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Original Paper

Simulation model for Francis and Reversible Pump Turbines

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Abstract

When simulating the dynamic behaviour of a hydro power plant, it is essential to have a good representation of the turbine behaviour. The pressure transients in the system occurs because the flow changes, which the turbine defines. The flow through the turbine is a function of the pressure, the speed of rotation and the wicket gate opening and is, most often described in a performance diagram or Hill diagram. In the Hill diagram, the efficiency is drawn like contour lines, hence the name. A turbines Hill diagram is obtained by performance tests on scaled model in a laboratory.

However, system dynamic simulations have to be performed in the early stage of a project, before the turbine manufacturer has been chosen and the Hill diagram is known. Therefore one have to rely on diagrams for a turbine with similar speed number. The Hill diagram is drawn through measured points, so for using the diagram in a simulation program, one have to iterate in the diagram based on curve fitting of the measured points.

This paper describes an alternative method. By means of the Euler turbine equation, it is possible to set up two differential equations which represents the turbine performance with good enough accuracy for the dynamic simulations. The only input is the turbine's main geometry, the runner blade in- and outlet angle and the guide vane angle at best efficiency point of operation (BEP). In the paper, simulated turbine characteristics for a high head Francis turbine, and for a reversible pump turbine are compared with laboratory measured characteristics.

Keywords: Hydropower, Turbines, Characteristics, Simulation

1. Introduction

The Euler turbine equation describes how the hydraulic power is transformed to mechanical rotational power to the turbine shaft. The transformation is due to the reaction force as a consequence of the velocity vectors changing both in direction and multitude. Seen from the hydraulic systems side, the turbine is a perfect throttle, throttling the whole head and subtracting all the hydraulic energy. In the differential equation describing the flow the turbine represents a hydraulic loss.

In a hydro system with reaction turbines, the water is a continuum from upper reservoir through the turbine and to the lower reservoir. Therefore, the turbine speed of rotation effects the flow and vice versa. In high head turbines, when the speed of rotation increases the flow decreases. There is a so-called pumping effect due to the centripetal force. The mentioned throttling effect is therefore a function of the speed of rotation.

A Francis turbine has a fixed geometry deigned to be optimal in one point of operation only, the so-called Best Efficiency Point, BEP. At BEP, the transformation of hydraulic power to rotational mechanic power is perfect, if not for friction loss. Disregarding the friction loss, at this point the hydraulic efficiency is 1.0. In the differential equation for angular speed of rotation, the Euler turbine equation defines the torque.

2. Euler's turbine equation

The turbine runner's task is to transform the available hydraulic power, proportional to the multiple of flow and head to mechanical rotating power. Euler's turbine equation states that the transformation is dependent on the change of velocity vectors from the inlet, index 1, to the outlet, index 2, of the runner as the Euler equation 2.1 describes:

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$$\eta_h \rho g Q H = \rho Q(u_1 c_{u1} - u_2 c_{u2}) = \omega T \tag{2.1}$$

where u_1 and u_2 is the turbine runner's peripheral speed at the inlet and the outlet respectively, and c_{u1} and c_{u2} is the water's absolute velocity's component in the peripheral speed's direction. The peripheral speeds, u_1 and u_2 are proportional to the inlet and outlet diameters respectively, the proportional constant being the angular speed of rotation ω. In Fig. 1, the velocity vectors, or the velocity triangles are shown.

The hydraulic efficiency is a measure of how good the runner is capable of transforming the hydraulic power to mechanical rotational power. The hydraulic efficiency pr. definition:

$$\eta_h = \frac{\omega T}{\rho g H Q} = \frac{\omega(u_1 c_{u1} - u_2 c_{u2})}{gH} \tag{2.2}$$

For a perfectly designed runner, assuming no friction loss, the hydraulic efficiency is 1.0, but only at one particular operational point, the so-called Best Efficiency Point, BEP. At BEP, the runner angles are perfect for transforming the hydraulic power to rotating mechanical power. For a Francis turbine, the runner blades are fixed. For a particular runner, the ideal velocity vectors will only apply for a given head, flow, speed of rotation and wicket gate position.

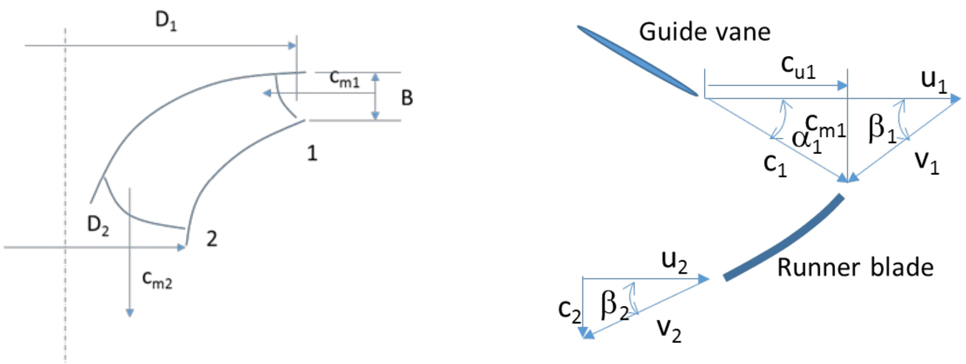

Fig. 1 Main geometry and velocity triangles at inlet and outlet of a Francis runner, At BEP, angle α_1 corresponds with the guide vane angle, angle β_1 corresponds with the runner blade inlet angle and angle β_2 with the outlet runner blade angle.

