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Computational procedures for exponential life model incorporating Bayes and shrinkage techniques

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Abstract. It is well known that using any additional information in the estimation of unknown parameters with new sample of observations diminishes the sampling units needed and minimizes the risk of new estimators. There are many rational reasons to assure that the existence of additional information in practice and there exists many practical cases in which additional information is available in the form of target value (initial value) about the unknown parameters. This article is described the problem of how the prior initial value about the unknown parameters can be utilized and combined with classical Bayes estimator to get a new combination of Bayes estimator and prior value to improve the properties of the new combination. In this article, two classes of Bayes-shrinkage and preliminary test Bayes-shrinkage estimators are proposed for the scale parameter of exponential distribution. The bias, risk and risk ratio expressions are derived and studied. The performance of the proposed classes of estimators is studied for different choices of constants engaged in the estimators. The comparisons, conclusions and recommendations are demonstrated.

Key Words: *exponential failure model, complete data, censored data, Bayes-shrinkage, preliminary test, risk, relative risk*

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1. INTRODUCTION

1.1 The uses of the exponential life model

It is well known that in many issues of censored data of continuous random variable there are many functions/parameters like "time of failure" and "average life" are represented by the exponential distribution. In addition, there are wide applications of exponential model in many fields and specialties. In fact, there are many reasons of using exponential model in numerous problems and especially in reliability, life testing and evaluation problems.

Some of particular reasons which show the usefulness of exponential model are the following: it is well known that the Bathtub curve for the exponential model (or the reliability curve) has three featured phases. These phases are called "debugging phase", "the chance failure phase" and "the wear-out phase". Exponential model seems to be a suitable model for modeling failures in second phase "the chance failure phase" since most of individual components/systems are dissipated maximum lifetimes in this phase. In addition, for the case of constant failure rate property of many variables, the exponential model is the most appropriate to model the inter arrival times.

Today, even though not widely defended, the unsupported assumption that most reliability engineering problems can be modeled well by the exponential distribution is still widely held. In a quest for simplicity and solutions that we can grasp, derive and easily communicate, many practitioners have embraced simple equations derived from the underlying assumption of an exponential distribution for reliability prediction, accelerated testing, reliability growth, maintainability and system reliability analyses.

Indeed, there are numerous and important applications of exponential distribution (see e.g. Davis (1952), Mann, et al. (1974) and Davison (2003)) in the fields of reliability estimation, life testing, quality control, maintenance, warranty and in most of specialties (engineering, physics, chemistry, biology, forestry, metrology, hydrology, medicine, pharmacy, economics, management, quality control, geology, geography and astronomy).

1.2 Shrinkage and preliminary-test procedures

It is very famous and well recognized by numerous experimenters in any applied fields that there are some prior information, experimental observations and estimation values about many practical problems. In addition, many experiments in each field were repeated by different or similar researchers in different times, i.e. it is possible to get some target values regarding many problems in each field due to past experiments.

Utilizing such experimental information in new experiments/studies will be beneficial in many directions. The sampling units may be utilized, the sampling costs may be reduced and some good statistical properties could be achieved.

Statistically speaking denote by θ_0 to any available prior estimate (or initial value) from previous experiments. Let us assume that the random sample of $X_1, X_2, \ldots, X_n$ from the probability distribution function $F(X \mid \theta)$, where θ is an unknown parameter of the random variable X and $F(X|\ .)$ has an exact and explicit form.

In accordance with the above ideas of availability of previous experiments and based on to Katti (1962), Thompson (1968), Mehta and Srinivasan (1971), Kambo et al. (1990, 1992) and Al-Hemyari and Al-Ali (2013) there are some rational/logical reasons to assure the availability of θ_0. Some of which are:

(i) "The prior value θ_0 of θ in many practical problems exists",

(ii) "We believe θ_0 is close to the true value of θ_o," or

(iii) "We fear that θ_0 may be near the true value of θ, i.e., something bad happens if $\theta_0 \cong \theta$, and we do not know about it".

On the bases of the above arguments, Huntsberger (1955) was the first to considered the following type of weighting shrinkage estimator,

$$\tilde{\theta}_H = \{\phi(\hat{\theta})\hat{\theta} + (1-\phi(\hat{\theta}))\theta_0\}, \tag{1}$$

where $\hat{\theta}$ is any good estimator of θ, θ_0 is any prior value of θ, $\phi(\hat{\theta})(0 \le \phi(\hat{\theta}) \le 1)$, be any weighting function of $\hat{\theta}$ and $1-\phi(\hat{\theta})$ assigning the level of dependability in θ_0. Let the weight function $\phi(\hat{\theta})$ of $\tilde{\theta}_H$ is splited into two portions and chosen accordance to the result of testing θ_0 in a small region R as follows,

$$\phi(\hat{\theta}) = \begin{cases} \varphi(\hat{\theta}), if\ \hat{\theta} \in R, \\ 1, if\ \hat{\theta} \notin R. \end{cases} \tag{2}$$

In this case $\varphi(\hat{\theta})(0 \le \varphi(\hat{\theta}) \le 1)$ is the new shrinkage weight function. Substituting $\varphi(\hat{\theta})$ given above in the estimator $\tilde{\theta}_H$ (equation (1)) leads to the Thompson (1968) type estimator $(\tilde{\theta}_T)$, which is called the preliminary test shrinkage estimator and given by,

$$\tilde{\theta}_T = \begin{cases} \varphi(\hat{\theta})(\hat{\theta}-\theta_0)+\theta_0, & if \quad \hat{\theta}_1 \in R \\ \hat{\theta}, \quad if \quad \hat{\theta}_1 \notin R \end{cases} \tag{3}$$

where R is preliminary test region constructed on space of θ and based on θ_0. It is worth to mentioning that if $\varphi(\hat{\theta}) = k$, k is constant such that $0 \le k \le 1$, then $\tilde{\theta}_T$ construed the first attempt of preliminary test shrinkage estimator which was proposed by Thompson. Also, the author extended the proposed estimator to binomial, Poisson and normal parameters.
Following Hunteburger (1955) and Thompson (1968), it is worth to mentioning that Mehta and Srinivasan (1971), Pandey (1988, 1983), Chiou (1992a, 1992b, 1990, 1987), Kambo et al. (1990,1992), Kourouklis (1994), Lemmer (2006), Al-Hemyari (2010), Al-Hemyari and Jehel (2011), Al-Hemyari and Al-Ali (2013), Al-Hemyari et al. (2013), Al-Hemyari and Al-Dabagh (2014), Al-Hemyari and Al-Dolami (2014) have studied estimators (1) and (3) for parameters and reliability function of exponential, Weibull and normal models by proposing different weight functions.

2. BAYES AND BAYES-SHRINKAGE PROCEDURES

In this section, the Bayes estimator and the Bayes-shrinkage estimator are reviewed and discussed.

2.1 The MLE and the Bayes procedures

The one-parameter exponential model is defined by,

$$f(x\,|\,\theta) = (1/\theta)\exp\{-(\frac{x}{\theta})\},\quad x \geq 0,\quad \theta > 0, \tag{4}$$

where θ being the characteristic life, acts as a scale parameter. Let $\hat{\theta}$ be an estimator of θ computed from a random samples $X_1, X_2, \ \ldots \ , X_n$ of size n taken from (4).

Define the MLE $\hat{\theta}$ of θ based on n observations by,

$$\hat{\theta} = S\,/\,n,\quad S = \sum_{i=1}^{n_1} X_i, \tag{5}$$

where $\hat{\theta}$ is a function of the complete sufficient statistics S and the statistics $T(\hat{\theta}) = 2n\hat{\theta}/\theta$ is distributed as chi-square random variable. Sarhan and Greenberg (1962), Mann et al. (1974), Sinha (1986) and Bain and Engelhardt (1991) and several other authors have studied the problem of the classical estimation of the parameter θ.
For the problem of assuming the random property of the unknown parameters, numerous papers have studied the Bayes estimators of the parameters of statistical distribution including exponential model as indicated in Section 1.
Consider the class of prior distributions of θ (see Sinha (1986)) given by,

$$g(\theta) \propto \theta\ \ \exp\{-(\frac{\sigma}{\theta})\},\quad \sigma,\ \ \kappa \geq 0,\quad \theta > 0. \tag{6}$$

Remark 1. It may be worth to mentioning that in the class of prior distributions given in (6) if $\kappa = C,\ \ C > 0$ and $\sigma = 0$ it will tends to a general class of priors. If $\kappa = 1$ and $\sigma = 0,$ then $g(\theta)$ tends to Jeffrey's prior.

