\overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t)\\ &\quad + (1-(-1)^m t^{-m})\frac{d}{dt}\overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t).\end{aligned}$$

Since $V_L(\mathrm{i}) = (\sqrt{2})^{n(L)-1}$ for any ribbon knot L [13], where $n(L)$ is the number of components of L, we know that $\overline{V}_{D_K\cup D_K^*(\infty,\ell)}(\mathrm{i}) = 1$ $(\ell = 0,\infty)$. Then we have

$$\begin{aligned}[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} &= [\frac{d}{dt}((-1)^m t^{-m}))]_{t=\mathrm{i}} + [(-1)^m t^{-m}\frac{d}{dt}V_{D_K\cup D_K^*(\infty,0)}(t)]_{t=\mathrm{i}}\\ &\quad + [\frac{d}{dt}(1-(-1)^m t^{-m})]_{t=\mathrm{i}}\\ &\quad + [(1-(-1)^m t^{-m})\frac{d}{dt}V_{D_K\cup D_K^*(\infty,\infty)}(t)]_{t=\mathrm{i}}.\end{aligned}$$

Since $[\dfrac{d}{dt}((-1)^m t^{-m}))]_{t=\mathrm{i}} + [\dfrac{d}{dt}(1-(-1)^m t^{-m})]_{t=\mathrm{i}} = 0$, we have

$$\begin{aligned}[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} &= [(-1)^m t^{-m}\frac{d}{dt}V_{D_K\cup D_K^*(\infty,0)}(t)]_{t=\mathrm{i}}\\ &\quad + [(1-(-1)^m t^{-m})\frac{d}{dt}V_{D_K\cup D_K^*(\infty,\infty)}(t)]_{t=\mathrm{i}}.\end{aligned}$$

To prove the theorem, we show that

$$[\frac{d}{dt}V_{D_K\cup D_K^*(\infty,0)}(t)]_{t=\mathrm{i}},\ [\frac{d}{dt}\overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t)]_{t=\mathrm{i}} \in 4\mathbb{Z}$$

as follows. First we need the following lemma.

Lemma 5.2. *Let $a(t)$ be a symmetric Laurent polynomial. If $a(1) = 0$, then $a(t)$ can be divided by $(t^{1/2} - t^{-1/2})^2$.*

Proof. By the assumption, we know that $a(t)$ can be divided by $t-1$ by the factor theorem. Since $a(t)$ is symmetric, $a(t)$ can be also divided by $t^{-1}-1$. □

By the fact that $V_L(1) = (-2)^{n(L)-1}$ for any link L [10], we know that

$$V_{D_K\cup D_K^*(\infty,0)}(1) = \overline{V}_{D_K\cup D_K^*(\infty,\infty)}(1) = 1.$$

Then by Lemma 5.2, we know that

$$V_{D_K\cup D_K^*(\infty,0)}(t) - 1 \ \text{ and } \ \overline{V}_{D_K\cup D_K^*(\infty,\infty)}(t) - 1$$

can be divided by $(t^{1/2} - t^{-1/2})^2$. Thus each of $V_{D_K \cup D_K^*(\infty,0)}(t)$ and $\overline{V}_{D_K \cup D_K^*(\infty,0)}(t)$ has a form of $(t^{1/2} - t^{-1/2})^2 w(t) + 1$ where $w(t)$ is a certain symmetric Laurent polynomial. Then we have

$$\frac{d}{dt}\overline{V}_{D_K \cup D_K^*(\infty,\ell)}(t)]_{t=\mathrm{i}} = 2(w(\mathrm{i}) - [\frac{d}{dt}w(t)]_{t=\mathrm{i}}) \ \ (\ell = 0, \infty).$$

Note that $w(\mathrm{i}) = 0$. Actually by the fact that $V_L(t) = (-\sqrt{2})^{n(L)-1}$ for any proper link L with trivial Arf invariant [12] and the fact that a symmetric union is a proper ribbon link [7] with trivial Arf invariant, we know that $\overline{V}_{D_K \cup D_K^*(\infty,\ell)}(\mathrm{i}) = 1$. So we have $(-2)w(\mathrm{i}) = \overline{V}_{D_K \cup D_K^*(\infty,\ell)}(\mathrm{i}) - 1 = 0$. Thus $w(\mathrm{i}) = 0$. Now it is easily seen that $[\frac{d}{dt}w(t)]_{t=\mathrm{i}} \in 2\mathbb{Z}$ since $w(t)$ is a symmetric Laurent polynomial. Thus $[\frac{d}{dt}\overline{V}_{D_K \cup D_K^*(\infty,\ell)}(t)]_{t=\mathrm{i}} \in 4\mathbb{Z}$. Then we know that $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}}$ has a form $a + b\mathrm{i}$ where $a, b \in 4\mathbb{Z}$.

