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A generalized Hollander-Proschan test for NBUE alternative based on U-statistics approach

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Abstract. In this paper, we introduce U-statistics approach to generalized Hollander-Proschan test for new better than used (NBUE) alternative. We prove, the proposed test is equivalent to test was introduced by Anis and Mitra (2011) and includes test was introduced by Hollander Proschan (1975). Also, the asymptotic properties are studied. The powers of our test are estimated. The Pitman asymptotic efficiencies of proposed test are also calculated. Finally, the test is applied to some real data.

Key Words: Asymptotic normality, Hollander-Proschan test, NBUE class of life distributions, pitman asymptotic efficiency, U-Statistics

1. INTRODUCTION

The concept of aging is very useful in comparisons between various replacement policies. In Barlow and Proschan (1964) comparisons are made between the age and the block replacement policies when lifetimes have the increasing failure rate (IFR) property. Many other comparisons are given in Barlow and Proschan (1981) under the weaker assumptions that lifetimes are new better than used (NBU).

In this paper we interested to the new better than used in expectation (NBUE) family of distributions which is important in the study of replacement policies. In particular, Marshall and Proschan (1972) have shown that the average waiting time between any two consecutive failures when no planned replacement policies are adopted is smaller than or equal to the similar quantity when an age replacement policy is adopted if and only if the life distribution is NBUE. Moreover, this average waiting time is the same under both policies if the system life is exponential. If the average waiting time between the consecutive failures is an important criterion in deciding whether to adopt an age replacement policy over the failure replacement policy for a given system, then a reasonable way to decide would be to test whether the life distribution of the given system is exponential. Rejection of the exponentiality hypothesis on the basis of the observed data would suggest as favoring the adoption of age replacement plan.

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Motivated from these recent works, we develop a new test procedure for testing exponentiality against NBUE alternatives. This paper is organized as follows. In section 2, we present a test statistic based on U-statistics for testing H_0 : F is exponential against H_1 : F is NBUE and not exponential. Also, the asymptotic properties are studied. In section 3, Pitman asymptotic efficiency (PAE) of the test for several common distributions is evaluated to assess the performance of our test. In section 4, the power estimates for sample size n=10, 20, 30 are also calculated. Finally, applications using real data are also presented in section 5.

2. THE PROPOSED NON-PARAMETRIC TEST

Suppose the lifetime X of a component has a distribution function *F* which is unknown to us. Available to us are independent observations on *n* components; i.e. we have at our disposal a random sample $X_1, X_2, \ldots, X_n$ from the distribution *F*. Studies on F as exponential versus that it belongs to a nonparametric class of life distributions have continued over the past three decades or more. Of the most common and practical are the increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU) and new better than used in expectation (NBUE). Properties and applications of these aging classes can be found, in Bryson and Siddiqui (1969), Barlow and Proschan (1981), Rolski (1975), and Stoyan (1983).
In this paper we focus on testing a null hypothesis H_0 and its alternative H_1, where H_0 : F is exponential versus H_1 : *F* belongs to the class NBUE and F is not exponential.

Definition 1. The random variable *X* is said to be NBUE if $e_F(0) \geq e_F(x) \quad x > 0$, where $e_F(0) = \mu$ and $e_F(x) = \frac{1}{\overline{F}(x)} \int_x^\infty t dF(t) - x$.

Anis and Mitra (2011) proposed the following measure as the deviation of a NBUE distribution from exponentiality the following measure defined as

$$\gamma_j(F) = \int_0^\infty [\overline{F}(t)]^j [e_F(0) - e_F(t)] dF(t), \tag{1}$$

where j is a positive real number. They substituted the unknown distribution function F by its empirical distribution function F_n in (1) and obtained the following test statistics

$$\gamma_j(F_n) = \sum_{k=1}^n X_{(k)} \left[\frac{-1}{j(j+1)n} + \frac{1}{j}\left\{\left(\frac{n-k+1}{n}\right)^{j+1} - \left(\frac{n-k}{n}\right)^{j+1}\right\}\right], \tag{2}$$

Where $X_{(1)} \leq \ldots \leq X_{(n)}$ denotes the order statistics based on the random sample $X_1, X_2, \ldots, X_n$.
Now, we introduce a simple test based on U-statistics. We need to prove the following theorem.