It is well known that the posterior density function of the parameter θ (see, Sinha (1986), pages 145-146) using $g(\theta),$ is given by,

$$g(\theta\,|\,T) = \frac{1}{\Gamma(n+\kappa-1)}\theta^{-(n+\kappa)}(S+\sigma)^{n+\kappa-1}\exp-(\frac{S+\sigma}{\theta}),\quad \theta > 0. \tag{7}$$

The expression of the Bayes estimator of the parameter θ under the squared error loss function is denoted by $\hat{\theta}_0$ and defined by,

$$\hat{\theta}_0 = \int_0^{\infty} \theta g(\theta\,|\,T)\,d\theta. \tag{8}$$

The expression of the Bayes estimator of the parameter θ as a result of the integration of the above equation can be displayed (see e.g. Bernardo and Smith (2000) and Sinha (1986)) by,

$$\hat{\theta}_0 = (S+\sigma)/(n+\kappa-2) = (n\hat{\theta}+\sigma)/(n+\kappa-2), \quad \sigma,\kappa > 0. \tag{9}$$

2.2 The classical Bayes-shrinkage procedure

Lemmer (1981) was suggested to shrink the unbiased linear estimator $\hat{\theta}$ of θ to ordinary Bayes estimator $\hat{\theta}_0$. If $\hat{\theta}_0$ was used instead of the prior value θ_0 in Huntsberger (1955) type estimator $\tilde{\theta}_H$, we will get the general type of Bayes-shrinkage estimator $\tilde{\theta}_L$ and defined by,

$$\tilde{\theta}_L = \{\varphi(\hat{\theta}_0)\hat{\theta} + (1-\varphi(\hat{\theta}_0))\hat{\theta}_0\}, \tag{10}$$

where $\phi(\hat{\theta}_0)(0 \le \phi(\hat{\theta}_0) \le 1)$, is a shrinkage weighting function assigning the level of dependability in $\hat{\theta}$ and $1-\phi(\hat{\theta}_0)$ assigning the level of dependability in the Bayes estimator $\hat{\theta}_0$. If we assume that the shrinkage weighting function $\phi(\hat{\theta}_0)$ is constant such that $0 \le k \le 1$, then $\tilde{\theta}_L$ tends to the first version of Bayes-shrinkage estimator which was proposed by Lemmer (1981).

It may be worth to be mentioned that Lemmer extended the proposed Bayesian shrinkage estimator to the parameters of binomial, Poisson and normal parameters. Following Lemmer (1981), Pandey and Upadhyay (1987, 1985a, 1985b, 1985c), Upadhyay and Singh (1992) and Yang et al. (2013) studied the Bayes-shrinkage estimator for the exponential parameter in different contexts.

The Bayesian shrinkage estimator has been adapted in various other estimation problems by Soland (1968), Pandey and Upudhyoy (1987, 1985a, 1985b, 1985c), Pandey and Singh (1989), Chiou (1993), Bennet et al. (2006), Prakash and Singh (2009), Zhao, et al. (2010), Lanping (2011), Shanubhouge and Jiheel (2013), and Al-Hemyari and Al-Dabag (2014).

The purpose of this paper is to modified the Bayes-shrinkage estimator and proposed the preliminary test Bayes-shrinkage estimator to the scale parameter of exponential distribution based on complete and censored data and using the weight functions $\phi(\hat{\theta}_0)$ and $\varphi(\hat{\theta}_0)$ and the region R. More specifically, the following classes of estimators are proposed and studied:

$$\tilde{\theta}_1 = \varphi(\hat{\theta}_0)(\hat{\theta}_0 - \theta_0) + \theta_0, \tag{11}$$

and

$$\tilde{\theta}_2 = \begin{cases} \varphi(\hat{\theta}_0)(\hat{\theta}_0 - \theta_0) + \theta_0, & if \quad \hat{\theta} \in R, \\ \hat{\theta}_0, & if \quad \hat{\theta} \notin R. \end{cases} \tag{12}$$

The classical and Bayes estimators are discussed in section 2. The computation of proposed estimators for the scale parameter of exponential distribution are derived and studied in sections 3 and 4 respectively. Two estimators are originated from the proposed classes are also studied for specific choices of $\phi(\hat{\theta}_0)$ and $\varphi(\hat{\theta}_0)$ and the region R in sections 3 and 4. In section 5, the proposed estimators are extended to the case of censored samples. The performance of the proposed classes is studied numerically based on the bias ratio, risk and the relative risk expressions. The rest of the paper (sections 6 and 7) is concentrated to the simulation results and discussion and summary, conclusions and recommendation.

3. THE MODIFIED BAYES–SHRINKAGE PROCEDURE

Just like the idea of modeling or formulization the Bayes estimators, it is well known that combining any additional information about the unknown parameters and a sample of observations in the estimation process of the unknown parameters will improve the statistical properties of the final estimators.

As explained earlier, the prior initial value may be arise for a number of reasons, and such prior value were combined numerously with the classical estimators to introduce the concepts of ordinary shrinkage estimators as well as the Bayes estimator was combined with classical estimator to introduce the concepts of Bayes–shrinkage estimators.

To some extent, combination of the Bayes estimator and prior initial value θ_0 is a reasonable choice with rational reason, satisfies same ideas of ordinary shrinkage estimators and Bayes estimators and it may be improved the estimation properties of the new combination.

Sections 1 and 2 present the problem of utilizing initial prior information θ_0 of the unknown parameter θ of statistical distributions in the new estimation problems, is based on a combination of the classical and Bayes estimator.

In this section, a modified Bayes-shrinkage procedure for the parameter θ is applied for the case of exponential distribution and studied.

3.1 The computation of the proposed Bayes–shrinkage procedure

The general Bayes –shrinkage estimators is denoted by $\tilde{\theta}_0$ and to be derived in this section. It is worth to be mentioning that the suggested class of new Bayes-shrinkage estimators is a modification of Lemmer Bayes-shrinkage estimator, and also is a general case of ordinary shrinkage estimators. The proposed class of Bayes–shrinkage estimators is confined of Bayes estimator and prior information for the scale parameter θ and given by,

$$\tilde{\theta}_0 = \{\phi(\hat{\theta}_0)\hat{\theta}_0 + (1-\phi(\hat{\theta}_0))\theta_0\} \tag{13}$$

where $\hat{\theta}_0$ is the Bayes estimator of θ, θ_0 is the prior value of θ, and $\phi(\hat{\theta}_0)$ $(0 \le \phi(\hat{\theta}_0) \le 1)$, represents any weighting function of $\hat{\theta}_0$ specifying the degree of belief in $\hat{\theta}_0$.

Remark 2. The main differential characteristics of the proposed classes of Bayes–shrinkage estimators $\tilde{\theta}_0$ (equation (13)) from the existing Bayes–shrinkage estimators are: the proposed class of estimators, which is a combination of Bayes estimator and prior information rather than two classical estimators (i.e. Bayes estimator and MLE) and the use of a modified suitable exponential weighting function.
The bias of $\tilde{\theta}_0$ is defined by,

$$B(\tilde{\theta}_0 \mid \theta) = \int_{\hat{\theta}=0}^{\infty} \tilde{\theta}_0 f(\hat{\theta} \mid \theta) d\hat{\theta} - \theta$$

$$= \int_{\hat{\theta}=0}^{\infty} [\phi(\hat{\theta}_0)\hat{\theta}_0 + (1-\phi(\hat{\theta}_0))\theta_0] f(\hat{\theta} \mid \theta) d\hat{\theta} - \theta \tag{14}$$

Remark 3. When $\theta = \theta_0$ we have

$$B(\tilde{\theta}_0 \mid \theta_0) = \int_{\hat{\theta}=0}^{\infty} [\varphi(\hat{\theta}_0)(\hat{\theta}_0 - \theta_0)] f(\hat{\theta} \mid \theta) d\hat{\theta}. \tag{15}$$

The risk expression (RE) of $\tilde{\theta}_0$ is defined by,

$$RE(\tilde{\theta}_0 \mid \theta) = MSE(\hat{\theta}_0 \mid \theta) / MSE(\tilde{\theta}_0 \mid \theta). \tag{16}$$

where

$$MSE(\tilde{\theta}_0 \mid \theta) = \int_{\hat{\theta}=0}^{\infty} [\ \phi(\hat{\theta}_0)\hat{\theta}_0 + (1-\phi(\hat{\theta}_0))\theta_0 - \theta\ ]^2 f(\hat{\theta} \mid \theta) d\hat{\theta}, \tag{17}$$

and $MSE(\hat{\theta}_0 | \ \theta)$ is the MSE expression of $\hat{\theta}_0$.

Remark 4. When $\theta = \theta_0$ we have

$$RE(\tilde{\theta}_0 \mid \theta_0) = MSE(\hat{\theta}_0 \mid \theta) / \int_{\hat{\theta}=0}^{\infty} [\ \phi(\hat{\theta}_0)(\hat{\theta}_0 - \theta_0)\ ]^2 f(\hat{\theta} \mid \theta) d\hat{\theta}. \tag{18}$$

In the following section, a particular choice of $\phi(\hat{\theta}_0)$ for the Bayes estimator is chosen in accordance with form of exponential model. In addition, the Bayes-shrinkage estimator ${}_1\tilde{\theta}_0$ of the proposed class of estimators for the scale parameter θ of the exponential distribution is studied, when a prior guess value θ_0 of the scale parameter is available.

3.2 The computation of the estimator ${}_1\tilde{\theta}_0$

In this section, the proposed Bayes-shrinkage estimators is considered when θ is the scale parameter of the exponential distribution, and when a special weight function $\phi_1(\hat{\theta}_0)$ is used (i.e. in equation (13)) and a prior guess value θ_0 of the scale parameter is available from the past experience. The resulting estimator is denoted by ${}_1\tilde{\theta}_0$.