In the case when m is even, both of $[(-1)^m t^{-m}]_{t=\mathrm{i}}$ and $[1 - (-1)^m t^{-m}]_{t=\mathrm{i}}$ are integers. Thus we have

$$[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} \in 4\mathbb{Z}.$$

We denote the *complex conjugate* of a complex number x by $\overline{x}$. Then we can easily see the following lemma.

Lemma 5.3. *For any Laurent polynomial $a(t)$, we have*

$$[\frac{d}{dt}a(t)]_{t=\mathrm{i}} = \overline{[\frac{d}{dt}a(t^{-1})]_{t=\mathrm{i}}}.$$

If $\overline{K}$ is amphicheiral, then $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} = [\frac{d}{dt}V_{\overline{K}}(t^{-1})]_{t=\mathrm{i}}$ since $V_{\overline{K}}(t) = V_{\overline{K}}(t^{-1})$ ([9], p. 29). So by Lemma 5.3, we know that $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}}$ is an integer. So we have $[\frac{d}{dt}V_{\overline{K}}(t)]_{t=\mathrm{i}} \in 4\mathbb{Z}$. This completes the proof. □

Example 5.4. Let K_m be a knot in Figure 3 again. Then by Theorem 1.1, we have

$$\begin{aligned}\frac{d}{dt}V_{\overline{K}}(t) &= [(-1)^m t^{-m}\frac{d}{dt}V_{3_1\sharp 3_1^*}(t) + (1 - (-1)^m t^{-m})\frac{d}{dt}V_{O^2}(t)]_{t=\mathrm{i}} \\ &= \mathrm{i}^m \cdot 4 + (1 - \mathrm{i}^m) \cdot 0 = 4\mathrm{i}^m,\end{aligned}$$

where O^2 is a 2-component trivial link and $3_1\sharp 3_1^*$ is a connected sum of the knot 3_1 ([9], p. 5) and the mirror image 3_1^*. Thus by Theorem 1.6, if m is an odd integer, we know that K_m cannot be expressed as $D_K \cup D_K^*(\infty, n)$ where n is even.

6. The minimal twisting number

In this section, we introduce the minimal twisting number for a knot with a symmetric union presentation.

Definition 6.1. We call the number k of $D_K \cup D_K^*(\infty, m_1, \ldots, m_k)$ the *twisting number* of the symmetric union. The *minimal twisting number* of a knot $\overline{K}$ with a symmetric union presentation is the smallest number of the twisting numbers of all symmetric union presentations for $\overline{K}$. We denote it by $\mathrm{tw}(\overline{K})$.

By the definition, we have the following.

Proposition 6.2. *The minimal twisting number is an invariant of a knot with a symmetric union presentation.*

Remark 6.3. Let K be a knot with a symmetric union presentation. If $\mathrm{tw}(K) = 0$, then K is a connected sum of a knot and its mirror image. If $\mathrm{tw}(K) = 1$, then K is a symmetric union of Kinoshita-Terasaka type [6]. The minimal twisting number is not additive under connected sum in general since $\mathrm{tw}(K \sharp K^*) = 0$ for any knot K.

Now we consider the following problem.

Problem. Is there a knot K with a symmetric union presentation which satisfies $\mathrm{tw}(K) \geq 2$?

Example 6.4. For each knot K in the set of knots $\{6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}, 10_{140}, 10_{153}\}$, we have $\mathrm{tw}(K) = 1$. (See [7] for the knots and their symmetric union presentations.)

Remark 6.5. Lamm found symmetric union presentations for all two-bridge ribbon knots [8] and actually, we know that the minimal twisting number of a two-bridge ribbon knot is equal to either one or two according to the result.

By Theorem 1.3, we know that if $\overline{K}$ is a knot with a symmetric union presentation $D_K \cup D_K^*(\infty, m)$, which satisfies $V_K(t) \neq 1$, then

$$\frac{t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1})}{t^m + (-1)^m}$$

is reducible polynomial over $\mathbb{Z}$. By using this property, we show the following.