Theorem 1. Let X is the NBUE random variable with distribution function F, then

$$\gamma_j(F) = \frac{1}{j} E[X_{(1:J+1)}] - \frac{\mu}{j(j+1)} \tag{3}$$

Proof.

We can write $\gamma_j(F)$ as

$$\gamma_j(F) = \int_0^\infty e_F(0)[\overline{F}(t)]^j\, dF(t) - \int_0^\infty e_F(t)[\overline{F}(t)]^j\, dF(t)$$

$$= \int_0^\infty \mu[\overline{F}(t)]^j\, dF(t) - \int_0^\infty \int_t^\infty x[\overline{F}(t)]^{j-1}\, dF(x)dF(t) + \int_0^\infty t[\overline{F}(t)]^j\, dF(t)$$

Use Fubini's theorem, we get

$$\gamma_j(F) = \frac{(j+1)}{j} \int_0^\infty t[\overline{F}(t)]^j\, dF(t) - \frac{\mu}{j(j+1)} \tag{4}$$

Suppose that the distribution $X_{(1:n)} = Min(X_1,..., X_n)$ is given by $F_{x_{(1:n)}}(x) = [\overline{F}(x)]^n$

Hence,

$$E[X_{(1:n)}] = \int_0^\infty nt[\overline{F}(t)]^{n-1}\, dF(t)$$

When n=j+1,

$$E[X_{(1:j+1)}] = \int_0^\infty (j+1)t[\overline{F}(t)]^j\, dF(t)$$

We therefore re-write (4) as

$$\gamma_j(F) = \frac{1}{j} E[X_{(1:j+1)}] - \frac{\mu}{j(j+1)} \tag{5}$$

Now, we using the U-statistics approach to find the estimator of $\gamma_j(F)$ as follows: Let, $U_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased estimator for μ. The estimator of $E[X_{(1:j+1)}]$ based on U-statistics is given by $U_{j+1} = \binom{n}{j+1}^{-1} \sum_{1\le i\le n} h(X_{i,1},..., X_{i,j+1})$, where $h(X_{i,1},..., X_{i,j+1}) = Min(X_1,..., X_{j+1})$ is the symmetric kernel. Hence, an unbiased estimator of $\gamma_j(F)$ is

$$\hat{\gamma}_j(F_n) = \frac{1}{j} U_{j+1} - \frac{1}{j(j+1)} U_1$$

After simplification, we can write the above expression as

$$\hat{\gamma}_j(F_n) = \frac{\Gamma(n-j)\Gamma(j+1)}{j\Gamma(n+1)} \sum_{i=1}^{n} [(n-i)^j X_i - \frac{1}{nj(j+1)} \sum_{i=1}^{n} X_i \tag{6}$$

To make the test scale invariant, let $\gamma_j^*(F)=\dfrac{\gamma_j(F)}{\mu}$, which can be estimated by

$$\hat{\gamma}_j^*(F_n)=\frac{\hat{\gamma}_j(F_n)}{\overline{X}}.$$

Note that the test statistic in (5) is equivalent to test proposed by Anis and Mitra (2011) to measure the deviation of a NBUE distribution from exponentiality. Also, it is important to note the test statistic in (5) is equivalent to test proposed by Hollander and Proschan (1975) when j=1. But only different in multiplicative factor appeared in the denominator. In fact we have $\hat{\gamma}_1(F_n)=\dfrac{1}{n(n-1)}\sum_{i=1}^{n}(\dfrac{3n}{2}-2i+\dfrac{1}{2})X_i$ instead of $\hat{\gamma}_1(F_n)=\dfrac{1}{n^2}\sum_{i=1}^{n}(\dfrac{3n}{2}-2i+\dfrac{1}{2})X_i$.

The following theorem summarizes the asymptotic normality of $\hat{\gamma}_j(F_n)$.

