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A new test of exponentiality against NDVRL

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Abstract. In this paper, the problem of testing exponentiality against net decreasing variance residual lifetime (NDVRL) classes of life distributions is investigated. For this property a nonparametric test is presented based on kernel method. The test is presented for complete and right censored data. Furthermore, Pitman's asymptotic relative efficiency (PARE) is discussed to assess the performance of the test with respect to other tests. Selected critical values are tabulated. Some numerical simulations on the power estimates are presented for proposed test. Finally, numerical examples are presented for the purpose of illustrating our test.

Key Words: Asymptotic normality, exponentiality, kernel function, right censored data, variance residual lifetime

1. INTRODUCTION

Let X be a non-negative continuous random variable with distribution function $F(x)$ and survival function $\overline{F}(x)=1-F(x)$. Mean residual lifetime $\mu(x)=E[X-x\mid X\geq x]=(1/\overline{F}(x))\int_x^\infty \overline{F}(u)du, x\geq 0$ and the variance residual lifetime $\sigma^2(x)=Var[X-x\mid X\geq x]=(1/\overline{F}(x))(2\int_x^\infty\int_y^\infty \overline{F}(u)dudy-\mu^2(x)), x\geq 0$. We are interested in life distributions for which $\sigma^2(x), x\geq 0$ is decreasing and use the notation (DVRL).

The problem of testing exponentiality against (DVRL) was investigated by Kanwar and Madhu (1991), Sen and Jain (1991), Gupta (2006), Abu-Youssef (2007, 2009) and recently by Al-Zahrani (2012). Our goal in this paper is to propose a new test for exponentiality against NDVRL distributions using nonparametric kernel method.

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For our propose, we need to redefine the variance residual lifetime $\sigma^2(x)$ in the following form $\sigma^2(x)=(2\overline{F}(x)G(x)-\nu^2(x))/\overline{F}^2(x)$ where, $G(x)=\int_x^\infty \nu(u)du, x\geq 0$, and $\nu(x)=\int_x^\infty \overline{F}(u)du, x\geq 0$.

Definition 1. A lifetime distribution F is said to have NDVRL if its conditional variance does not exceed initial variance, i.e., $\sigma^2(x)\leq\sigma^2(0)=\sigma^2$, for all $x>0$.

We make use of the property of the VRL stated in the above definition in order to build new test statistic for exponentiality. Testing exponentiality against various classes of lifetime distributions has been studied and used in different fields. For information, we refer to Ahmad (1992, 2001), Al-Zahrani and Stoyanov (2008), Hollander and Proschan (1972), and Marshall and Proschan (1972).
This paper is organized as follows. In section 2, a non-parametric test based on kernel function will be proposed to construct new test for exponetiality versus the class NDVRL. Also, present Monte Carlo null distribution critical points for sample size $n=5(5)50$. In section 3, Pitman's asymptotic relative efficiency (PARE) used to asses the performance of test and make a comparison with other available tests. In section 4, the power estimation of the test is given for some well known alternatives. A non-parametric test introduced in the right censored data in section 5. Finally, numerical examples are presented for the purpose of illustrating our test.in section 6.

2. TESTING AGAINST THE NDVRL CLASS

The test depends on random sample $X_1,...,X_n$ from a population with distribution function F. We test H_0 : F is exponential against H_1 : $F\in$ NDVRL and not exponential. In order to test H_0 against H_1 we may use the following measure of departure from

$$\Delta_F=\int_0^\infty f(x)[\overline{F}(x)\sigma^2-2\overline{F}(x)G(x)+\nu^2(x)]dF(x) \tag{1}$$

Note that under $H_0:\Delta_F=0$, while under $H_1:\Delta_F>0$.