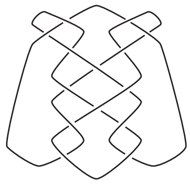
Theorem 6.6. $\mathrm{tw}(10_{42}) = 2$.

Proof. Since 10_{42} has a symmetric union presentation as in Figure 4, we know that $\mathrm{tw}(10_{42}) \leq 2$.

Suppose 10_{42} has a symmetric union presentation $D_K \cup D_K^*(\infty, m)$.

The Jones polynomial of 10_{42} is as follows.

$$V_{10_{42}}(t) = -t^{-5} + 3t^{-4} - 6t^{-3} + 10t^{-2} - 12t^{-1} + 14 - 13t + 10t^2 - 7t^3 + 4t^4 - t^5.$$



10_{42}

FIGURE 4

Since $[\frac{d}{dt}(V_{9_{27}}(t))]_{t=-1} = 8$, we have $|m| \leq 1$ and by the primeness of 10_{42}, we know that $m = \pm 1$. Let $H_{\overline{K}}^m(t) = \dfrac{t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1})}{t^m + (-1)^m}$ for a knot $\overline{K}$. By Theorem 1.3, we know that $H_{10_{42}}^{\pm 1}(t)$ has a form $V_K(t)V_K(t^{-1})$ for some knot K. By calculation, we have

$$\begin{aligned} H_{10_{42}}^{1}(t) = & -t^{-5} + 3t^{-4} - 7t^{-3} + 10t^{-2} - 12t^{-1} + 15 \\ & - 12t + 10t^2 - 7t^3 + 3t^4 - t^5, \\ H_{10_{42}}^{-1}(t) = & -t^{-5} + 4t^{-4} - 6t^{-3} + 10t^{-2} - 13t^{-1} + 13 \\ & - 13t + 10t^2 - 6t^3 + 4t^4 - t^5. \end{aligned}$$

Now we show that $H_{10_{42}}^{\pm 1}(t)$ cannot be $V_K(t)V_K(t^{-1})$ for any knot K. Suppose $H_{10_{42}}^{1}(t)$ has a form $f(t)f(t^{-1})$ for some Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$. Then we may assume that $f(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ $(a_0a_5 \neq 0)$. In this case, we have that following five equations.

(1) $a_5a_0 = -1$,
(2) $a_5a_1 + a_4a_0 = 3$,
(3) $a_5a_2 + a_4a_1 + a_3a_0 = -7$,
(4) $a_5a_3 + a_4a_2 + a_3a_1 + a_2a_0 = 10$,
(5) $a_5a_4 + a_4a_3 + a_3a_2 + a_2a_1 + a_1a_0 = -12$,
(6) $a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 15$.

By the equation (6), we know that $|a_i| \leq 3$ $(i = 1, 2, 3, 4, 5)$ and by the equation (1), we know that $(a_0, a_5) = (1, -1)$ or $(-1, 1)$.

Suppose that $(a_0, a_5) = (-1, 1)$. Then by the equation (3), we have $a_2 + a_4a_1 - a_3 = -7$ and by the equation (2), we have $a_1 = a_4 + 3$. So $(a_1, a_4) = (0, -3)$ or $(1, -2)$ or $(2, -1)$ or $(3, 0)$. If $(a_1, a_4) = (1, -2)$ or $(2, -1)$, then we have

$$\begin{cases} a_2 - a_3 = -5 \\ a_2^2 + a_3^2 = 8 \end{cases}$$

by the equations (3) and (6). Then $a_3 \notin \mathbb{Z}$.

If $(a_1, a_4) = (0, -3)$ or $(3, 0)$, then we have

$$\begin{cases} a_2 - a_3 = -7 \\ a_2^2 + a_3^2 = 4 \end{cases}$$

by the equations (3) and (6). Then $a_3 \notin \mathbb{Z}$.

Suppose that $(a_0, a_5) = (1, -1)$. Then by the equation (2), we have $a_4 = a_1 + 3$. So $(a_1, a_4) = (0, 3)$ or $(-1, 2)$ or $(-2, 1)$ or $(-3, 0)$. By the equation (3), we have $-a_2 + a_4a_1 + a_3 = -7$. If $(a_1, a_4) = (-1, 2)$ or $(-2, 1)$, then we have

$$\begin{cases} a_2 - a_3 = 5 \\ a_2^2 + a_3^2 = 8 \end{cases}$$

by the equations (3) and (6). Then $a_3 \notin \mathbb{Z}$.