In fact, the performance of ${}_1\tilde{\theta}_0$ is depends on two factors. The first is the closeness of θ_0 to θ, and the second factor is the way of selection of $\phi_1(\hat{\theta}_0)$. Indeed, the value of θ_0 usually came from the previous experiment and the experimenter have no information about the real value of θ. Moreover, $\phi_1(\hat{\theta}_0)$ should not be based on unknown θ. In order to gain some improvement in the performance of ${}_1\tilde{\theta}_0$, the function $\phi_1(\hat{\theta}_0)$ has to be carefully selected.

Following Mehta and Srinivasan (1971) and Al-Hemyari and Al-Dabag (2014), the estimator ${}_1\tilde{\theta}_0$ uses the following modified weighting function $\phi_1(\hat{\theta}_0)$,

$$\phi_1(\hat{\theta}_0) = (1 - \mu \exp(-\beta \hat{\theta} \theta_0)), \quad 0 \le \mu \le 1, \quad \beta \ge 0. \tag{19}$$

Remark 5. The choice $\mu = 0$, and with any choice of β, lead to $\phi_1(\hat{\theta}_0) = 1$; if $\mu = 1$ and $\beta = 0$, then $\phi_1(\hat{\theta}_0) = 0$; and any other choices for $0 \le \mu \le 1$ and $\beta \ge 0$ lead to $0 \le \phi_1(\hat{\theta}_0) \le 1$.

Using $\phi_1(\hat{\theta}_0)$ (as given in equation (19)) in equation (13), the Bayes shrinkage estimator ${}_1\tilde{\theta}_0$ takes the following formula,

$${}_1\tilde{\theta}_0 = (\hat{\theta} + [(\sigma / n) / ((n + \kappa - 2) / n)]) + \mu(\theta_0 - \hat{\theta} + [(\sigma / n) / ((n + \kappa - 2) / n)]) \exp(-\beta \hat{\theta} \theta_0). \tag{20}$$

Again using Eq. (14) and Eq. (16), the bias ratio (bias of ${}_1\tilde{\theta}_0$ relative to θ) and the risk ratio expressions of classical estimator $\hat{\theta}$ or the Bayes estimator $\hat{\theta}_0$ relative to the risk of ${}_1\tilde{\theta}_0$ (or the efficiency of ${}_1\tilde{\theta}_0$ relative to $\hat{\theta}$ or $\hat{\theta}_0$) are given respectively by:

$$B({}_1\tilde{\theta}_0; \theta \mid \hat{\theta}) = (\frac{1}{[(n + \kappa - 2) / n]} - 1) + \frac{(\sigma / n)}{[(n + \kappa - 2) / n]} \theta^{-1} + (\mu\lambda - \mu \frac{(\sigma / n)}{[(n + \kappa - 2) / n]} \theta^{-1}) \times$$
$$\times (\frac{n^n}{(n + \beta\lambda\theta^2)^n}) - (\frac{\mu \quad n^{n+1}}{[(n + \kappa - 2) / n](n + \beta\lambda\theta^2)^{n+1}}), \tag{21}$$

$$RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta})=[RE(\hat{\theta}\,|\,\theta)/RE({}_1\tilde{\theta}_0\,|\,\theta)]=1/\{\frac{n+1}{[(n+\kappa-2)/n]}+\frac{2n\theta^{-1}(\sigma/n)}{[(n+\kappa-2)/n]}(\frac{1}{[(n+\kappa-2)/n]}-1)$$
$$+n(1-\frac{2}{[(n+\kappa-2)/n]})+\frac{n\theta^{-2}(\sigma/n)^2}{[(n+\kappa-2)/n]^2}+2\mu[(\frac{\lambda}{[(n+\kappa-2)/n]}-2\frac{(\sigma/r)}{[(n+\kappa-2)/n]}\theta^{-1}$$
$$+\frac{1}{[(n+\kappa-2)/n]})\frac{n^{n+2}}{(n+\beta\lambda\theta^2)^{n+1}}+(\frac{\lambda(\sigma/n)\theta^{-1}}{[(n+\kappa-2)/n]}-\lambda-\frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2}\theta^{-2}+\frac{(\sigma/n)}{[(n+\kappa-2)/n]}\theta^{-1})\times$$
$$\times(\frac{n^{n+1}}{(n+\beta\lambda\theta^2)^n})-\frac{(n+1)n^{n+2}}{[(n+\kappa-2)/n]^2(n+\beta\lambda\theta^2)^{n+2}}]+\mu^2[\frac{(n+1)n^{n+2}}{[(n+\kappa-2)/r]^2(n+2\beta\lambda\theta^2)^{n+2}}+(\lambda^2-2\lambda\times$$
$$\times\frac{(\sigma/n)\theta^{-1}}{[(n+\kappa-2)/n]}+\frac{(\sigma/n)^2\theta^{-2}}{[(n+\kappa-2)/n]^2})\frac{n^{n+1}}{(n+2\beta\lambda\theta^2)^r}+\frac{2}{[(n+\kappa-2)/n]}(\frac{\sigma\theta^{-1}}{n}-n)\frac{n^{n+2}}{(n+2\beta\lambda\theta^2)^{n+1}}]\},$$
(22)

And

$$RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta}_0)=[RE(\hat{\theta}_0\,|\,\theta)/RE({}_1\tilde{\theta}_0\,|\,\theta)]=\{\frac{1}{[(n+\kappa-2)/n]^2}+\frac{1}{[(n+\kappa-2)/n]^2}[(1-[(n+\kappa-2)/n])$$
$$+\theta^{-2}(\sigma/n)]^2\}(1/\{\frac{n+1}{[(n+\kappa-2)/n]^2})+\frac{2\theta^{-1}(\sigma/n)}{[(n+\kappa-2)/n]}(\frac{1}{[(n+\kappa-2)/n]}-1)+(1-\frac{2}{[(n+\kappa-2)/n]})$$
$$+\frac{\theta^{-2}(\sigma/n)^2}{[(n+\kappa-2)/n]^2}+2\mu[(\lambda-2\theta^{-1}(\mu/n)+1)(\frac{n^{n+1}}{[(n+\kappa-2)/n](n+d\lambda\theta^2)^{r+1}})+(\frac{\lambda(\sigma/n)\theta^{-1}}{[(n+\kappa-2)/n]}-\lambda$$
$$-\frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2}\theta^{-2}+\frac{(\sigma/n)}{[(n+\kappa-2)/n]}\theta^{-1})\frac{n^n}{(n+\beta\lambda\theta^2)^n}-\frac{(n+1)n^{n+1}}{[(n+\kappa-2)/n]^2(n+\beta\lambda\theta^2)^{n+2}}]+\mu^2\times$$
$$\times[\frac{(n+1)n^{n+1}}{[(n+\kappa-2)/n]^2(n+2\beta\lambda\theta^2)^{n+2}}(\lambda^2-\frac{2\lambda(\sigma/n)\theta^{-1}}{[(n+\kappa-2)/n]}+\frac{(\sigma/n)\theta^{-2}}{[(n+\kappa-2)/n]^2})(\frac{n^n}{(n+2\beta\lambda\theta^2)^n})$$
$$+\frac{2(\theta^{-1}(\mu/n)-\lambda)}{[(n+\kappa-2)/n]}(\frac{n^n}{(n+2\beta\lambda^2\theta^2)^{n+1}})]\},$$
(23)

where $RE(\hat{\theta}_0|\ \theta)$ is the risk expression of $\hat{\theta}_0$.

4. THE PRELIMINARY TEST BAYES–SHRINKAGE PROCEDURE

4.1 The Computation of the proposed Bayes–shrinkage procedure

The Bayes–shrinkage estimators $\tilde{\theta}_0$ was proposed in last section and the general expressions of the bias ratio and relative risk were derived. A special case of Bayes-shrinkage estimator uses an exponential weighting function is studied and the related expressions were derived.

As it was mentioned in section 3 that the performance of $\tilde{\theta}_0$ were depends on two factors. Although, the second factor was treated in section 2 but the first factor which is related to the closeness of θ_0 to θ will be received more attention in this section and should not be neglected.

In this section, we particularly paid more attention to test the closeness of θ_0 to θ and then use the prior value θ_0 and concentrated in Bayes-type estimators to get new class of estimators which may valuably improve the performance of new class of estimators through the associate risk as it's a key feature of estimators' performance.
In fact, the preliminary test procedure was used numerously in ordinary shrinkage estimators to test the closeness of θ_0 to θ. In this section, following the ordinary shrinkage estimators the preliminary test procedure will be used to test the closeness of θ_0 to θ in $\widetilde{\theta}_0$ leading to a new class of preliminary test Bayes–shrinkage estimators denoted by $\widetilde{\theta}_{0p}$.

