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APPROXIMATE QUARTIC LIE $*$-DERIVATIONS

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Abstract. We will show the general solution of the functional equation $f(x+ay)+f(x-ay)+2(a^2-1)f(x) = a^2f(x+y)+a^2f(x-y)+2a^2(a^2-1)f(y)$ and investigate the stability of quartic Lie $*$-derivations associated with the given functional equation.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam. Afterwards, the result of Hyers was generalized by Aoki [1] for additive mapping and by Rassias [14] for linear mappings by considering a unbounded Cauchy difference. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. For further information about the topic, we also refer the reader to [10], [8], [2] and [3].

Recall that a Banach $*$-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. Jang and Park [9] investigated the stability of $*$-derivations and of quadratic $*$-derivations with Cauchy functional equation and the Jensen functional equation on Banach $*$-algebra. The stability of $*$-derivations on Banach $*$-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see [12] and [19], respectively. Also, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see [6].

Rassias [13] investigated stability properties of the following quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y)\,. \tag{1.1}$$

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It is easy to see that $f(x) = x^4$ is a solution of (1.1) by virtue of the identity

$$(x+2y)^4 + (x-2y)^4 + x^4 = 4(x+y)^4 + 4(x-y)^4 + 24y^4 . \tag{1.2}$$

For this reason, (1.1) is called a quartic functional equation. Also Chung and Sahoo [4] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function $f : \mathbb{R} \to \mathbb{R}$ is a solution of (1.1) if and only if $f(x) = A(x, x, x, x)$, where the function $A : \mathbb{R}^4 \to \mathbb{R}$ is symmetric and additive in each variable.

In this paper, we deal with the following functional equation:

$$\begin{aligned} f(x+ay) + f(x-ay) + 2(a^2-1)f(x) \\ = a^2 f(x+y) + a^2 f(x-y) + 2a^2(a^2-1)f(y) \end{aligned} \tag{1.3}$$

for all $x , y \in X$ and an integer $a(a \neq 0 , \pm 1)$. We will show the general solution of the functional equation (1.3), define a quartic Lie $*$-derivation related to equation (1.3) and investigate the Hyers-Ulam stability of the quartic Lie $*$-derivations associated with the given functional equation.

2. A Quartic Functional Equation

In this section let X and Y be real vector spaces and we investigate the general solution of the functional equation (1.3). Before we proceed, we would like to introduce some basic definitions concerning n-additive symmetric mappings and key concepts which are found in [16] and [18]. A function $A : X \to Y$ is said to be *additive* if $A(x+y) = A(x) + A(y)$ for all $x , y \in X$. Let n be a positive integer. A function $A_n : X^n \to Y$ is called *n-additive* if it is additive in each of its variables. A function A_n is said to be *symmetric* if $A_n(x_1 , \cdots , x_n) = A_n(x_{\sigma(1)} , \cdots , x_{\sigma(n)})$ for every permutation $\{\sigma(1) , \cdots , \sigma(n)\}$ of $\{1 , 2 , \cdots , n\}$. If $A_n(x_1 , x_2 , \cdots , x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x , x , \cdots , x)$ and $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$. such a function $A^n(x)$ will be called a *monomial function* of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_n = y$ in $A_n(x_1 , x_2 , \cdots , x_n)$ will be denoted by $A^{s,n-s}(x , y)$.