Let $X_1,...,X_n$ be a random sample from F and let $\overline{F}_n(x)=\frac{1}{n}\sum_{j=1}^n I(X_j>x)$, is the empirical distribution of survival function $\overline{F}(x)$, $d\overline{F}_n(x)=\frac{1}{n}$, $\nu(x)$ is estimated by $\hat{\nu}_n(x)=\frac{1}{n}\sum_{j=1}^n (X_j-x)I(X_j>x)$, $\hat{G}_n(x)=\frac{1}{2n}\sum_{j=0}^n (X_j-x)^2 I(X_j>x)$ and pdf $f(x)$ is estimated by $\hat{f}_n(x)=\frac{1}{na_n}\sum_{j=1}^n k(\frac{x-X_j}{a_n})$ where $k(.)$ be a known pdf, symmetric and bounded with mean 0 and finite variance (σ_k^2). Let a_n be a sequence of reals such that $a_n \to 0$ and $na_n \to \infty$ $n \to \infty$. Hence, the estimate of Δ_F is proposed by

$$\hat{\Delta}_{F_n} = \int_0^\infty \hat{f}_n(x)[\overline{F}_n(x)\sigma^2 - 2\overline{F}_n(x)\hat{G}_n(x) + \hat{v}_n^2(x)]d\hat{F}_n(x) \tag{2}$$

According to the classical Glivenko-Cantelli theorem, see Serfling (1980), F_n converges to F in almost surely, when the sample size n tends to infinity. Clearly, Δ_F is continuous integral functional of F and also of v and G, where v and G are themselves defined in terms of F. Hence $\hat{v}_n \to v$ and $\hat{G}_n \to G$ as $n \to \infty$. The continuity of Δ_F as a function of F means that if $\widetilde{F}$ is a distribution function which in some sense is close to F, e.g. $\| \widetilde{F} - F \| < \varepsilon$ for a small $\varepsilon > 0$, then $\widetilde{\Delta}_F$ which is based on $\widetilde{F}$ is close to Δ_F in the sense that $\| \widetilde{\Delta}_F - \Delta_F \| < \varepsilon_1$ for a small $\varepsilon_1 > 0$. These properties suggest to write the following as an estimator of Δ_F :

$$\hat{\Delta}_{F_n} = \frac{1}{n^4 a_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} k(\frac{X_i - X_l}{a_n})[S_n^2 + (X_j - X_i)(X_k - X_i)]I(X_j > X_i)I(X_k > X_i) \tag{3}$$

To make the test statistic in (3) scale invariant, we take

$$\hat{\Delta}^*_{F_n} = \frac{\hat{\Delta}_{F_n}}{\overline{X}^2}. \tag{4}$$

in order to use the U-statistic procedure we set

$$\phi_n(X_1, X_2, X_3, X_4) = \frac{1}{a_n}K(\frac{X_1 - X_4}{a_n})[S_n^2 + (X_2 - X_3)(X_3 - X_1)]I(X_2 > X_1)I(X_3 > X_1). \tag{5}$$

Here X_1, X_2, X_3, X_4 are four independent lifetimes each with distribution function F. We define the symmetric kernel as $\eta(X_1, X_2, X_3, X_4) = \frac{1}{4!}\sum_R \phi_n(X_1, X_2, X_3, X_4)$, where R represents the sum over all arrangements of $X_1, X_2, X_3,$ and X_4. Then $\hat{\Delta}_{F_n}$ is equivalent to the U-statistic $U_n = \frac{1}{\binom{n}{4}}\sum_{i \neq j \neq k \neq l}\phi_n(X_i, X_j, X_k, X_l)$.

Theorem 1. *Let $n^4 a_n \to 0$ as $n \to \infty$,* suppose that f has bounded second derivative and if $V^2 = Var[\phi(X_1)] < \infty$ where $\phi(X_1)$ is given by (6), then $\sqrt{n}(\hat{\Delta}^*_{F_n} - \Delta^*_F)$ is asymptotically normal with mean 0 and variance V^2, where $\Delta^*_F = \Delta_F / \mu^2$ *and*

$$V^2 = \frac{1}{\mu^4}Var\{2f(X_1)[\sigma^2\overline{F}^2(X_1) + [\int_{X_1}^\infty y dF(y)]^2 - \overline{F}(X_1)\int_{X_1}^\infty y^2 dF(y)] + \int_0^{X_1} f^2(y)[2\sigma^2\overline{F}(y)$$
$$+2X_1\int_y^\infty u dF(u) - X_1^2]\overline{F}(y) - \int_y^\infty u^2 dF(u)]dy\}.$$

Under H_0, $V_0^2 = \frac{9}{14}$.