According to the Euler turbine equation, the head difference between inlet and outlet, marked 1) and 2) in Fig. 1 is given by:

$$g(H_1 - H_2) = u_1 c_{u1} - u_2 c_{u2} \tag{2.3}$$

Applying the Cosines sentence on the inlet velocity triangle, see Figure 1, gives:

$$v_1^2 = u_1^2 + c_1^2 - 2u_1 c_1 \cos\alpha_1 = u_1^2 + c_1^2 - 2u_1 c_{u1} \tag{2.4}$$

$$u_1 c_{u1} = \frac{1}{2}c_1^2 - \frac{1}{2}v_1^2 + \frac{1}{2}u_1^2 \tag{2.5}$$

And accordingly for the outlet triangle:

$$u_2 c_{u2} = \frac{1}{2}c_2^2 - \frac{1}{2}v_2^2 + \frac{1}{2}u_2^2 \tag{2.6}$$

Implemented in equation 2.3 gives:

$$g(H_1 - H_2) = \frac{1}{2}(c_1^2 - c_2^2) - \frac{1}{2}(v_1^2 - v_2^2) + s\omega^2 \tag{2.7}$$

Where:

$$s = \frac{1}{8}D_1^2\left(1 - \frac{D_2^2}{D_1^2}\right) \tag{2.8}$$

The turbine represents a throttle in the system as it transforms the hydraulic power to rotational mechanical power. The throttling is also a function of the angular speed of rotation, which is defined by the turbine geometry, eq 2.8. When s is positive, i.e. $D_1>D_2$, the flow will decrease and if s is negative, i.e. $D_1<D_2$, the flow will increase as the speed of rotation increases as illustrated in Fig. 2.

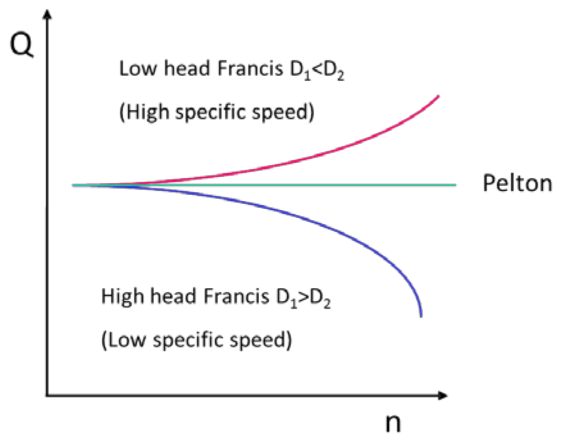

Fig. 2 Illustration of how the flow, Q, changes with the speed of rotation, n for Francis turbines of high and low specific speed.

Introducing the opening degree of the turbine, κ, defined by:

$$\kappa = \frac{\dfrac{Q}{\sqrt{2gH}}}{\dfrac{Q_R}{\sqrt{2gH_R}}} \tag{2.9}$$

where the subscript $_R$ denotes the design point of the turbine, and solving the equation with respect to the head H gives:

$$H = H_R \left(\frac{Q}{\kappa Q_R} \right)^2 \tag{2.10}$$

which in the design point is the head difference between runner inlet and outlet as expressed in equation 2.7, hence:

$$gH_R \left(\frac{Q}{\kappa Q_R} \right)^2 = \frac{1}{2}(c_{1R}^{\ 2} - c_{2R}^{\ 2}) - \frac{1}{2}(v_{1R}^{\ 2} - v_{2R}^{\ 2}) + s\omega_R^{\ 2} \tag{2.11}$$

or:

$$\frac{1}{2}(c_{1R}^{\ 2} - c_{2R}^{\ 2}) - \frac{1}{2}(v_{1R}^{\ 2} - v_{2R}^{\ 2}) = gH_R \left(\frac{Q}{\kappa Q_R} \right)^2 - s\omega_R^{\ 2} \tag{2.12}$$

Implemented in equation 2.7 gives:

$$g(H_1 - H_2) = gH_R \left(\frac{Q}{\kappa Q_R} \right)^2 + s(\omega^2 - \omega_R^{\ 2}) \tag{2.13}$$

3. The differential equations for turbine and system

The turbine implemented in a system, as illustrated in Fig. 3, can be represented by two differential equations, the momentum equation and the torque equation:

$$I_h \frac{dQ}{dt} = gH - gH_t \tag{3.1}$$

$$I_p \frac{d\omega}{dt} = T - T_G \tag{3.2}$$

where T_G is the generator torque.

I_h is the hydraulic inertia trough the turbine, and I_p is the polar moment of inertia of the rotating masses (generator).