Theorem 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1.3) if and only if f is of the form $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of the* 4*-additive symmetric mapping $A_4 : X^4 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.3). Letting $x = y = 0$ in the equation (1.3), we have

$$f(0) = 2a^2(a^2 - 1)f(0)\,,$$

that is, $f(0) = 0$. Putting $x = 0$ in the equation (1.3), we get

$$f(ax) + f(-ay) = a^2 f(y) + a^2 f(-y) + 2a^2(a^2 - 1)f(y) \tag{2.1}$$

for all $y \in X$. Replacing y by $-y$ in the equation (2.1),we obtain

$$f(ax) + f(-ay) = a^2 f(y) + a^2 f(-y) + 2a^2(a^2 - 1)f(-y) \tag{2.2}$$

for all $y \in X$. Combining two equations (2.1) and (2.2), we have $f(y) = f(-y)$, for all $y \in X$. That is, f is even. We can rewrite the functional equation (1.3) in the form

$$\begin{aligned} &f(x) + \frac{1}{2(a^2-1)} f(x+ay) + \frac{1}{2(a^2-1)} f(x-ay) \\ &-\frac{a^2}{2(a^2-1)} f(x+y) - \frac{a^2}{2(a^2-1)} f(x-y) - a^2 f(y) = 0 \end{aligned}$$

for all $x\,, y \in X$ and an integer $a(a \neq 0\,, \pm 1)$. By Theorem 3.5 and 3.6 in [18], f is a generalized polynomial function of degree at most 4, that is, f is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.3}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3, 4$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$. Substituting (2.3) into the equation (1.3) we have

$$\begin{aligned} & A^4(x+ay) + A^3(x+ay) + A^2(x+ay) + A^1(x+ay) \\ & + A^4(x-ay) + A^3(x-ay) + A^2(x-ay) + A^1(x-ay) \\ & + 2(a^2-1)[A^4(x) + A^3(x) + A^2(x) + A^1(x)] \\ = \;& a^2[A^4(x+y) + A^3(x+y) + A^2(x+y) + A^1(x+y) \\ & + A^4(x-y) + A^3(x-y) + A^2(x-y) + A^1(x-y)] \\ & + 2a^2(a^2-1)[A^4(y) + A^3(y) + A^2(y) + A^1(y)] \end{aligned}$$

for all $x, y \in X$. Note that

$$\begin{aligned}
&A^4(x+ry)+A^4(x-ry)=2A^4(x)+12r^2A^{2,2}(x,y)+2r^4A^4(y)\,,\\
&A^3(x+ry)+A^3(x-ry)=2A^3(x)+6r^2A^{1,2}(x,y)\,,\\
&A^2(x+ry)+A^2(x-ry)=2A^2(x)+2r^2A^2(y)\,,\\
&A^1(x+ry)+A^1(x-ry)=2A^1(x)\,.
\end{aligned}$$

Since $a \neq 0, \pm 1$, we have

$$A^3(y)+A^2(y)+A^1(y)=0 \tag{2.4}$$

for all $y \in X$. Thus

$$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0(x)=A^4(x)$$

for all $x \in X$.

Conversely, assume that $f(x)=A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of a 4-additive symmetric mapping $A_4 : X^4 \to Y$. Note that

$$\begin{aligned}
&A^4(qx+ry)\\
&=q^4A^4(x)+4q^3rA^{3,1}(x,y)+6q^2r^2A^{2,2}(x,y)+4qr^3A^{1,3}(x,y)+r^4A^4(y)\\
&c^sA^{s,t}(x,y)=A^{s,t}(cx,y)\,,\quad c^tA^{s,t}(x,y)=A^{s,t}(x,cy)
\end{aligned}$$

where $1 \leq s,t \leq 3$ and $c \in \mathbb{Q}$. Thus we may conclude that f satisfies the equation (1.3). □

3. Quartic Lie ∗-derivations

Throughout this section, we assume that A is a complex normed ∗-algebra and M is a Banach A-bimodule. We will use the same symbol $||\cdot||$ as norms on a normed algebra A and a normed A-bimodule M. A mapping $f : A \to M$ is a *quartic homogeneous mapping* if $f(\mu a)=\mu^4 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $f : A \to M$ is called a *quartic derivation* if

$$f(xy)=f(x)y^4+x^4f(y)$$

holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x,\, y]$ will denote the commutator $xy-yx$. We say that a quartic homogeneous mapping $f : A \to M$ is a quartic Lie derivation if

$$f([x,\, y])=[f(x),\, y^4]+[x^4,\, f(y)]$$

for all $x,\, y \in A$. In addition, if f satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called the *quartic Lie $*$-derivation.*