If $(a_1, a_4) = (0, 3)$ or $(-3, 0)$, then we have

$$\begin{cases} a_3 - a_2 = -7 \\ a_2^2 + a_3^2 = 4 \end{cases}$$

by the equations (3) and (6). Then $a_3 \notin \mathbb{Z}$.

From the above, we have contradiction and we know that $H^1_{10_{42}}(t)$ does not have a form $f(t)f(t^{-1})$ for some Laurent polynomial $f(t)$. Next suppose that $H^{-1}_{10_{42}}(t)$ has a form $f(t)f(t^{-1})$ for some polynomial $f(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ $(a_0 \cdot a_5 \neq 0)$. Then we have that following five equations.

(1) $a_5a_0 = -1$,
(2) $a_5a_1 + a_4a_0 = 4$,
(3) $a_5a_2 + a_4a_1 + a_3a_0 = -6$,
(4) $a_5a_3 + a_4a_2 + a_3a_1 + a_2a_0 = 10$,
(5) $a_5a_4 + a_4a_3 + a_3a_2 + a_2a_1 + a_1a_0 = -13$,
(6) $a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 13$.

By using the same argument above, we have $|a_i| \leq 3$ $(i = 1, 2, 3, 4, 5)$ and $(a_0, a_5) = (1, -1)$ or $(-1, 1)$. Then in each case, we can show that $a_3 \notin \mathbb{Z}$. Thus we know that $H^{-1}_{10_{42}}(t)$ cannot be $V_K(t)V_K(t^{-1})$ for any knot K. This completes the proof. □

Remark 6.7. We can also see that $\mathrm{tw}(9_{27}) = \mathrm{tw}(10_{99}) = \mathrm{tw}(10_{123}) = 2$. (See [7] for the knots and their symmetric union presentations.)

7. On the amphicheirality of symmetric unions

In this section, we consider symmetric unions related to the following problem.

Problem (*Kirby Problem* 1.88(C) (Jones)). Is there a non-trivial knot with the same Jones polynomial as the unknot?

Symmetric unions often appear when we construct a non-trivial knot with trivial *Alexander polynomial* (See [9] for the definition). For example, a knot with symmetric union presentation $D_O \cup D_O^*(\infty, m_1, \ldots, m_k)$ with even integers $m_1, \ldots, m_k$ has trivial Alexander polynomial [7]. (Here O is the unknot.) Then it is natural to expect to have a non-trivial knot with trivial Jones polynomial from among such symmetric unions.

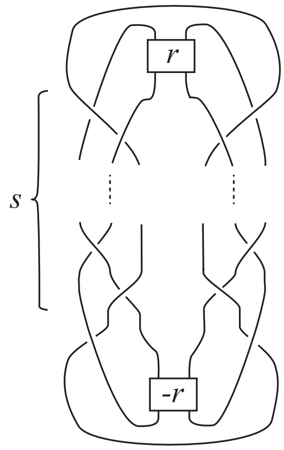


FIGURE 5

By using Theorem 1.3, we have the following.

Proposition 7.1. *If there exists a non-trivial, amphicheiral knot with a symmetric union presentation* $D_O \cup D_O^*(\infty, m)$, *then the Jones polynomial is trivial and, in particular, the answer to the above problem is affirmative.*

Remark 7.2. For example, 10_{153} has a presentation $D_O \cup D_O^*(\infty, m)$ (See [7]), but it is not amphicheiral since it has non-trivial Jones polynomial.

Finally, we give more examples of symmetric unions concerning Theorem 1.3.

Example 7.3. Let $K_{r,s}$ $(r, s \geq 1)$ be a knot with a symmetric union presentation as described in Figure 5. We denote the partial knot of the symmetric union by K_c. Then we can find that $K_{r,s}$ is amphicheiral and the Jones polynomial of $K_{r,s}$ is the same as that of $K_c \sharp K_c^*$ by making use of a result of Watson [14]. Actually, we know that the Khovanov homology of $K_{r,s}$ is equivalent to that of $K_c \sharp K_c^*$ by the result.

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