Proof. We use Theorem A of Serfling (1980), states that $\sqrt{n}(U_n-\theta)\xrightarrow{D} N(0,V^2)$, (i. e. convergence in distribution) where $V^2<\infty$ and $V^2=Var[\phi(X_1)]$. We can define $\phi(X_1)$ as

$$\begin{aligned}\varphi(X_1)&=E[\varphi_n(X_1,X_2,X_3,X_4)\mid X_1]+E[\varphi_n(X_2,X_1,X_3,X_4)\mid X_1]\\&\quad+E[\varphi_n(X_2,X_3,X_1,X_4)\mid X_1]+E[\varphi_n(X_2,X_3,X_4,X_1)\mid X_1]\end{aligned}\tag{6}$$

and

$$E[\phi_n(X_1,X_2,X_3,X_4)\mid X_1]=f(X_1)[\sigma^2\overline{F}^2(X_1)+(\int_{X_1}^{\infty}ydF(y))^2-\overline{F}(X_1)\int_{X_1}^{\infty}y^2dF(y)]\tag{7}$$

$$\begin{aligned}E[\varphi_n(X_2,X_1,X_3,X_4)\mid X_1]&=\sigma^2\int_0^{X_1}f(y)\overline{F}(y)dF(y)+X_1\int_0^{X_1}\int_y^{\infty}uf(y)dF(u)dF(y)\\&\quad-X_1\int_0^{X_1}yf(y)\overline{F}(y)dF(y)+\int_0^{X_1}\int_y^{\infty}uf(y)dF(u)dF(y)-\int_0^{X_1}\int_y^{\infty}u^2f(y)dF(u)dF(y)\end{aligned}\tag{8}$$

$$\begin{aligned}E[\varphi_n(X_2,X_3,X_1,X_4)\mid X_1]&=\sigma^2\int_0^{X_1}f(y)\overline{F}(y)dF(y)+X_1\int_0^{X_1}\int_y^{\infty}uf(y)dF(u)dF(y)\\&\quad+X_1\int_0^{X_1}yf(y)\overline{F}(y)dF(y)-\int_0^{X_1}\int_y^{\infty}uf(y)dF(u)dF(y)-X_1^2\int_0^{X_1}\int_y^{\infty}f(y)dF(u)dF(y)\end{aligned}\tag{9}$$

observe that $E[\phi_n(X_2,X_3,X_4,X_1)\mid X_1]$ has the same representation as (7). Thus by adding the sum of twice of right hand side of (7), (8), and (9) we get:

$$\begin{aligned}\phi(X_1)&=\{2f(X_1)[\sigma^2\overline{F}^2(X_1)+[\int_{X_1}^{\infty}ydF(y)]^2-\overline{F}(X_1)\int_{X_1}^{\infty}y^2dF(y)]\\&\quad+\int_0^{X_1}f^2(y)[2\sigma^2\overline{F}(y)+2X_1\int_y^{\infty}udF(u)-X_1^2]\overline{F}(y)-\int_y^{\infty}u^2dF(u)]dy\}.\end{aligned}$$

By taking the variance of $\phi(X_1)$ we get V^2. If the null hypothesis H_0 is true, $\phi_0(X_1)=\frac{1}{27}(-8+8e^{-3X_1}+24X_1-9X_1^2)$ then $\Delta_F=0$, and the the null variance $V_0^2=\frac{9}{14}$ by direct calculation. The theorem is provided.

We have simulate the upper percentile points for 95^{th}, 98^{th} and 99^{th} using the program Mathematica 9. Table 1 gives these percentile points of statistic $\hat{\Delta}_{F_n}$ in (3) and Figure 1 shows the relation between critical values and sample size.

- **The algorithm for getting the upper percentile points for 95^{th}, 98^{th} and 99^{th} to statistic $\hat{\Delta}_{F_n}$.**

(i) Perform 10^4 simulation run for samples of sizes $n=5(5)50$.