Example 3.1. Let $A = \mathbb{C}$ be a complex field endowed with the map $z \mapsto z^* = \bar{z}$ (where $\bar{z}$ is the complex conjugate of z). We define $f : A \to A$ by $f(a) = a^4$ for all $a \in A$. Then f is quartic and

$$f([a,\, b]) = [f(a),\, b^4] + [a^4,\, f(b)] = 0$$

for all $a \in A$. Also,

$$f(a^*) = f(\bar{a}) = \bar{a}^4 = \overline{f(a)} = f(a)^*$$

for all $a \in A$. Thus f is a quartic Lie $*$-derivation.

In the following, $\mathbb{T}^1$ will stand for the set of all complex units, that is,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} \,|\, |\mu| = 1\}\,.$$

For the given mapping $f : A \to M$, we consider

$$\begin{aligned} \Delta_\mu f(x,y) &:= f(\mu x + s\mu y) + f(\mu x - s\mu y) - s^2\mu^4 f(x+y) - s^2\mu^4 f(x-y) \\ &\quad +2\mu^4(s^2-1)f(x) - 2\mu^4 s^2(s^2-1)f(y)\,, \end{aligned} \tag{3.1}$$

$$\Delta f(x,y) := f([x,\, y]) - [f(x),\, y^4] - [x^4,\, f(y)]$$

for all $x,\, y \in A$, $\mu \in \mathbb{C}$ and $s \in \mathbb{Z}\,(s \neq 0\,, \pm 1)$.

Theorem 3.2. *Suppose that $f : A \to M$ is an even mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \to [0,\, \infty)$ such that*

$$\widetilde{\phi}(a,b,x,y,z) := \sum_{j=0}^{\infty} \frac{1}{|s|^{4j}} \phi(s^j a, s^j b, s^j x, s^j y, s^j z) < \infty \tag{3.2}$$

$$||\Delta_\mu f(a,b)|| \le \phi(a,b,0,0,0) \tag{3.3}$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \le \phi(0,0,x,y,z) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} \,|\, 0 \le \theta \le \frac{2\pi}{n_0}\}$ and all $a,b,x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $b \in A$ the mapping $r \mapsto f(rb)$ from $\mathbb{R}$ to M is continuous, then there exists a unique quartic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(b) - L(b)|| \le \frac{1}{2|s|^4} \widetilde{\phi}(0,b,0,0,0)\,. \tag{3.5}$$

Proof. Let $a = 0$ and $\mu = 1$ in the inequality (3.3), we have

$$||f(b) - \frac{1}{s^4}f(sb)|| \leq \frac{1}{2|s|^4}\phi(0, b, 0, 0, 0) \tag{3.6}$$

for all $b \in A$. Using the induction, it is easy to show that

$$||\frac{1}{s^{4t}}f(s^t b) - \frac{1}{s^{4k}}f(s^k b)|| \leq \frac{1}{2|s|^4}\sum_{j=k}^{t-1}\frac{\phi(0, s^j b, 0, 0, 0)}{|s|^{4j}} \tag{3.7}$$

for $t > k \geq 0$ and $b \in A$. The inequalities (3.2) and (3.7) imply that the sequence $\{\frac{1}{s^{4n}}f(s^n b)\}_{n=0}^{\infty}$ is a Cauchy sequence. Since M is complete, the sequence is convergent. Hence we can define a mapping $L : A \to M$ as

$$L(b) = \lim_{n\to\infty}\frac{1}{s^{4n}}f(s^n b) \tag{3.8}$$

for $b \in A$. By letting $t = n$ and $k = 0$ in the inequality (3.7), we have

$$||\frac{1}{s^{4n}}f(s^n b) - f(b)|| \leq \frac{1}{2|s|^4}\sum_{j=0}^{n-1}\frac{\phi(0, s^j b, 0, 0, 0)}{|s|^{4j}} \tag{3.9}$$

for $n > 0$ and $b \in A$. By taking $n \to \infty$ in the inequality (3.9), the inequalities (3.2) implies that the inequality (3.5) holds.