The preliminary test Bayes–shrinkage estimators $\widetilde{\theta}_{0p}$ is defined by: based on n observations of model (4), compute the classical estimator $\hat{\theta}$, and based on equation (9) compute the Bayes estimator $\hat{\theta}_0$. For the problem of testing the closeness of θ_0 to θ, a preliminary test region (R) has to be formulated on the bases of θ_0. If $\hat{\theta} \in R$, i.e. θ_0 is accepted, the Bayes- shrinkage estimator takes the form $\varphi(\hat{\theta}_0)(\hat{\theta}_0 - \theta_0) + \theta_0$ in this step. If $\hat{\theta} \notin R$, i.e. θ_0 is rejected, and then the Bayes- shrinkage estimator consist of $\hat{\theta}_0$ only. Thus, the class of preliminary test Bayes-shrinkage estimators $\widetilde{\theta}_{0p}$ takes the form,

$$\tilde{\theta}_{0p} = \begin{cases} \varphi(\hat{\theta}_0)\hat{\theta}_0 + (1-\varphi(\hat{\theta}_0))\theta_0, & if \quad \hat{\theta} \in R, \\ \hat{\theta}_0, & if \quad \hat{\theta} \notin R. \end{cases} \tag{24}$$

where $\varphi(\hat{\theta}_0)$ is defined in (1.2).

Remark 6. It is noteworthy that considerable differential characteristics of the proposed classes of preliminary test Bayes-shrinkage estimators $\widetilde{\theta}_{0p}$ (Eq. (24)) from the existing preliminary test Bayes-shrinkage estimators are: the proposed class of estimators, which is also a combination of Bayes estimator and prior information rather than two classical estimators (i.e. Bayes estimator and MLE) that use a modified suitable exponential weighting function and two choices of test region.

Remark 7. If $\hat{\theta} \in R$ with probability 1, the Bayes-shrinkage estimators $\widetilde{\theta}_{0p}$ (Eq. (24)) tends to $\widetilde{\theta}_0$ (EQ. (13)), i.e. $\widetilde{\theta}_{0p}$ is a special case of $\widetilde{\theta}_0$.

In order to study the behavioral pattern of $\widetilde{\theta}_{0p}$, the bias and risk expressions of $\widetilde{\theta}_{0p}$ are derived for any $\hat{\theta}$, $\varphi(\hat{\theta}_0)$ and R. The bias expression of $\widetilde{\theta}_{0p}$ is defined by,

$$B(\tilde{\theta}_{0p} \mid \theta; R) = \int_0^\infty \tilde{\theta}_{0p} f(\hat{\theta} \mid \theta) d\hat{\theta} - \theta$$
$$= \int_R [\varphi(\hat{\theta}_0)\hat{\theta}_0 + (1-\varphi(\hat{\theta}_0))\theta_0 - \theta] f(\hat{\theta} \mid \theta) d\hat{\theta} + \int_{\bar{R}} (\hat{\theta}_0 - \theta) f(\hat{\theta} \mid \theta) d\hat{\theta}. \quad (25)$$

The risk expression of $\tilde{\theta}_{0p}$ is,

$$RE(\tilde{\theta}_{0p} \mid \theta; R) = MSE(\hat{\theta}_0 \mid \theta) / MSE(\tilde{\theta}_{0p} \mid \theta; R). \quad (26)$$

where

$$MSE(\tilde{\theta}_{0p} \mid \theta; R) = \int_0^\infty (\tilde{\theta}_{0p} - \theta)^2 f(\hat{\theta} \mid \theta) d\hat{\theta} = \int_R [\varphi(\hat{\theta}_0)\hat{\theta}_0 + (1-\varphi(\hat{\theta}_0))\theta_0 - \theta]^2 f(\hat{\theta} \mid \theta) d\hat{\theta}$$
$$+ \int_{\bar{R}} (\hat{\theta}_0 - \theta)^2 f(\hat{\theta} \mid \theta) d\hat{\theta}, \quad (27)$$

and $MSE(\hat{\theta}_{0p} | \theta)$ is the MSE expression of $\hat{\theta}_{0p}$.

4.2 Two choices for region R

It is obviously that the class of estimators $\tilde{\theta}_{0p}$ (Eq. (24)) contains the Bayes estimator, prior value, shrinkage function $\varphi(\hat{\theta}_0)$ and the region R. In addition, the performance of $\tilde{\theta}_{0p}$ is naturally depends on $\varphi(\hat{\theta}_0)$ and R. In Section 3, shrinkage function $\varphi(\hat{\theta}_0)$ is selected and to complete the issue, two choices of the region R are proposed in this section.

In order to test the closeness of θ_0 to θ, the hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \neq \theta_0$ is developed. It is worth to be mentioning that there are many approaches to carry a test for the above hypothesis. One of common approach of testing the above hypothesis is the preliminary test acceptance region were used by numerous papers.

For the above purpose, denote the test statistic for testing the hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \neq \theta_0$ by $T(\hat{\theta})$. Assume that the above test has a level of significance α, then $T(\hat{\theta})$ has the lower $L_{1-\alpha/2}$ and upper $U_{\alpha/2}$ 100(α /2) percentile points. It is well known that the preliminary test region (denoted by R_1) for $T(\hat{\theta})$ is defined by,

$$R_1 = \{T(\hat{\theta}) \in [L_{1-\alpha/2},\ U_{\alpha/2}]\}. \quad (28)$$

Undoubtedly, the statistics $T(\hat{\theta}) = 2n\hat{\theta}/\theta$ is distributed as chi-square random variable. Denoting the lower and upper 100(α /2) percentile points of the chi-square distribution by $\chi^2_{1-\alpha/2,2n}$ and $\chi^2_{\alpha/2,2n}$ respectively, and $2n$ are denoted to the degrees of freedom. This implies that the region R_1 has the following form,

$$R_1 = [(\theta_0 / 2n)\chi^2_{1-\alpha/2,2n},\ (\theta_0 / 2n)\chi^2_{\alpha/2,2n}]. \quad (29)$$

Following Al-Hemyari and Al-Dolimi (20-14), and Al-Hemyari and Al-Dabag (2014), it seems sensible to take in account the differences between θ_0 and θ in construction of the region R. Denote this choice R_2 which has the following formula,

$$R_2 = \{\theta : (\theta - \theta_0)^2 \le MSE(\hat{\theta} \mid \theta)\}. \tag{30}$$

Some algebraic derivations lead to the following interval R_2,

$$R_2 = [Max(0, \theta_0 - \theta(1/\sqrt{n})),\ \ \theta_0 + \theta(1/\sqrt{n})]. \tag{31}$$

Special cases of preliminary test Bayes–shrinkage procedure is studied in subsection 4.3 and the expressions of bias ratio and risk are derived.

4.3 The computation of the estimator ${}_1\widetilde{\theta}_{0p}$

The general class of the preliminary test Bayes–shrinkage estimators for any parameter θ is proposed in section 4.2. In this section, the proposed class of Bayes-shrinkage estimators is considered where θ is the scale parameter of the exponential distribution and when a prior guess value θ_0 of θ is available from the past experience. The estimator will uses the same weight function given in equation (19) of section 3.2 with the regions R_i, $i = 1,2$ derived in section 4.3 and denoting the resulting estimator by ${}_1\widetilde{\theta}_{0p}$, and given by,

$${}_1\tilde{\theta}_{0p} = \begin{cases} (1 - \mu \exp(-\beta\hat{\theta}\theta_0))\hat{\theta}_0 + (\mu \exp(-\beta\hat{\theta}\theta_0))\theta_0, & if \quad \hat{\theta} \in R_i, \\ \hat{\theta}_0, \quad if \quad \hat{\theta} \notin R_i, \quad i = 1,2. \end{cases} \tag{32}$$

Using the same algebraic calculations of equations (22), the bias ratio (bias| θ) of ${}_1\widetilde{\theta}_{op}$ is given by,

$$B({}_1\tilde{\theta}_{0p}; \theta \mid \hat{\theta}_0; R_i) = \mu\lambda n^n (n+\beta\lambda)^{-n} - \frac{\mu}{f_2}[n^{n+1} 2^{-(n+1)} (n+\beta\lambda)^{-(n+1)}](F_{2n+2}(b^*_i) - F_{2n+2}(a^*_i))$$
$$+ n^n 2^{-n} \theta^{-1} (\sigma/n)(n+\beta\lambda)^{-n} (F_{2n}(b^*_i) - F_{2n}(a^*_i)) + (\frac{1}{[(n+\kappa-2)/n]} - 1)\theta^{-1} \frac{(\sigma/n)}{[(n+\kappa-2)/n]}, \tag{33}$$

where

$$F(\hat{\theta} \mid R_i) = \int_{R_i} \frac{1}{\Gamma(n)} (\frac{n}{\theta})^n (\hat{\theta})^{n-1} e^{-n(\hat{\theta}/\theta)} d\hat{\theta}. \tag{34}$$

Since under H_0, $T(\hat{\theta})$ has chi-square distribution with $2n$ degrees, it can be seen that

$$F(\hat{\theta} \mid R_i) = \int_{a_i}^{b_i} \frac{(n/\theta)^n (\hat{\theta})^{(n+j-1)-1}}{\Gamma(r+n-1)} \exp(-n(\hat{\theta}/\theta)) d\hat{\theta}, \quad j = 1,2,3; \quad i = 1,2, \tag{35}$$
$$= F_d(b^*_i) - F_d(a^*_i),$$

where $R_i = [a_i,\ b_i],\ a_i < b_i,\ \ R_i^* = [a_i^*, b_i^*],\ a_i^* < b_i^*,\ a_i^* = 2na_i/\theta,\ b_i^* = 2nb_i/\theta,$ and $F_d(.)$ is the distribution function of a chi-square random variable with *d* degrees of freedom.