(ii) Calculate $\sqrt{\frac{14n}{9}}\hat{\Delta}_{F_n}$ and reject if these values exceed Z_α the standard normal variable.

(iii) Calculate the upper percentile points for 95^{th}, 98^{th} and 99^{th}.

Table 1. Crtical value of $\hat{\Delta}_{F_n}$

n	95^{th}	98^{th}	99^{th}
5	0.701	0.834	0.923
10	0.479	0.573	0.636
15	0.400	0.476	0.528
20	0.350	0.416	0.462
25	0.303	0.362	0.402
30	0.280	0.340	0.377
35	0.265	0.315	0.348
40	0.247	0.294	0.326
45	0.235	0.279	0.309
50	0.221	0.263	0.291

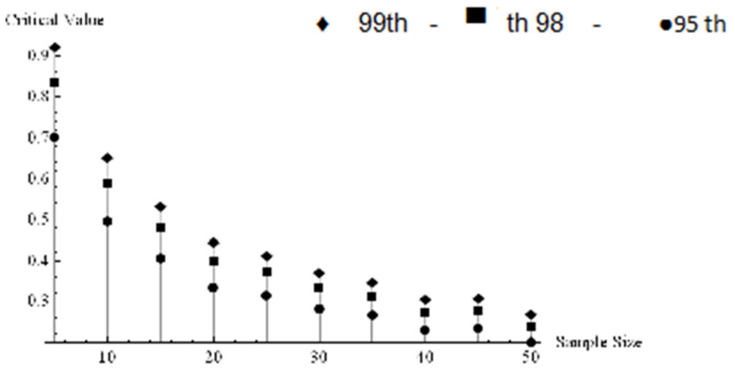

Figure 1. The Relation Between Sample Size and Critical Values

3. PITMAN'S ASYMPTOTIC RELATIVE EFFICIENCY (PARE)

Since the above test $\hat{\Delta}_{F_n}$ in (3) is new we compare our test to other classes. Here we choose the test K^* was presented by Hollander and Proschan (1972) for NBUE class and the test $\delta^*_{F_n}$ was presented by Al-Zahrani (2012) for NDVRL class. The comparison is achieved by using Pitman asymptotic relative efficiency, which can be defined as follows: Let T_{n_1} and T_{n_2} be two test statistics for testing $H_0 : F_\theta \in F_{\theta_n} ; \theta_n = \theta + \frac{k}{\sqrt{n}}$, where k an arbitrary constant, then the asymptotic relative efficiency of T_{n_1} relative to T_{n_2} can be defined as

$$e(T_{n_1}, T_{n_2}) = \frac{\mu_1'(\theta_0)/\sigma_1(\theta_0)}{\mu_2'(\theta_0)/\sigma_2(\theta_0)}$$

where $\mu_i'(\theta_0) = [lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{n_i})]_{\theta\to\theta_0}$ and $\sigma_i^2(\theta_0) = lim_{n\to\infty} Var(T_{n_i}), i = 1,2$. The efficiencies of $\hat{\Delta}_{F_n}$ are calculated for the Weibull distribution and the linear failure rate distribution in Table 2 as follows:

Table 2. PAE of $\hat{\Delta}_{F_n}$, K^* and $\delta^*_{F_n}$

Distribution	$\hat{\Delta}_{F_n}$	K^*	$\delta^*_{F_n}$
Linear failure rate family	0.6222	0.871	0.5976
Weibull family	1.8962	1.200	1.5793

In Table 3, we give PARE of the $\hat{\Delta}_{F_n}$, K^* and $\delta^*_{F_n}$ tests for which their efficiencies are mentioned in Table 2.

Table 3. PARE of $\hat{\Delta}_{F_n}$ with respect to K^* and $\delta^*_{F_n}$

Relative efficiency	Linear failure rate family	Weibull family
$e(\hat{\Delta}_{F_n}, K^*)$	0.7141	1.5801
$e(\hat{\Delta}_{F_n}, \delta^*_{F_n})$	1.0411	1.200

These calculations in Table 3 clearly indicate that our test is well comparable with other tests.