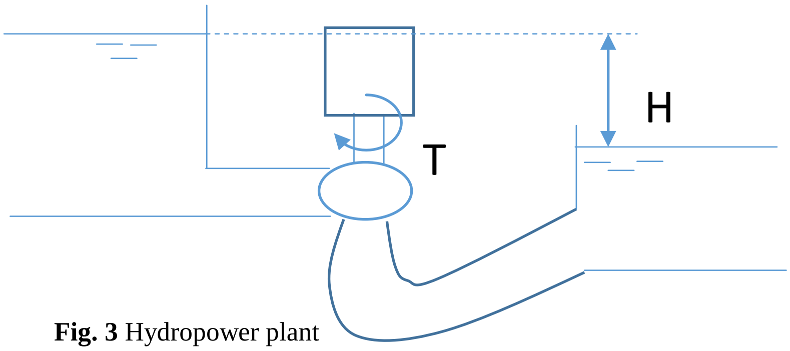

Fig. 3 Hydropower plant

Globally, the turbine is defined from the inlet flange of spiral casing to the outlet of the draft tube. As the flow is entering the spiral casing and goes through the wicket gate, before entering the turbine runner, some of the available head has been transformed to velocity energy. The flow is also given a swirl, and through the runner the swirl will be transformed to torque. The runner outlet is designed so that the swirl is zero before entering the draft tube. It is, however, only the extraction of the power through the turbine runner that matters in the momentum equation, hence:

$$gH_t = g(H_1 - H_2) = gH_R \left(\frac{Q}{\kappa Q_R}\right)^2 + s(\omega^2 - \omega_R{}^2) \tag{3.3}$$

The torque, T, is defined by the Euler equation:

$$T = \rho Q(r_1 c_{1u} - r_2 c_{2u}) \tag{3.4}$$

Examining the velocity diagram, the equation may be transformed to:

$$T = \rho Q(r_1 c_1 \cos\alpha_1 + r_2 A_z \cot\beta_2 c_1 \sin\alpha_1 - r_2{}^2\omega) \tag{3.5}$$

Examining the equation, the first two terms in the parenthesis are functions of the absolute velocity and the geometrical properties. With reference to the deduction in Appendix A the equation has the form:

$$T = \rho Q(t_s - r_2{}^2\omega) \tag{3.6}$$

Where $\rho Q t_s$ is the start torque, i.e. at $\omega = 0$.

4. Dimensionless equations

The equations can be made dimensionless by implementing the dimensionless properties defined as follows: Flow: $q = Q/Q_R$, Head: $h = H/H_R$ and angular speed of rotation: $\tilde{\omega} = \omega/\omega_R$, Dimensionless starting torque: $m_s = t_s/t_R$ where t_s is the specific torque when the angular speed of rotation, $\omega = 0$, and t_R is the rated specific starting torque. The dimensionless equations are:

Momentum:
$$T_{wt} \frac{dq}{dt} = h - \left(\frac{q}{\kappa}\right)^2 - \sigma(\tilde{\omega}^2 - 1) \tag{4.1}$$

Torque: $$T_a \frac{d\omega}{dt} = \frac{1}{h} q(m_s - \psi\tilde{\omega}) - \eta_G \tag{4.2}$$

In eq 4.1, the turbine head is:

$$h_t = \left(\frac{q}{\kappa}\right)^2 - \sigma(\tilde{\omega}^2 - 1) \tag{4.3}$$

η_G in eq 4,2 is the generator efficiency, and the turbine efficiency is:

$$\eta_h = \frac{1}{h}(m_s - \psi\tilde{\omega})\tilde{\omega} \tag{4.4}$$

T_{wt} and T_a are time constants representing the hydraulic and rotating inertia, respectively, see eq 4.1 and 4.2

The throttling dependency of angular speed of rotation given by s (eq.2.8) is made dimensionless by:

$$\sigma = s\frac{\omega_R}{gH_R} \tag{4.5}$$

The torque equation, eq. 3.5, seems to be a rather complicated expression, but by making the equation dimensionless, see Appendix A, the dimensionless starting torque m_s, will be:

$$m_s = \xi\frac{q}{\kappa}(\cos\alpha_1 + \tan\alpha_{1R}\sin\alpha_1) \tag{4.6}$$

BEP is when $\alpha_1 = \alpha_{1R}$ and $q = \kappa = 1.0$, hence $m_{sR} = \frac{\xi}{\cos\alpha_{1R}}$

When the equation is made dimensionless, two dimensionless figures, ψ and ξ, come forth. These are, for a particular turbine, constants defined at BEP as follows:

$$\psi = \frac{{u_{2R}}^2}{gH_R} \qquad \xi = \frac{u_{1R}c_{1R}}{gH_R} \tag{4.7}$$

5. The efficiency

The hydraulic (Euler) efficiency is pr definition $T\omega/(\rho gQH)$, hence in dimension less term:

$$\eta_h = \frac{q(m_s - \psi\tilde{\omega})\tilde{\omega}}{qh} = \frac{1}{h}(m_s - \psi\tilde{\omega})\tilde{\omega} \tag{5.1}$$

Examining the hydraulic efficiency for q=0 and $\tilde{\omega}=1$, $c_{u1} \to c_1$ and $c_{2u} \to u_2$ hence:

$$\eta_h = \frac{1}{gH}(u_1c_1 - {u_2}^2) \tag{5.2}$$

With rated head and $\kappa=1$, the hydraulic efficiency $\eta_h = \xi - \psi$ and not equal to zero. This is in contradiction to all practical experience. It is therefore reasonable to formulate an incipient loss, or rather an incipient efficiency according to the equation below:

$$\eta_i = 1 - (q-1)^2 \tag{5.3}$$

This is a parabola where the peak point is at (1,1), i.e. at $q=q_R=1$, the incipient efficiency is 1.0 assuming no incipient loss at rated flow, and at q=0, the efficiency is zero, see Fig. 4, left figure.