The expressions of the risk ratios of MLE or the Bayes estimators relative to ${}_1\widetilde{\theta}_{op}$ (or the efficiency of ${}_1\widetilde{\theta}_{0p}$ relative to $\hat{\theta}_0$ or $\hat{\theta}_0$) were achieved by continuing the same mode of the bias ratio expression which yields respectively to:

$$RR({}_1\tilde{\theta}_{0p};\theta\,|\,\hat{\theta};R_i) = (\theta^2/n)(1/\{\frac{2\mu}{[(n+\kappa-2)/n]}\{[(\theta_0+\theta)-2(\sigma/n)]2^{-(n+1)}n^{n+1}\theta\ (n+\beta\lambda)^{-(n+1)}(F_{2n+2}(b^*_i)$$
$$-F_{2n+2}(a^*_i)) - \frac{2^{-(n+1)}n^{n+1}(2n+2)}{[(n+\kappa-2)/n]^2}(n+\beta\lambda)^{-(n+2)}\theta^2(F_{2n+4}(b^*_i)-F_{2n+4}(a^*_i)) + ((\theta_0+\theta)\frac{(\sigma/n)}{[(n+\kappa-2)/n]}$$
$$-\frac{(\sigma/n)}{[(n+\kappa-2)/n]^2} - \theta\theta_0)2^{-n}n^n(n+\beta\lambda)^{-n}(F_{2n+2}(b^*_i)-F_{2n+2}(a^*_i))\} + \mu^2\{(\theta_0^2 - 2\frac{(\sigma/n)\theta_0}{[(n+\kappa-2)/n]}$$
$$+\frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2})n^n2^{-n}(n+2\beta\lambda)^{-n}(F_{2n}(b^*_i)-F_{2n}(a^*_i)) + \frac{2\theta}{[(n+\kappa-2)/n]}((\sigma/n) - \frac{\theta_0}{[(n+\kappa-2)/n]})2^{-(n+2)}\times$$
$$\times n^{n+1}(n+2\beta\lambda)^{-(n+1)}(F_{2n+2}(b^*_i)-F_{2n+2}(a^*_i)) + \frac{(n+1)2^{-(n+2)}n^{n+1}}{[(n+\kappa-2)/n]^2}(n+2\beta\lambda)^{-(n+2)}\theta^2(F_{2n+4}(b^*_i)-F_{2n+4}(a^*_i))\}$$
$$+\frac{(n+1)\theta^2}{n[(n+\kappa-2)/n]^2} + \frac{2(\sigma/n)\theta}{[(n+\kappa-2)/n]}(\frac{1}{[(n+\kappa-2)/n]}-1) + (1-\frac{2}{[(n+\kappa-2)/n]})\theta^2 + \frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2}\}), \tag{36}$$

and

$$RR({}_1\tilde{\theta}_{0p};\theta\,|\,\hat{\theta}_0;R_i) = \{\frac{1}{[(n+\kappa-2)/n]^2} + \frac{1}{[(n+\kappa-2)/n]^2}[(1-[(n+\kappa-2)/n]) + \theta^{-2}(\sigma/n)]^2\}(1/\{2\mu\{[\theta_0$$
$$+\theta - 2(\sigma/n)]2^{-(n+1)}n^{n+1}\theta[(n+\kappa-2)/n]^{-1}(n+\beta\lambda)^{-(n+1)}(F_{2n+2}(b^*_i)-F_{2n+2}(a^*_i)) - \frac{(n+1)2^{-(n+1)}n^{n+1}}{[(n+\kappa-2)/n]^2}\times$$
$$\times(n+\beta\lambda)^{-(n+2)}\theta^2(F_{2n+4}(b^*_i)-F_{2n+4}(a^*_i)) + (\frac{(\theta_0+\theta)(\sigma/n)}{[(n+\kappa-2)/n]} - \frac{(\sigma/n)}{[(n+\kappa-2)/n]^2} - \theta\theta_0)2^{-n}n^n(n+\beta\lambda)^{-n}\times$$
$$\times(F_{2n+2}(b^*_i)-F_{2n+2}(a^*_i))\} + \mu^2\{(\theta_0^2 - 2\frac{\theta_0(\sigma/n)}{[(n+\kappa-2)/n]} + \frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2})n^n2^{-n}(n+2\beta\lambda)^{-n}(F_{2n}(b^*_i)$$
$$-F_{2n}(a^*_i)) + \frac{2}{[(n+\kappa-2)/n]}((\sigma/n) - \frac{\theta_0}{[(n+\kappa-2)/n]})2^{-(n+2)}n^{n+1}\theta(n+2\beta\lambda)^{-(n+1)}(F_{2n+2}(b^*_i)-F_{2n+2}(a^*_i))$$
$$+\frac{2^{-(n+2)}n^{n+1}}{[(n+\kappa-2)/n]^2}(n+2\beta\lambda)^{-(n+2)}(n+1)\theta^2(F_{2n+4}(b^*_i)-F_{2n+4}(a^*_i))\} + \frac{(n+1)\theta^2}{n[(n+\kappa-2)/n]^2} + \frac{2\theta(\sigma/n)}{[(n+\kappa-2)/n]}\times$$
$$\times(\frac{1}{[(n+\kappa-2)/n]}-1) + (1-\frac{2}{[(n+\kappa-2)/n]})\theta^2 + \frac{(\sigma/n)^2}{[(n+\kappa-2)/n]^2}\}). \tag{37}$$

5. OTHER SAMPLING PROCEDURES

In sections 3 and 4, the Bayes-shrinkage and preliminary Bayes-shrinkage estimators were derived respectively for complete data. Indeed, the censored samples are extensively used in life testing problems and models. In this section, the proposed estimators ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ are extended to the censored sampling schemes.

In fact, the proposed estimators are based mainly on the Bayes estimator $\hat{\theta}_0$ and θ_0, where $\hat{\theta}_0$ (see, equation (9)) is a function of $\hat{\theta}$. Thus, we need only to study the modification of $\hat{\theta}$ in each of the following schemes:

i) Right censored samples: Let the smallest $r(\leq n)$ ordered statistics $X_{(1)} \leq X_{(2)} \leq \ \ldots \ \leq X_{(r)}$ of a random observations were followed the exponential probability distribution function $F(X \mid \theta)$. The MLE $\hat{\theta}$ of θ based on $r(\leq n)$ ordered statistics is defined by,

$$\hat{\theta} = S / r, \quad S = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}. \tag{38}$$

It is well known that the statistics $2r\hat{\theta}/\theta$ is distributed as chi-square distribution with $2r$ degrees of freedom. Then, replacing $2n$ by $2r$ in the expressions of the estimators ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ derived in sections 3 and 4 will create the righteous estimators for the above modifications.

ii) Left censored samples: Let the largest $n-b+1(\leq n)$ ordered statistics $X_{(b)} \leq X_{(b+1)} \leq \ \ldots \ \leq X_{(n)}$ of a random observations were followed the exponential probability distribution function $F(X \mid \theta)$. The MLE $\hat{\theta}$ of θ based on $n-b+1(\leq n)$ ordered statistics is defined by,

$$\hat{\theta} = S / n, \quad S = (b\beta_{b,n} + b - n)X_{(a)} + \sum_{i=b+1}^{n} X_{(i)}, \tag{39}$$

where $\beta_{b,n} = \sum_{i=1}^{b} (1/(n-i+1))^{-1}$. Then, since the distribution of the statistics $2r\hat{\theta}/\theta$ is almost like chi-square distribution with $2n$ degrees of freedom (see Sarhan and Greenberg (1962)). Thus, computing $\hat{\theta}$ as given in equation (39) in place of equation (9) will create the righteous estimators for the left censored scheme.

iii) Doubly censored samples: The estimators ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ derived in sections 3 and 4 may be used for doubly censored scheme. Let the doubly censored scheme of $b-c+1(\leq n)$ order statistics $X_{(b)} \leq X_{(b+1)} \leq \ \ldots \ \leq X_{(c)}, \ 1 \leq b \leq c \leq n$ of a random observations were followed the exponential probability distribution function $F(X \mid \theta)$. The MLE $\hat{\theta}$ of θ based on $1 \leq b \leq c \leq n$ ordered statistics is defined by,

$$\hat{\theta} = S/b, \quad S = (b\beta_{b,n} + b - n)X_{(b)} + \sum_{i=b+1}^{c-1} X_{(i)} + (n-c+1)X_{(c)}. \tag{40}$$

Indeed, the distribution of the statistics $2b\hat{\theta}/\theta$ is almost like chi-square with $2b$ degrees of freedom. Thus, computing $\hat{\theta}$ as given in equation (40) in place of equation (9) will also create the righteous Bayes-shrinkage estimators for the doubly censored scheme.

iv) Censored sample with replacement: Assume that a test is performed of a random sample of size n having exponential distribution with the mean θ. Assume that the test is performed with replacement and to be terminated if there are r failures $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$. Then, the MLE estimator of θ (see Mann et al. (1974)) is defined by,

$$\hat{\theta} = nS/r, \quad S = \sum_{i=1}^{r} X_{(i)} - X_{(i-1)}. \tag{41}$$

Computing $\hat{\theta}$ as given in equation (41) in place of equation (9) will also create the righteous Bayes-shrinkage estimators for censored sample with replacement scheme. Since, the distribution of the statistics $2r\hat{\theta}/\theta$ is known to have chi-square distribution with $2r$ degrees of freedom.