4. THE POWER ESTIMATES OF THE TEST STATISTIC

The power estimate of the test statistic $\hat{\Delta}_{F_n}$ in (3) is considered for the significant level at 95^{th} upper percentile in Table 4, for three of the most commonly used alternatives are

- (i) Linear failure rate family: $\overline{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, x \geq 0, \theta \geq 0$.
- (ii) Makeham family: $\overline{F}_2(x) = e^{-x-\theta(x-1+e^{-x})}, x \geq 0, \theta \geq 0$
- (iii) Weibull family: $\overline{F}_3(x) = e^{-x^{\theta}}, x \geq 0, \theta \geq 0$

These distributions are reduced to exponential distribution for appropriate values of θ.

Table 4. Power Estimates of $\hat{\Delta}_{F_n}$

Distribution	θ	$n=10$	$n=20$	$n=30$
Linear failure rate family $\overline{F}_1$ (L.F.R)	1 2 3	0.977 0.972 0.968	0.989 0.978 0.974	0.988 0.982 0.978
Makeham family $\overline{F}_2$ (M.F)	1 2 3	0.983 0.976 0.972	0.990 0.983 0.979	0.993 0.987 0.983
Weibull family: $\overline{F}_3$	2	0.978	0.985	0.989

The power estimates values in Table 4 shows clearly are increasing as n increasing and decreasing as θ increasing. The higher value of the power estimate indicates that the test statistic is more able to detect departure from exponentiality towards the NDVRL property as we see in Table 4.

5. TEST Of NDVRL CLASSES Of LIFE DISTRIBUTIONS IN CASE OF RIGHT *CENSORED DATA*

In this section, a test statistic proposed to test H_0 versus H_1 with randomly right censored samples. In the censoring model, let $X_1,...,X_n$ be independent and identically distributed according to a continuous life distribution F. Let $Y_1,...,Y_n$ be independent and identically distributed according to a continuous censoring distribution G. Also $X's$ and $Y's$ are independent in randomly right censored model, we observe the pairs (Z_i,δ_i), $i=1,2,...,n$ where $Z_i = min(X_i,Y_i)$ and $\delta_i = 1$ if $Z_i = X_i$, $\delta_i = 0$ if $Z_i = Y_i$.

Let $Z_{(0)} < Z_{(1)} < ... < Z_{(n)}$ denote the ordered $Z's$ and δ_i is the δ corresponding to $Z_{(i)}$, respectively . using the Kaplan and Meier (1958) estimator in the case of censored data $(Z_{(i)},\delta_i)$, $i=1,2,...,n$, then the proposed test statistic is given by (3) can be written using right censored data as:

$$\hat{\Delta}^C_{F_n} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\hat{f}(x)[S_n^2 + (Z_{(j)} - Z_{(i)})(Z_{(k)} - Z_{(i)})][\prod_{p=1}^{i-2} C_p^{\delta_p} - \prod_{p=1}^{i-1} C_p^{\delta_p}][\prod_{q=1}^{j-2} C_q^{\delta_q} - \prod_{q=1}^{j-1} C_q^{\delta_q}] \tag{10}$$

where

$\hat{f}(x) = \sum_{l=1}^{n}\delta_l k(x - Z_{(l)})$ and

$S_n^2 = 2\sum_{i=1}^{n} Z_{(i)}(Z_{(i)} - Z_{(i-1)})\prod_{j=1}^{i-1} C_j^{\delta_j} - (\sum_{j=1}^{l}\prod_{k=1}^{j-1} C_k^{\delta_k}(Z_{(j)} - Z_{(j-1)}))^2$

$$l = i + j \text{ if } Z_{(i)} + Z_{(j)} < Z_{(n)}, \text{ l= n if } Z_{(i)} + Z_{(j)} > Z_{(n)}$$

We have simulate the upper percentile points for 95^{th} , 98^{th} and 99^{th} using the program Mathematica 9. Table 5 gives these percentile points of statistic $\hat{\Delta}^{C}_{F_n}$ and Figure 2 shows the relation between the sample size and critical values in the case of censored data.