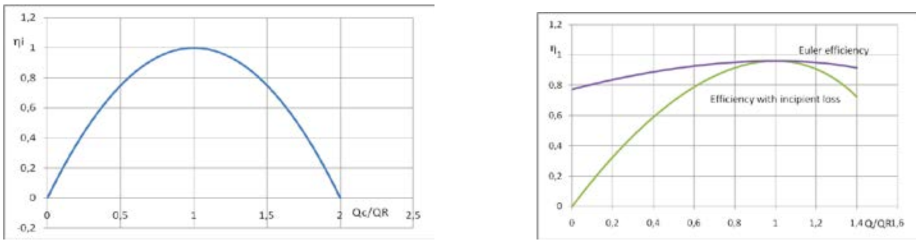

Fig. 4 Incipient efficiency as a function of flow

The hydraulic efficiency will then be:

$$\eta_h = \eta_i \frac{1}{h}(m_s - \psi\omega)\omega \tag{5.4}$$

For $\tilde{\omega}$=1, i.e. the synchronous speed of rotation, Fig. 4, wright, shows the result.

Using geometry defined term for σ, (see eq. 4.5) implies that the runner's main geometry is according to the Euler equation, which is not always the case, besides; it is a question of how to define the main dimension of the runner, which is actual a 3D geometry. Assuming that the turbine has BEP at q=1, h=1, and $\tilde{\omega}$=1, and that the derivative dη/dω of equation 4.4 is equal to zero at BEP, the relationships between σ ,ψ, ξ and η_{hR} are:

$$\sigma = \frac{\eta_{hR} - \psi}{\eta_{hR} + \psi} \tag{5.5}$$

$$\xi = (\eta_{hR} + \psi)\cos\alpha_{1R} \tag{5.6}$$

This is derived in Appendix B

The turbine characteristics and efficiency may now be simulated by the two differential equations (4.1) and (4.2), starting with ω = 0, then setting η_G <0. (if η_G=0, it will end at run-away-speed.)

Simulations of the turbine characteristics of a high head Francis turbine with this model, which is strictly in accordance with the Euler equation, the peak efficiency at BEP will be η_{hR} = 1, assuming no friction loss.

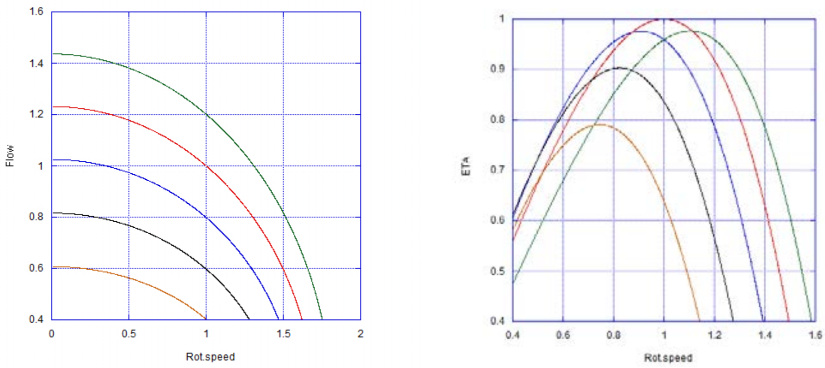

Fig. 5 Flow vs speed of rotation and efficiency vs rotational speed according to the Euler equation.

For high head Francis turbines, and especially for RPTs, the speed-flow characteristics are in fact much steeper. This indicates that it is not only the diameter ratio that decides the turbine characteristics.

The flow dependency of the speed of rotation is often referred to as "pumping effect" because as the speed increases, the turbine will act like a pump, increasing the throttling through the turbine, and even turn the flow direction if the speed of rotation is high enough.

There are reasons to model high head Francis turbines, and especially RPTs in accordance with the Euler turbine equation, but with an added term based on the pump equation in order to include the enhanced “pumping effect” due to the geometry of the runner.

6. The pump equation

A centrifugal pump and a Francis turbine is in fact the same kind of machine, both obeying the Euler equation. Assuming rotation free inlet, the theoretical head of a pump may be expressed by the Euler equation:

$$gH_{th\infty} = u_2(u_2 - c_{m2}) = u_2(u_2 - \frac{Q}{\pi B_2 D_2 \tan\beta_2}) \qquad (6.1)$$

(In pump literature, $H_{th\infty}$ is “the theoretical head with infinite number of blades”, which means that the fluid follows the runner blades with no slip.)

At a given speed of rotation, i.e. for a given u_2, The QH-characteristics will be ascending if the outlet angle $\beta_2 > 90^o$ and descending if $\beta_2 < 90^o$, i.e. forward or backward leant runner blades as illustrated in Fig. 6.