Remark 8. The proposed classes of Bayes–shrinkage estimators $\tilde{\theta}_0$ and preliminary test Bayes-shrinkage estimators $\tilde{\theta}_{0p}$ can be extended to estimate the parameters of Pareto, gamma, chi-square and Weibull distributions in the cases of completed and censored data.

6. SIMULATION RESULTS AND DISCUSSION

For the problem of studying the properties of the proposed estimators, the bias ratio, risk and risk ratio expressions of the proposed estimators ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ were derived. In addition, few remarks were demonstrated to explain the behavioral pattern of each estimators in sections 3 and 4. In fact, it is difficult to provide a comprehensive theoretical study due to the complexity of the bias ratio, risk and risk ratio expressions of ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$.

In order to study the behavioral pattern of each estimator with respect to each constant involved in ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$, a simulation study was provided in place of a theoretical study for the bias ratio and risk ratio of ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ with respect to two classical estimators (MLE and Bayes estimators).The performance of ${}_1\tilde{\theta}_o$ and ${}_1\tilde{\theta}_{op}$ with respect to μ, β, λ, θ_0, σ, κ and n was also studied. Some of the simulation results are shown in Figures 1 to 12.

6.1 Simulation results and discussion of ${}_1\tilde{\theta}_0$

In order to observe the performance of the proposed Bayes-shrinkage estimator ${}_1\tilde{\theta}_0$, and to compare the risk of classical estimator $(\hat{\theta})$ or Bayes estimator $(\hat{\theta}_0)$ relative to the risk of Bayes-shrinkage estimator ${}_1\tilde{\theta}_0$ (or the efficiency of ${}_1\tilde{\theta}_0$ relative to $\hat{\theta}_0$ or $\hat{\theta}_0$) $(RR({}_1\tilde{\theta}_0;\theta\,|\,.))$, the constants $\sigma = 0,2,4,6$, $\kappa = 1,2,3$, $\mu = 0.004, 0.01, 0.05, 0.1, 0.5,$ $\beta = 0.004, 0.05, 0.5, 1,$ $n = 15(5)30,$ $\lambda = 0.1(0.1)1(1)5$ and $\theta_0 = 2$ have chosen and tried. Some of these computations are shown in Graphs 1 to 7, 10 and 11.

In Graphs 1-7 we presented some sample values of the risk ratio $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$. The $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is a decreasing function of λ, and the risk ratio of ${}_1\tilde{\theta}_0$ is higher than that of the MLE $\hat{\theta}$ and Bayes estimator $\hat{\theta}_0$ for the range of $0.1 \le \lambda \le 5$, this means that the estimator $\tilde{\theta}_0$ is more efficient than the classical estimators. Also, it is observed that generally $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is higher (i.e. it is more efficient) than $RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta})$ for the same range of λ. For this reason and to save space the results of $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ are discussed and some of them are reported.

Figures 1-7 shows that $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is an increasing function of n and a decreasing function of κ, i.e. $n = 30$ and $\kappa = 1$ give higher value of the relative risk than other values of n and κ. Also, $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is higher when $\sigma = 2$ than other values of σ. In addition, for fixed n, σ and κ, $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is higher when $\mu = 0.5$ than other values of μ. Finally, for fixed n, σ, κ and μ, $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ is a decreasing function of β, i.e. $\beta = 0.004$ gives higher value of the relative risk than other values of β. Thus, the constants $n = 30, \sigma = 2, \kappa = 1, \beta = 0.004$ and $\mu = 0.5$ are recommended. Indeed, the $RR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$ for the choices $n = 30, \sigma = 2, \kappa = 1, \beta = 0.004$ and $\mu = 0.5$ is much higher than the classical estimators (as much as 21.88-23.53 times).

Figures 10 and 11 provided some values of $BR({}_1\tilde{\theta}_0;\theta\,|\hat{\theta}_0)$. It is really interesting to observe that the bias ration of $\tilde{\theta}_0$ for the choices $n = 30, \sigma = 2, \kappa = 1, \beta = 0.004$ and $\mu = 0.5$ is reasonable and smaller than other values of the constants.

In fact, comparing the results of ${}_1\tilde{\theta}_0$ demonstrated in Graphs 1-7 with classical estimators in accordance with the risk ratio of ${}_1\tilde{\theta}_0$ relative to the MLE $\hat{\theta}$ or the Bayes estimator $\hat{\theta}_0$ shows that the proposed estimator $\tilde{\theta}_0$ is more efficient than the classical estimators for the range of $0.1 \le \lambda \le 5$. Moreover, the proposed estimator $\tilde{\theta}_0$ is more efficient than the

Bayes-shrinkage estimators of Pandey and Upadhyay (1985c), Upadhyay and Singh (1992), and Yang et al.(2013).

6.2 Simulation results and discussion of $_1\tilde{\theta}_{0p}$

The performance of the Bayes-shrinkage estimator $_1\tilde{\theta}_0$ is studied and compared with respect to $\hat{\theta}$ and $\hat{\theta}_0$ in last section. In this section, the preliminary Bayes-shrinkage estimator $_1\tilde{\theta}_{0p}$ is studied for regions R_i, $i = 1,2$ and for the choices $\alpha = 0.01, 0.05, 0.1$, $\sigma = 0,2,4,6$, $\kappa = 1,2,3$, $\lambda = 0.1(0.1)1(1)5$, $n = 15(5)30$, $\lambda = 0.1(0.1)1(1)5$.
$\mu = 0.004, 0.01, 0.05, 0.1, 0.5$, $\beta = 0.004, 0.05, 0.5, 1$, $\theta_0 = 2$ and $\lambda = 0.1(0.1)1(1)5$.
Also, the risk ratio of $_1\tilde{\theta}_{0p}$ is compared with respect to the risk ratio of $\hat{\theta}$ and $\hat{\theta}_0$.
It is observed that the ratio $RR(_1\tilde{\theta}_{0p};\theta|\hat{\theta}_0;R_i)$ is higher than (i.e. they are more efficient) that of classical estimators ($\hat{\theta}_0$ and $\hat{\theta}$) for the range of $0.1 \le \lambda \le 5$, and generally $RR(_1\tilde{\theta}_{0p};\theta|\hat{\theta}_0;R_1)$ is higher (i.e. it is more efficient) than $RR(_1\tilde{\theta}_{0p};\theta|\hat{\theta}_0;R_2)$. Also, the risk ratio of estimator $_1\tilde{\theta}_{0p}$ with preliminary test region R_1 is a decreasing function of α.
Moreover, the performance of the estimator $_1\tilde{\theta}_{0p}$ is similar to that of $_1\tilde{\theta}_0$ for the constant $n, \sigma, \kappa, \beta, \mu$. Thus, some of the simulation values of $RR(_1\tilde{\theta}_{0p};\theta|\hat{\theta}_0;R_1)$ are presented in Graphs 8 and 9.
Finally, it is observed that $_1\tilde{\theta}_0$ is biased. Graph 12 shows some simulation values of $BR(_1\tilde{\theta}_{0p};\theta|\hat{\theta}_0;R_1)$.
In fact, comparing the results of $_1\tilde{\theta}_{0p}$ with classical $\hat{\theta}$, $\hat{\theta}_0$ and existing Bayes-shrinkage estimators in accordance with the risk ratio of $_1\tilde{\theta}_0$ relative to the MLE $\hat{\theta}$ or the Bayes estimator $\hat{\theta}_0$ shows that the proposed estimator $\tilde{\theta}_{0p}$ is also efficient than the similar estimators for the range of $0.1 \le \lambda \le 5$.