Theorem 2.

(i) As n $\to\infty$, $\sqrt{n}(\hat{\gamma}_j(F_n)-\gamma_j(F))$ is asymptotically normal with mean 0 and variance $\dfrac{(j+1)^2}{j^2}\sigma_j^2+\dfrac{1}{j^2(1+j)^2}\sigma^2-\dfrac{2}{j^2}\sigma_{1j}^2$,

where,

$$\sigma_j^2=Var[jX-jXF(x)+j\int_0^x ydF(y)],$$

$$\sigma^2=Var[X]$$

$$\sigma_{ij}^2=E[(jX-jXF(x)+j\int_0^x ydF(y))(X-\mu)].$$

(ii) Under H_0, the variance σ_0^2 is

$$\sigma_0^2=\frac{j^2(1+j)^4-6j(1+j)^2+12}{12j^2(1+j)^2}$$

Proof.

(i) Since,

$$\sqrt{n}(\hat{\gamma}_j(F_n)-\gamma_j(F))=\sqrt{n}(\frac{1}{j}U_{j+1}-\frac{1}{j(j+1)}U_1-\frac{1}{j}E(X_{(1:j+1)})+\frac{\mu}{j(j+1)})$$
$$=\sqrt{n}(\frac{1}{j}(U_{j+1}-E(X_{(1:j+1)}))-\frac{1}{j(j+1)}(U_1-\mu))$$

Using the central limit theorem for U-statistics Serfling (2001), we get $\sqrt{n}(U_{j+1}-E(X_{(1:j+1)})$ has limiting distribution $N(0,\dfrac{(j+1)^2}{j^2}\sigma_j^2)$ as $n\to\infty$, where σ_j^2

is the asymptotic variance of U_{j+1}. Also, the limiting distribution of $\frac{\sqrt{n}}{j(j+1)}(U-\mu)$ is $N(0,\frac{1}{j^2(1+j)^2}\sigma^2)$ as $n \to \infty$. Hence $\sqrt{n}(\hat{\gamma}_j(F_n)-\gamma_j(F))$ has limiting distribution $N(0,\frac{(j+1)^2}{j^2}\sigma_j^2+\frac{1}{j^2(1+j)^2}\sigma^2-\frac{2}{j^2}\sigma_{1j}^2)$.

(ii) Under H_0,

$$\mu_0 = E(\hat{\gamma}_j(F_n)) = 0,$$

$$\sigma_0^2 = \frac{j^2(1+j)^4 - 6j(1+j)^2 + 12}{12 j^2 (1+j)^2}.$$

Setting; $j=1$ then $\sigma_0^2 = \frac{1}{12}$, with agrees with results obtained in the Hollander and Proschan (1975).

Remark. Under H_0 the limiting distribution $\frac{\sqrt{n}\gamma_j^*(F_n)}{\sigma_0}$ is $N(0,1)$. Hence for large value of n, we reject the null hypothesis of exponentiality if $\frac{\sqrt{n}\gamma_j^*(F_n)}{\sigma_0} > Z_\alpha$ where Z_α is the upper $\alpha-$percentile of $N(0,1)$.

3. THE PITMAN ASYMTOTIC EFFICIENCY (PAE)

For the test suggested above the *PAE* is computed to assess the performance of our test, where

$$PAE(\gamma_j(F)) = \left| \frac{d}{d\theta} \frac{\mu'|_{\theta=\theta_0}}{\sigma_0} \right|$$

where, $\mu' = \frac{d}{d\theta}\gamma_j(F)$ and $\sigma_0 = \sqrt{\sigma_0^2}$

Now, we evaluate the PAE for some commonly used distribution in reliability. These are

1. Linear failure rate: $\overline{F}_\theta(x) = e^{-(x+\frac{1}{2}\theta x^2)}$, $x>0$, $\theta \ge 0$;
2. Makeham: $\overline{F}_\theta(x) = e^{-x-\theta(x-1+e^{-x})}$, $x>0$, $\theta \ge 0$;
3. Weibull: $\overline{F}_\theta(x) = e^{-x^\theta}$ $x>0$, $\theta>0$.