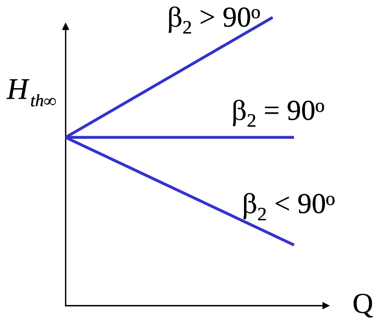

Fig. 6 Theoretical QH-characteristics of centrifugal pumps, dependent on the outlet angle β_2.

In order to get stabile pump characteristics, centrifugal pumps must have backward leant runner blades. This is also the case for a RPT, when running as a pump. However, when the speed of rotation changes direction in turbine mode of operation, the pump effect will be caused by forward lent blades.

In equation 6.1 it is assumed no rotation at the pump inlet. For this case, there is rotation at the inlet because of the guide vanes. This rotation term is included in equation 6.2 below.

$$gH_{th\infty} = u_2 c_{2u} - u_1 c_{1u} = u_2(u_2 - \frac{c_{2m}}{\tan\beta_2}) - u_1(u_1 - \frac{c_{1m}}{\tan\beta_1}) \qquad (6.2)$$

In general c_m = Q/Area, and examining the velocity diagrams:

$$gH_{th\infty} = u_2(u_2 - \frac{Q}{A_2 \tan\beta_2}) - u_1(u_1 - \frac{Q}{A_1 \tan\beta_1}) \qquad (6.3)$$

The pumping effect in a Francis turbine is caused by that the turbine will behave in the same way as a centrifugal pump. The water flows through the turbine from upper to lower reservoir and will give rotational power on the turbine shaft according to the Euler turbine equation. At the same time, the pump equation will also apply, but with the direction of the rotation as in turbine mode of operation. Seen from the pump equation, the flow will be negative until the flow really turns, which will happen if one increases the speed of rotation sufficiently. As pump regarded, this will be a pump with forward lent blades, i.e. with increasing characteristics, see Fig. 6.

Using notations common for turbine mode of operation, the pump equation will be:

$$gH_p = u_1(u_1 - \frac{Q}{A_1 \tan\beta_1}) - u_2(u_2 - \frac{Q}{A_2 \tan\beta_2}) \qquad (6.4)$$

Introducing the angular speed of rotation, in general $u = \omega \frac{D}{2}$:

$$gH_p = 2s_p\omega^2 - \omega\left(\frac{D_1}{2}\frac{Q}{A_1 \tan\beta_1} - \frac{D_2}{2}\frac{Q}{A_2 \tan\beta_2}\right) \quad (6.5)$$

Where s_p is a geometry dependent parameter formally defined by the inlet and outlet diameters in the same way as in eq. 2.8.

Implementing the pumping head to the turbine equation eq. 3.3:

$$gH_t = gH_n\left(\frac{Q}{\kappa Q_n}\right)^2 + s(\omega^2 - \omega_n^2) + 2s_p\omega^2 + \omega R_q Q \quad (6.6)$$

where:

$$R_q = \frac{D_1}{2}\frac{1}{A_1 \tan\beta_1} - \frac{D_2}{2}\frac{1}{A_2 \tan\beta_2} \quad (6.7)$$

Again, it is convenient to make the equation dimensionless by dividing the equation with gH_R and introducing $h=H/H_R$, $q=Q/Q_R$, and $\tilde{\omega}=\omega/\omega_n$:

$$h_t = \frac{1}{\kappa^2}q^2 + \sigma(\tilde{\omega}^2 - 1) + 2\sigma_p\tilde{\omega}^2 - \tilde{\omega}r_q q \quad (6.8)$$

where:

$$r_q = \frac{R_q\omega_R Q_R}{gH_R}$$

To obtain the design requirement that $\eta=\eta_R$ when $q=1$, $\kappa=1$, $h=1$ and $\omega=1$, $\sigma_p = ½\ \sigma$.

Implementing the pump term in the eq. 4.1 and 4.2 gives:

Momentum:
$$T_{wt}\frac{dq}{dt} = h - \frac{q|q|}{\kappa^2} - \sigma(\tilde{\omega}^2 - 1) - \sigma\tilde{\omega}^2 + r_q\omega|q| \quad (6.9)$$

Torque:
$$T_a\frac{d\omega}{dt} = \frac{1}{h}|q|(m_s - \psi\tilde{\omega}) - \eta_G \quad (6.10)$$

The absolute sign on q has to be there so that the equations are correct when the flow, q, turns to negative flow in 4^{th} quadrant.. The same goes for the equations for a Francis turbine, eq 4.1 and 4.2.

7. Q_{ED} – N_{ED} Characteristics

The definition of the unit speed, N_{ED} and the unit flow, Q_{ED}, respectively is:

$$N_{ED} = \frac{nD}{\sqrt{gH}} \text{ and } Q_{ED} = \frac{Q}{D^2\sqrt{gH}} \quad (7.1)$$

In these definitions, H is the actual head difference over the turbine which is the head defined by the Euler equation, see equations (4.3) for a turbine with Euler geometry and equation (6.8) for a turbine which differs from Euler geometry typical for a RPT.