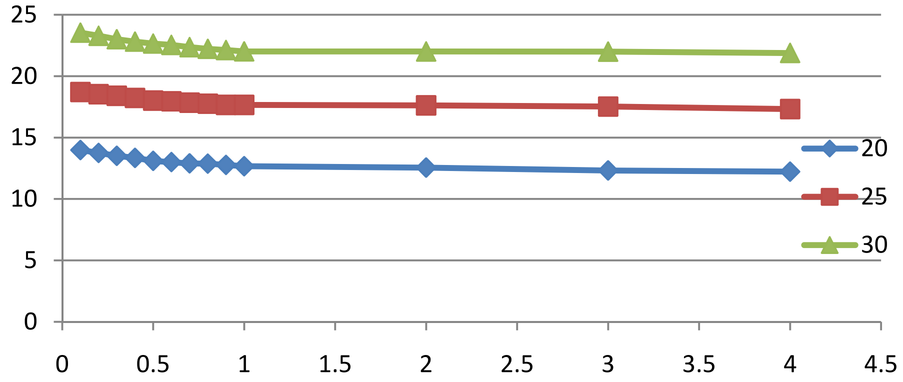

Figure 1. $RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta}_0)(\sigma = 2, \kappa = 1, \beta = 0.004, \mu = 0.5, \alpha = 0.05$, $n = 20,25,30$, $\lambda = 0.1(0.1)1(1)4.)$

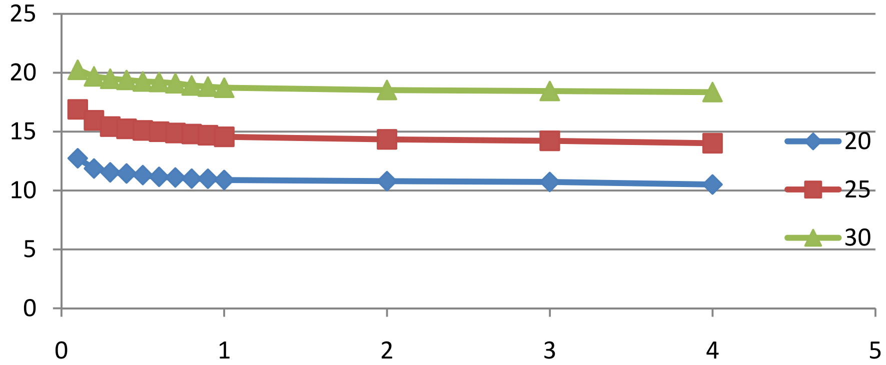

Figure 2. $RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta}_0)(\sigma = 2, \kappa = 2, \beta = 0.004, \mu = 0.5$, $n = 20,25,30$, $\lambda = 0.1(0.1)1(1)4.)$

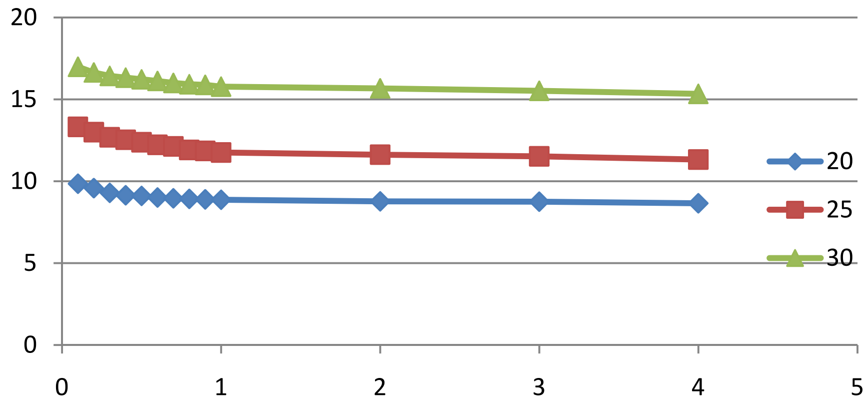

Figure 3. $RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta}_0)(\sigma = 2, \kappa = 3, \beta = 0.004, \mu = 0.5$, λ=0.1(0.1)1(1)4. $\lambda = 0.1(0.1)1(1)4.)$

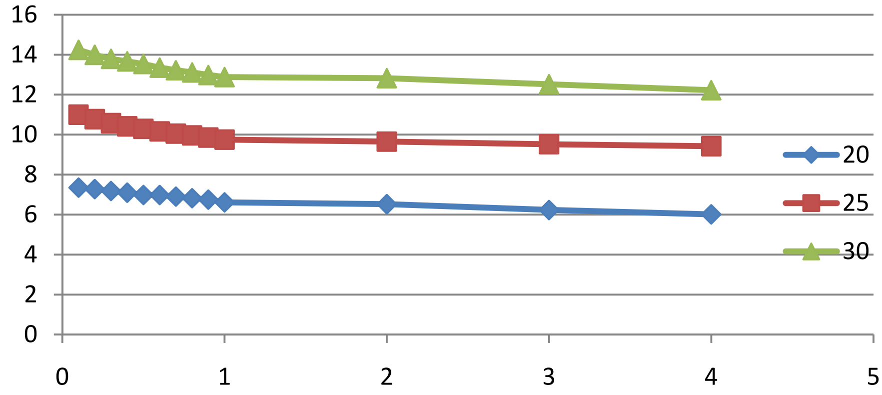

Figure 4. $RR({}_1\tilde{\theta}_0;\theta|\hat{\theta}_0)(\sigma = 2, \kappa = 4, \beta = 0.004, \mu = 0.5,$ λ=0.1(0.1)1(1)4.,
$\lambda = 0.1(0.1)1(1)4.)$

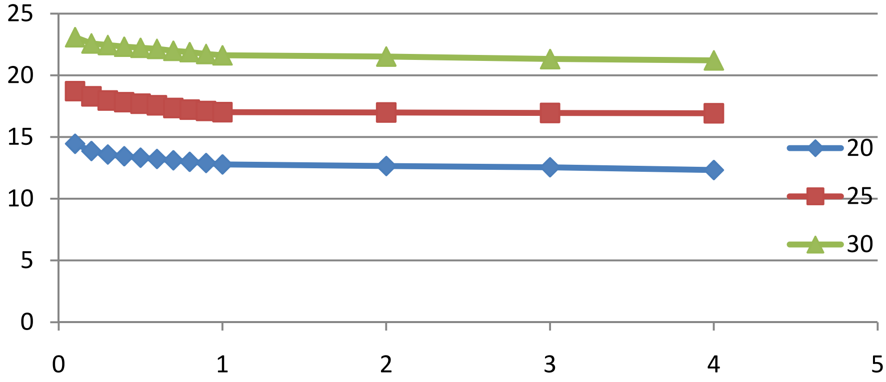

Figure 5. $RR({}_1\tilde{\theta}_0;\theta|\hat{\theta}_0)(\sigma = 4, \kappa = 1, \beta = 0.004, \mu = 0.5,$ λ=0.1(0.1)1(1)4.
$\lambda = 0.1(0.1)1(1)4.)$

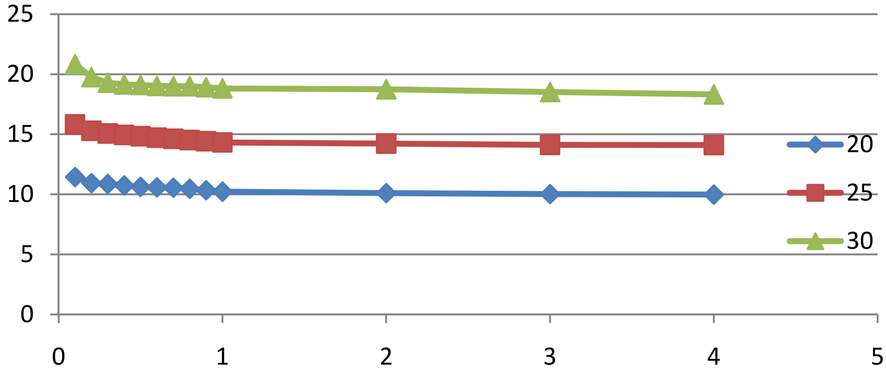

Figure 6. $RR({}_1\tilde{\theta}_0;\theta|\hat{\theta}_0)(\sigma = 4, \kappa = 2, \beta = 0.004, \mu = 0.5,$ λ=0.1(0.1)1(1)4.
$\lambda = 0.1(0.1)1(1)4.)$

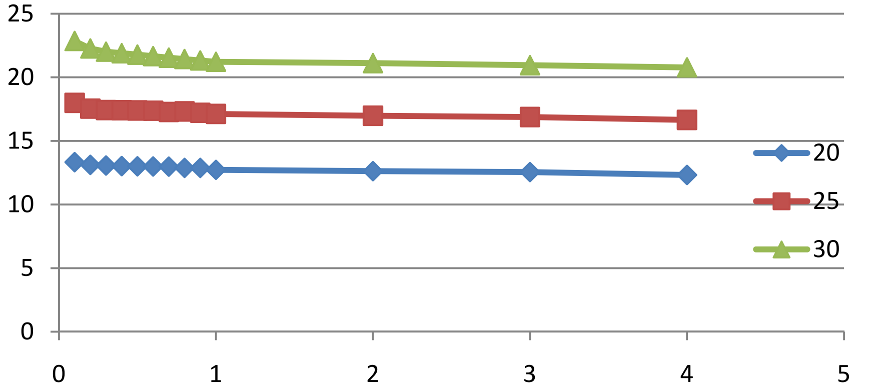

Figure 7. $RR({}_1\tilde{\theta}_0;\theta\,|\,\hat{\theta}_0)\,(\sigma = 6, \kappa = 1, \beta = 0.004, \mu = 0.5$, λ=0.1(0.1)1(1)4.
$\lambda = 0.1(0.1)1(1)4.)$