Under H_0, we evaluate PAE for the above distributions; we get the result in Table 2 as follows

Table 1. PAE of $\gamma_j(F)$

Distribution	PAE
Linear Failure Rate	$\dfrac{2\sqrt{3}j}{(1+j)\sqrt{12-6j(1+j)^2+j^2(1+j)^4}}$
Makeham	$\dfrac{2\sqrt{3}j}{(1+2j)\sqrt{12-6j(1+j)^2+j^2(1+j)^4}}$
Weibull	$\dfrac{2\sqrt{3}Log[1+j]}{\sqrt{12-6j(1+j)^2+j^2(1+j)^4}}$

The maximum values of PAE for the Weibull, linear failure rate and Makeham families are 1.27, 0.94 and 0.639 respectively at j= 0.9, 0.91 and 0.89. Table 2 shows that the Pitman asymptotic relative efficiency (PARE) of our test and generalized Hollander-Proschan test for NBUE alternative was introduced by Anis and Mitra (2011) ($\gamma^*{}_j(F)$) and test was introduced by Hollander Proschan (1975) ($\gamma_1(F)$).

Table 2. PARE of $\gamma_j(F)$ with respect to $\gamma^*{}_j(F)$ and $\gamma_1(F)$

Relative Efficiency	Linear Failure Rate	Makeham	Weibull
E($\gamma_j(F), \gamma^*{}_j(F)$)	0.96	2.211	1.04
E($\gamma_j(F), \gamma_1(F)$)	1.09	2.211	1.06

It is clear that our test statistic is more efficient for generalized Hollander-Proschan test for NBUE alternative was introduced by Anis and Mitra (2011) ($\gamma^*{}_j(F)$) and test was introduced by Hollander and Proschan (1975) ($\gamma_1(F)$).
Table 3 and Figure 1 show the efficiency values for three distributions for j=0.25(0.25)2. It is clear that PAR increases in j=0.25(0.25)1 and decreases in j=1.25(0.25)2.

Table 3. The efficiency values for three distributions for j=0.25(0.25)

j	PAR		
	Linear failure rate	Weibull	Makeham
0.25	0.2212	0.2458	0.1843
0.5	0.4523	0.55025	0.3392
0.75	0.7941	1.0370	0.5559
1	0.8660	1.2005	0.5773
1.25	0.5129	0.7487	0.3297
1.5	0.3146	0.4804	0.1966
1.75	0.2123	0.3376	0.1297
2	0.1529	0.2520	0.0917

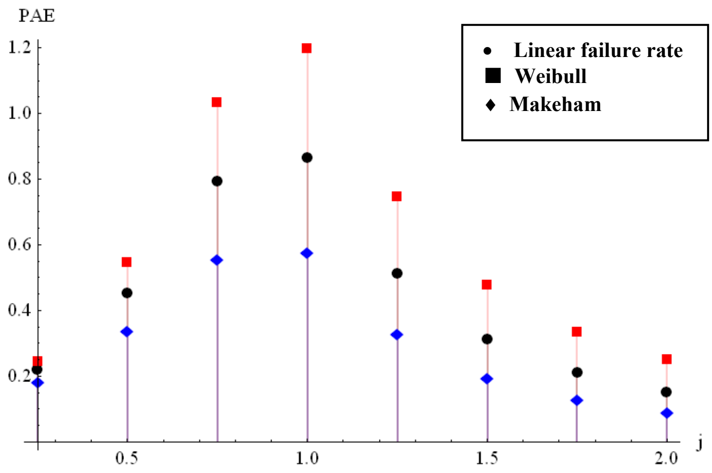

Figure 1. The efficiency values for three distributions for j=0.25 (0.25) 2

4. POWER ESTIMATES

The power estimate of the test statistic $\gamma^*_j(F)$ is useful in clarifying how much the test can detect the departure from exponentiality towards the class NBUE. The higher value of the power estimate indicates that the test statistic is more able to detect such a departure.