8. Simulations and verifications

8.1 High head Francis turbine

Main dimensions of the model runner:

D_1 = 630 mm	α_{1R}= 10^o
D_2 = 349 mm	β_1 = 77.4^o
B_1 = 60 mm	β_2 = 69.6^o

The geometry is according to the Euler turbine equation and is simulated with the turbine model strictly in accordance with Euler turbine equations, eq (4.1) and (4.2).

Figure 7 shows simulated turbine characteristics, N_{ED} – Q_{ED}, and efficiency, Eta. In Fig. 7, fare wright, simulated and measured characteristics are compared.

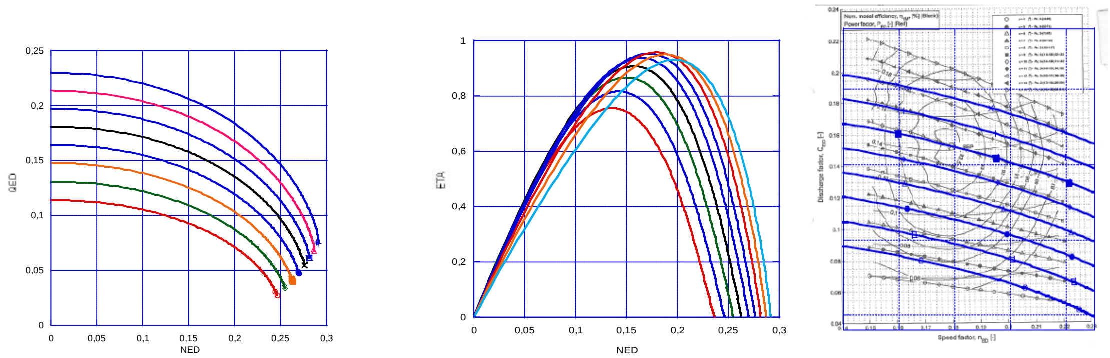

Fig. 7 Simulated turbine characteristics and efficiency where η_{hR}=0.96. Comparison between measured and simulated characteristics are shown on the figure on the fare end left

8.2 Reversible pump turbine

Main dimensions of the model runner:

D_1 = 630 mm	α_{1R}= 10^o
D_2 = 349 mm	β_1 = 11^o
B_1 = 59.6 mm	β_2 = 12.8^o

The main dimension differs from the Euler geometry in order to gain sufficient pumping power and the characteristics must be modelled with the correction of the Euler equations described by the equations (6.9) and (6.10). Figure 8 shows simulated turbine characteristics, N_{ED} – Q_{ED}, compared with measured characteristics, as well as the simulated relative efficiency for the same wicket gate openings.

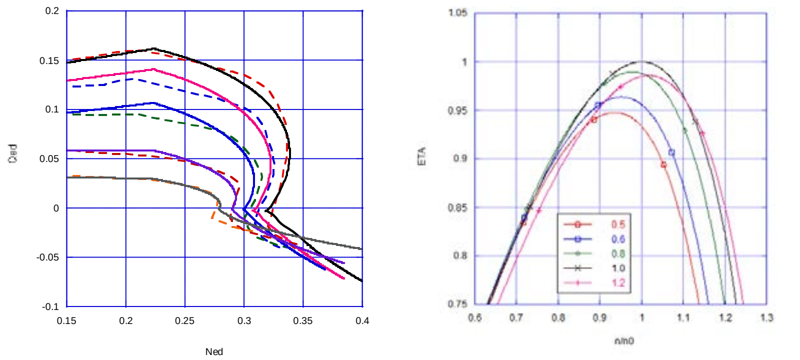

Fig. 8 Simulated and measured characteristics of Reversible Pump Turbine. Simulated efficiency at the left for different opening degrees ($\eta_R = 1$).

9. Conclusion

The model seems to represent the turbine characteristics with sufficient accuracy to be use for system dynamic simulations. It is convenient that the turbine is represented by ordinary differential equations, which can be solved incorporated in the equation system for the whole water power system directly. In the described model, it is assumed no frictional losses. It may easily be included in the equations, a tuning towards measured turbine characteristics and efficiency is then necessary.

Figure 9a shows a simulation of the performance of a RPT installed in the Water Power Laboratory as it goes from nominal speed to run-away speed of rotation. In the laboratory it is possible to apply a negative torque, forcing the RPT to turn the flow direction. When the flow turns, of course it gives a negative head, which means that the RPT acts as a pump, see Fig. 9b