- **The algorithm for getting the upper percentile points for 95^{th} , 98^{th} and 99^{th} to statistic $\hat{\Delta}^{C}_{F_n}$.**

(i) Perform 10^4 simulation run for samples of sizes $n = 5(5)25$.

(ii) Calculate $\sqrt{\frac{14n}{9}}\hat{\Delta}^{c}{}_{F_n}$ and reject if these values exceed Z_α the standard normal variable.

(iii) Calculate the upper percentile points for 95^{th} , 98^{th} and 99^{th}.

Table 5. Crtical Value of $\hat{\Delta}^{C}_{F_n}$

n	95^{th}	98^{th}	99^{th}
5	0.545	0.648	0.717
10	0.385	0.458	0.507
15	0.314	0.379	0.419
20	0.272	0.329	0.358
25	0.243	0.289	0.321

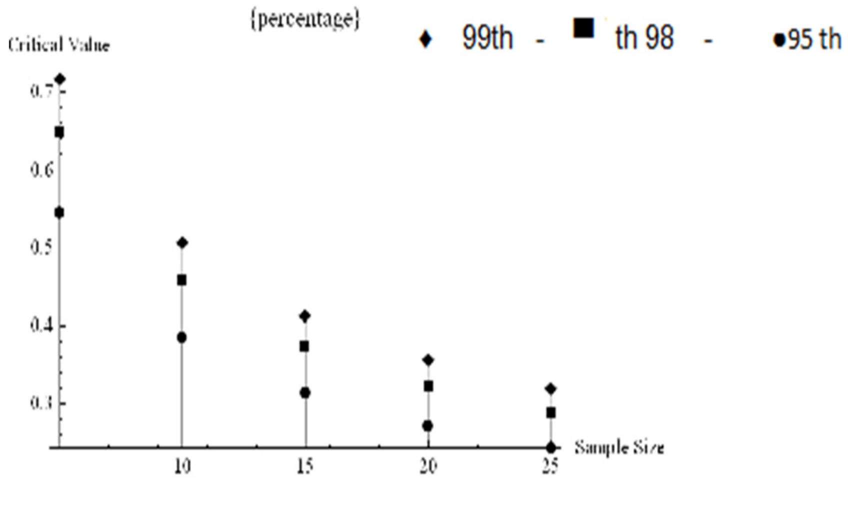

Figure 2. The Relation Between Sample Size and Critical Values (Right Censored Data)

6. APPLICATION

6.1 Application for complete data

Example 1. The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al (2004) the ordered life times (in days) are:

10; 14; 14; 14; 14; 14; 15; 17; 18; 20; 20; 20; 20; 20; 23; 23; 24; 26; 30; 30;
31; 40; 49; 51; 52; 60; 61; 67; 71; 74; 75; 87; 96; 105; 107; 107; 107; 116; 150.

It was found that the test statistic for the data set, $\sqrt{\frac{14n}{9}}\hat{\Delta}_{F_n}$ = -0.000005$< Z_\alpha$. Then we accept the null hypothesis of exponentiality.

6.2 Application right censored data

Example 2. The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al (2004) the ordered life times (in days) are:

(i) Non-censored data

10; 14; 14; 14; 14; 14; 15; 17; 18; 20; 20; 20; 20; 20; 23; 23; 24; 26; 30; 30;

31; 40; 49; 51; 52; 60; 61; 67; 71; 74; 75; 87; 96; 105; 107; 107; 107; 116; 150.

(ii) Censored data

30; 30; 30; 30; 30; 60; 150; 150; 150; 150; 150; 185.

It was found that the test statistic for the data set, $\sqrt{\frac{14n}{9}}\hat{\Delta}^c{}_{F_n} = -6.216x10^{-16} < Z_\alpha$. Then we accept the null hypothesis of exponentiality.

7. CONCLUSION

Testing exponentiality against the classes of life distributions has a good deal of attention. In this study, we derive a new test statistic based on a goodness of fit approach for testing the exponentiality against the NDVRL class of life distributions which are not exponential. This test is simple and has high relative efficiency for some commonly used alternatives. Critical values are tabulated for complete and right censored data.

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