Table 4. Power Estimates for $\gamma^*_j(F)$

Distribution	J	θ	n=10	n=20	n=30
Linear Failure Rate	1	0.5	0.8672	0.9474	0.9833
		0.75	0.8382	0.9144	0.9599
		1	0.8169	0.8852	0.9331
	1.5	0.5	0.8222	0.8928	0.9405
		0.75	0.8020	0.8624	0.9089
		1	0.7876	0.8389	0.8814
	2	0.5	0.8160	0.8838	0.9317
		0.75	0.7977	0.8556	0.9011
		1	0.7846	0.8338	0.8752
Makeham	1	0.5	0.8408	0.9177	0.9626
		0.75	0.8238	0.8951	0.9427
		1	0.8100	0.8749	0.9224
	1.5	0.5	0.8033	0.8646	0.9113
		0.75	0.7917	0.8458	0.8897
		1	0.7824	0.8301	0.8706
	2	0.5	0.7987	0.8572	0.9030
		0.75	0.7880	0.8397	0.8824
		1	0.7795	0.8251	0.8642

The power of the test statistics $\gamma^*{}_j(F)$ is considered for 5% percentile in Table 4 for two alternatives. These alternatives are:

1. Linear failure rate: $\overline{F}_\theta(x) = e^{-(x+\frac{1}{2}\theta x^2)}$, $x > 0$, $\theta \geq 0$;
2. Makeham: $\overline{F}_\theta(x) = e^{-x-\theta(x-1+e^{-x})}$, $x > 0$, $\theta \geq 0$;

For appropriate values of θ, these distributions can be reduced to the exponential distribution. The power estimate of the test statistic $\gamma^*{}_j(F)$, given in Table 4 shows the chance of detecting departure from exponentiality towards the NBUE property as θ decreases, or the sample size n increases for the linear failure rate and Makeham.

5. APPLICATION

Example 1. Bryson and Siddiqui (1969) have analyzed data which are survival times, in days from diagnosis, of patients suffering from chronic granulocytic leukemia. Here n = 43 and the order statistic $X_1 < X_2 < \ldots < X_{43}$ are: 7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455 , 468, 495, 497, 532, 571, 579, 581, 650, 702, 715, 779, 881 , 900, 930, 968, 1077, 1109, 1314, 1334, 1367, 1534, 1712, 1784 , 1877, 1886, 2045, 2056, 2260, 2429, 2509 .

We compute blew the value of test statistic and the associated p-value for different values of j in Table 5. We observe that the null hypothesis of exponentiality is rejecting for *j*=0.75, 1 and accepted for j=0.25, 0.5, 1.25.

Table 5. Results for Example 1

j	The test statistic	p-value
0.25	1.0767	0.1408
0.5	1.3210	0.0932
0.75	1.8038	0.0357
1	1.6860	0.0459
1.25	0.8987	0.1844

Example 2. In an experiment at the Florida State University to study the effect of methyl mercury poisoning on the life lengths of goldfish, goldfish were subjected to various dosages of methyl mercury Kochar (1985). At one dosage level, the ordered times in days to death are: 0.86, 0.88, 1.04, 1.24, 1.35, 1.41, 1.45, 1.65, 1.67, and 1.67.

We compute below the value of our test statistic and the associated p-value for different values of j and s in Table 6. We observe that the null hypothesis of exponentiality is accepted for all values of j.

Table 6. Results for Example 2

j	The test statistic	p-value
0.25	-0.9478	0.8284
0.5	-1.2424	0.8930
0.75	-1.7778	0.9622
1	-1.7188	0.9571
1.25	-0.9391	0.8262

5. CONCLUSION

In this paper, we introduce U- statistic approach to generalized Hollander-Proschan type test for NBUE which is introduces by Anis and Mitra (2011).This test includes the test statistic proposed by Hollander- Proschan (1975). We derived the asymptotic distribution of our test statistic and studied the efficiency to assess the performance of our test statistic. Also, the power estimates are calculated for different values of j, θ and n.

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