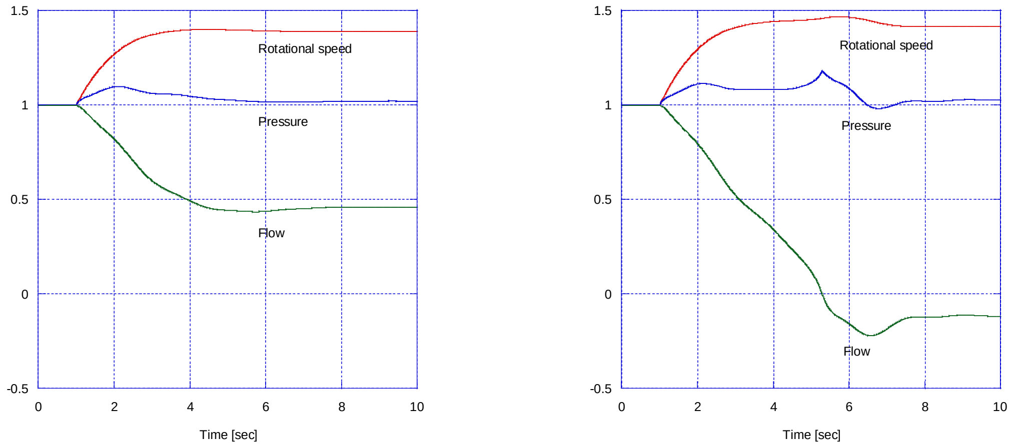

Fig. 9 Simulation of RPT performance as it goes from nominal speed to run-away-speed of rotation (9a). In Fig. 9b) a negative torque is applied to make the RPT turn the flow direction.

In a laboratory, the characteristics are measured by altering the speed of rotation and measure head, flow and torque. This is repeated for different wicket gate positions. This is not what happens in a hydropower plant. Here the head is given by the reservoir levels. The turbine gives a torque on the shaft which will result in a speed of rotation dependent on the electric torque. The flow will be dependent of the throttling effect of the runner, which is a function of the speed of rotation and the wicket gate position. The causality in laboratory tests and in a real hydropower plant is different. Hence, it is quite explainable that a S-shaped characteristic might occur.

Nomenclature

D_1	Inlet diameter of the turbine runner [m]	T	Torque [Nm]
D_2	Outlet diameter of the runner [m]	t_R	Specific rated torque [Nm/(m$_3$/s)]
B_1	Runner width at inlet [m]	t_s	Specific torque at ω=0 (starting torque) [Nm/(m3/s)]
r	Runner radius	ω	Angular speed of rotation [rad/s]
H	Head [m]	c	Absolute velocity [m/s]
H_t	Head difference over the runner [m]	u	Peripheral velocity [m/s]
Q	Flow [m$_3$/s]	v	Relative velocity [m/s]
$H_{th\infty}$	Theoretical pump head [m]	g	Acceleration of gravity [m/s^2]
α_1	Guide vane angle [^{o}]	ρ	Density of water [kg/m^3]
β_1	Runner inlet angle [^{o}]	h	Dimensionless head
β_2	Runner outlet angle [^{o}]	q	Dimensionless flow

κ	Guide vane opening degree	$\tilde{\omega}$	Dimensionless angular speed
		m_s	Dimensionless starting torque

Subscripts:

1 – Runner inlet	m – Meridional direction
2 – Runner outlet	t – Turbine
R – Rated	p – Pump
u – Peripheral direction	G - Generator

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Appendix A: The staring torque

The torque, T, is defined by the Euler equation by:

$$T = \rho Q(r_1 c_{1u} - r_2 c_{2u}) \quad \text{(A1)}$$

Examining the velocity diagram, the equation may be transformed to:

$$T = \rho Q(r_1 c_1 \cos\alpha_1 + r_2 A_z \cot\beta_2 c_1 \sin\alpha_1 - r_2^2 \omega) \quad \text{(A2)}$$

Examining the equation, the first two terms in the parenthesis, are functions of the absolute velocity and the geometrical properties. The equation has the form:

$$T = \rho Q(m_s - r_2^2 \omega) \quad \text{(A3)}$$

Where the specific starting torque is:

$$t_s = r_1 c_1 \cos\alpha_1 + r_2 A_z \cot\beta_2 c_1 \sin\alpha_1 \quad \text{(A4)}$$

The efficiency is Tω/ρgQH, hence:

$$\eta = \frac{1}{gH}(t_s - r_2^2 \omega)\omega \quad \text{(A5)}$$

Demanding that $\frac{d\eta}{d\alpha_1}=0$ when $\alpha_1=\alpha_{1R}$:

$$\frac{d\eta}{d\alpha_1}=\frac{1}{gH}\omega(-r_1c_1\sin\alpha_{1R}+r_2A_Z\cot\beta_2c_1\cos\alpha_{1R})=0 \tag{A6}$$

gives: $\cot\beta_2=\frac{r_1}{A_Zr_2}\tan\alpha_{1R}$ (A7)

Substituting for $\cot\beta_2$ in equation A4:

$$t_s=r_1c_1\cos\alpha_1-r_2A_z\frac{r_1}{A_Zr_2}c_1\tan\alpha_{1R}\sin\alpha_1 \tag{A8}$$

$$t_s=r_1c_1(\cos\alpha_1+\tan\alpha_{1R}\sin\alpha_1)$$

In the velocity diagram, c_{m1} is the flow divided on the perpendicular area A_z, and in general $c_{m1}=c_1\sin\alpha_1$. The connection between the opening degree k and the guide vane angle is: $\sin\alpha_1=\kappa\sin\alpha_{1R}$. Substituting in the equation for t_s and remembering that $Q=Q_nq$ gives:

$$t_s=\frac{r_1Q_R}{A_z\sin\alpha_{1R}}\frac{q}{\kappa}(\cos\alpha_1+\tan\alpha_{1R}\sin\alpha_1) \tag{A9}$$