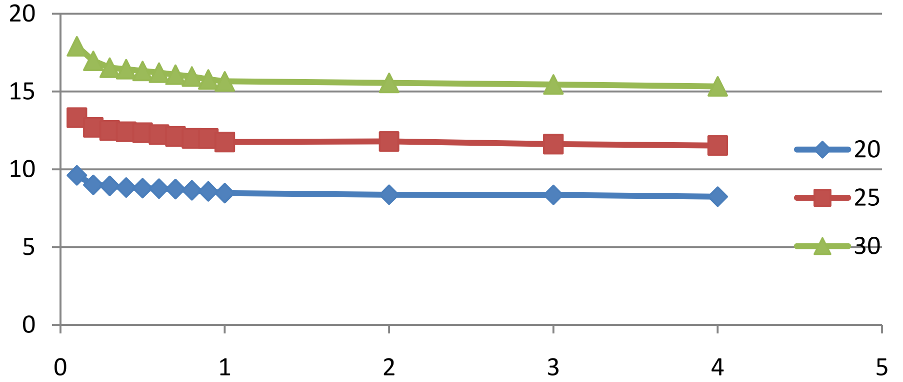

Figure 8. $RR({}_1\tilde{\theta}_{0p};\theta\,|\,\hat{\theta}_0;R_1)\,(\sigma = 2, \kappa = 1, \beta = 0.004, \mu = 0.5, \alpha = 0.01$,
λ=0.1(0.1)1(1)4., $\lambda = 0.1(0.1)1(1)4.)$

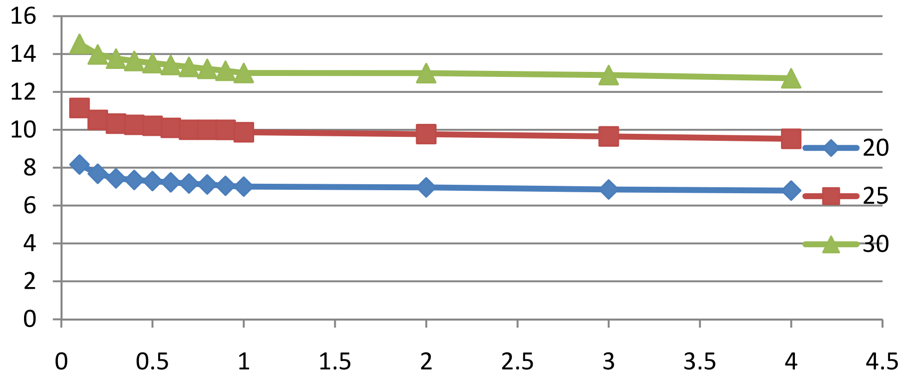

Figure 9. $RR({}_1\tilde{\theta}_{0p};\theta\,|\,\hat{\theta}_0;R_1)\,(\sigma = 2, \kappa = 1, \beta = 0.004, \mu = 0.5, \alpha = 0.05$,
$n = 20,25,30$, $\lambda = 0.1(0.1)1(1)4.)$

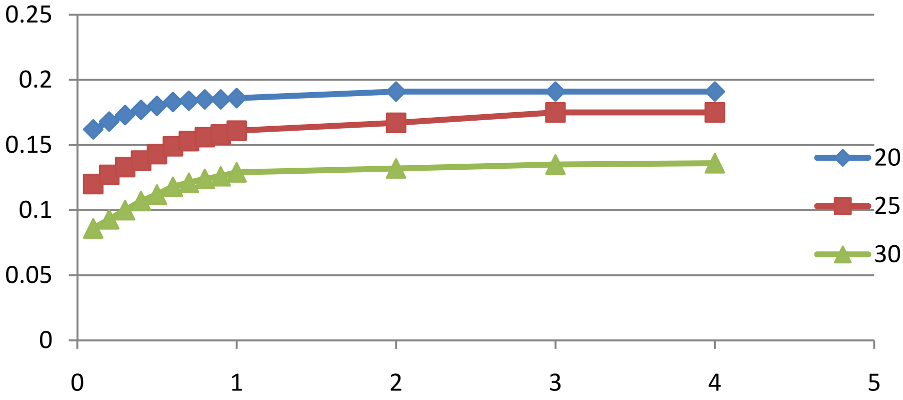

Figure 10. $BR(_1\tilde{\theta}_0 \,|\, \hat{\theta}_0)\,(\sigma = 2, \kappa = 1, \beta = 0.004, \mu = 0.5\,, n = 20{,}25{,}30\,,$
$\lambda = 0.1(0.1)1(1)4.)$

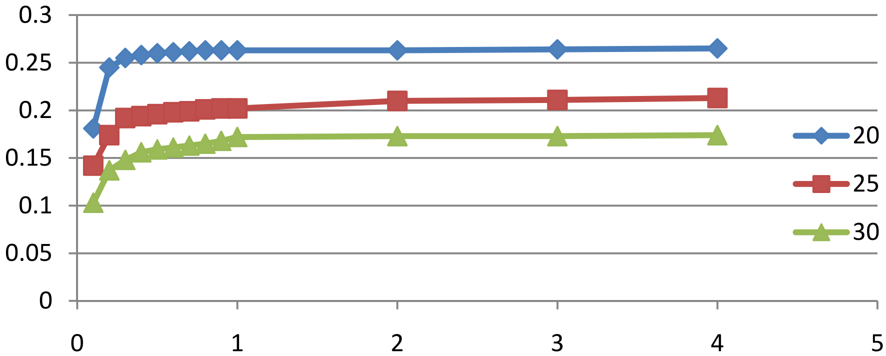

Figure 11. $BR(_1\tilde{\theta}_0 \,|\, \hat{\theta}_0)\,(\sigma = 2, \kappa = 2, \beta = 0.004, \mu = 0.5\,,\ \lambda{=}0.1(0.1)1(1)4.,$
$\lambda = 0.1(0.1)1(1)4.)$

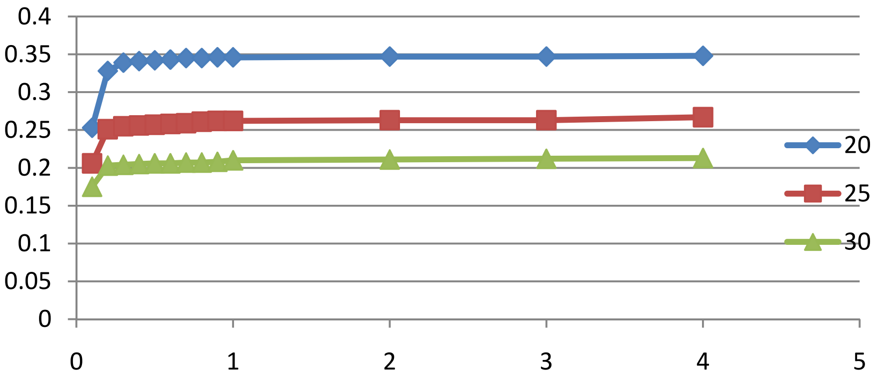

Figure 12. $BR(_1\tilde{\theta}_{0p} \,|\, \hat{\theta}_0)\,(\sigma = 2, \kappa = 1, \beta = 0.004, \mu = 0.5, \alpha = 0.01\,,\ \lambda{=}0.1(0.1)1(1)4.,$
$\lambda = 0.1(0.1)1(1)4.)$

7. SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

In this paper, two classes of Bayes-shrinkage and preliminary Bayes-shrinkage estimators are proposed studied and the expressions of bias ratio and risk ratio of each class for each parameter are derived. Two examples $({}_1\tilde{\theta}_0, {}_1\tilde{\theta}_{0p})$ for the scale parameter of exponential model with specific weight functions $(\phi(\hat{\theta}_0))$ and regions $R_i, i = 1,2$ are given and the performance of ${}_1\tilde{\theta}_0$ and ${}_1\tilde{\theta}_{0p}$ are studied using simulation. Some of the simulation results are demonstrated in Figures 1-12.

It is observed that the Bayes-shrinkage estimator ${}_1\tilde{\theta}_0$ is performed better than the MLE and Bayes estimator in the sense of smaller risk. The constants $n = 30, \sigma = 2, \kappa = 1, \beta = 0.004$ and $\mu = 0.5$ are recommended which achieved much smaller risk (much higher relative efficiency) than the classical estimators (as much as 21.88-23.53 times).

It is also seen that the Bayes-shrinkage estimator ${}_1\tilde{\theta}_{0p}$ with preliminary test region R_1 performed better than the MLE and Bayes estimator in the sense of smaller risk. The constants $n = 30, \sigma = 2, \kappa = 1, \beta = 0.004$ and $\mu = 0.5$ are recommended which achieved much smaller risk (much higher relative efficiency) than the classical estimators with a broader range of $0.1 \le \lambda = \theta_0 / \theta \le 5$ (as much as 12-15 times).

Therefore, the proposed estimators ${}_1\tilde{\theta}_0$ and ${}_1\tilde{\theta}_{0p}$ are recommended for the practical use if the prior initial value θ_0 of θ is available.

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