Introducing dimensionless specific torque $m_s=\frac{t_s}{t_R}$ where rated specific torque is given by:

$$\eta_R\rho gQ_RH_R=T_R\omega_R \text{ hence: } m_R=\frac{T_R}{\rho Q_R}=\frac{gH_R}{\omega_R}=\frac{gH_Rr_1}{u_{1R}} \tag{A10}$$

$$m_s=\frac{t_s}{t_R}=\xi\frac{q}{\kappa}(\cos\alpha_1+\tan\alpha_{1R}\sin\alpha_1) \tag{A11}$$

where $\zeta=\frac{u_{1R}c_{1R}}{gH_R}$ which is a machine constant

Appendix B Derivation of the self-governing parameter, σ, and the machine constant ξ

ξ and ψ are dimensionless machine constants defined by the velocity vectors at the BEP.

$$\xi=\frac{u_{1R}c_{1R}}{gH_R} \tag{B1}$$

$$\psi=\frac{u_{2R}^{\ 2}}{gH_R} \tag{B2}$$

The stationary operational points may be found by setting the differential equations equal to zero, hence:

$$h-\left(\frac{q}{\kappa}\right)^2-\sigma(\tilde{\omega}^2-1)=0 \tag{B3}$$

$$q(m_s-\psi)-1=0 \tag{B4}$$

Solving the equations for flow q:

$$q=\pm\sqrt{\left|h\kappa^2-\sigma\kappa^2(\tilde{\omega}^2-1)\right|} \tag{B5}$$

The expression under the root sign will be negative at high angular speed of rotation, hence the absolute sign, and the flow will then be negative.

The opening degree, κ, is defined:

$$\kappa = \frac{q}{q_R} \text{ when } \tilde{\omega} = 1. \qquad \text{(B6)}$$

The slope of the flow characteristic can be found by taking the derivative of the flow:

$$\frac{dq}{d\tilde{\omega}} = \frac{1}{2}(h\kappa^2 - \sigma\kappa^2(\tilde{\omega}^2 - 1))^{-\frac{1}{2}}(-2\sigma\kappa^2) \qquad \text{(B7)}$$

At $\tilde{\omega}$ = 1 and rated head, i.e. h=1 $\frac{dq}{d\tilde{\omega}} = -\sigma\kappa$ (B8)

Turbines are designed to have BEP at the rated values, i.e. at h = 1, κ = 1, q = 1 and $\tilde{\omega}$ = 1. This implies also that $\frac{d\eta}{d\omega} = 0$ at BEP.

The efficiency is found by dividing the torque multiplied by the angular speed of rotation divided by flow multiplied by head, hence:

$$\eta = \frac{1}{h}(m_s - \psi\tilde{\omega})\tilde{\omega} \qquad \text{(B9)}$$

As shown in Appendix A, the dimensionless starting torque is: $m_s = \xi\frac{q}{\kappa}(\cos\alpha_1 + \tan\alpha_{1R}\sin\alpha_1)$. At κ = 1, the guide vane angle $\alpha_1 = \alpha_{1R}$, hence:

$$m_s = \xi\frac{q}{\kappa}(\cos\alpha_{1R} + \tan\alpha_{1R}\sin\alpha_{1R}) = \xi(\frac{\cos\alpha_{1R}\cos\alpha_{1R}}{\cos\alpha_{1R}} + \frac{\sin\alpha_{1R}\sin\alpha_{1R}}{\cos\alpha_{1R}})$$

$$m_s = \xi\frac{q}{\kappa}\frac{1}{\cos\alpha_{1R}} \qquad \text{(B10)}$$

By inserting the expression for m_s in eq. (B9) and setting $\tilde{\omega}$=1, and at $\tilde{\omega}$=1 q=κ pr. definition, the efficiency is:

$$\eta = \xi\frac{1}{\cos\alpha_{1R}} - \psi \qquad \text{(B11)}$$

At BEP, $\eta = \eta_R$, and a connection between the two machine constants ψ and ξ can be found:

$$\xi = (\eta_R + \psi)\cos\alpha_{1R} \qquad \text{(B12)}$$

The derivative of eq. (B11), setting: $m_s = \xi q\frac{1}{\cos\alpha_{1r}}$ when κ = 1:

$$\frac{d\eta}{d\tilde{\omega}} = \xi q\frac{1}{\cos\alpha_{1R}} - 2\psi\tilde{\omega} + \tilde{\omega}\xi\frac{1}{\cos\alpha_{1R}}\frac{dq}{d\tilde{\omega}} \qquad \text{(B13)}$$

At optimum point: $\frac{d\eta}{d\tilde{\omega}} = 0$ and when $\tilde{\omega}$ = 1 $\frac{dq}{d\tilde{\omega}} = -\kappa\sigma$ and q=κ:

$$\xi \frac{1}{\cos\alpha_{1R}} - 2\psi + \xi \frac{1}{\cos\alpha_{1R}}(-\sigma) = 0 \tag{B14}$$

Inserting the expression for ξ, eq (B12) gives:

$$\sigma = \frac{\eta_R - \psi}{\eta_R + \psi} \tag{B15}$$