Now, we will show that the mapping L is a unique quartic Lie $*$-derivation such that the inequality (3.5) holds for all $b \in A$. We note that

$$\begin{aligned} ||\Delta_\mu L(a, b)|| &= \lim_{n\to\infty}\frac{1}{|s|^{4n}}||\Delta_\mu f(s^n a, s^n b)|| \\ &\leq \lim_{n\to\infty}\frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{4n}} = 0, \end{aligned} \tag{3.10}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. By taking $\mu = 1$ in the inequality (3.10), it follows that the mapping L is a quartic mapping. Also, the inequality (3.10) implies that $\Delta_\mu L(0, b) = 0$. Hence

$$L(\mu b) = \mu^4 L(b)$$

for all $b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. Let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$. Then $\mu = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Let $\mu_1 = \mu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$. Hence we have $\mu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$. Then

$$L(\mu b) = L(\mu_1^{n_0} b) = \mu_1^{4n_0}L(b) = \mu^4 L(b)$$

for all $\mu \in \mathbb{T}^1$ and $a \in A$. Suppose that ρ is any continuous linear functional on A and b is a fixed element in A. Then we can define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(r) = \rho(L(rb))$$

for all $r \in \mathbb{R}$. It is easy to check that g is cubic. Let

$$g_k(r) = \rho\Big(\frac{f(s^k rb)}{s^{4k}}\Big)$$

for all $k \in \mathbb{N}$ and $r \in \mathbb{R}$.

Note that g as the pointwise limit of the sequence of measurable functions g_k is measurable. Hence g as a measurable quartic function is continuous (see [5]) and

$$g(r) = r^4 g(1)$$

for all $r \in \mathbb{R}$. Thus

$$\rho(L(rb)) = g(r) = r^4 g(1) = r^4 \rho(L(b)) = \rho(r^4 L(b))$$

for all $r \in \mathbb{R}$. Since ρ was an arbitrary continuous linear functional on A we may conclude that

$$L(rb) = r^4 L(b)$$

for all $r \in \mathbb{R}$. Let $\mu \in \mathbb{C}\,(\mu \neq 0)$. Then $\frac{\mu}{|\mu|} \in \mathbb{T}^1$. Hence

$$L(\mu a) = L\Big(\frac{\mu}{|\mu|}|\mu| b\Big) = \Big(\frac{\mu}{|\mu|}\Big)^4 L(|\mu| b) = \Big(\frac{\mu}{|\mu|}\Big)^4 |\mu|^4 L(b) = \mu^4 L(b)$$

for all $b \in A$ and $\mu \in \mathbb{C}\,(\mu \neq 0)$. Since b was an arbitrary element in A, we may conclude that L is quartic homogeneous.

Next, replacing x, y by $s^k x, s^k y$, respectively, and $z = 0$ in the inequality (3.4), we have

$$\begin{aligned} ||\Delta L(x,y)|| &= \lim_{n\to\infty} ||\frac{\Delta f(s^n x, s^n y)}{s^{4n}}|| \\ &\leq \lim_{n\to\infty} \frac{1}{|s|^{4n}} \phi(0,0,s^n x, s^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. Hence we have $\Delta L(x,y) = 0$ for all $x, y \in A$. That is, L is a quartic Lie derivation. Letting $x = y = 0$ and replacing z by $s^k z$ in the inequality (3.4), we get

$$\Big|\Big|\frac{f(s^n z^*)}{s^{4n}} - \frac{f(s^n z)^*}{s^{4n}}\Big|\Big| \leq \frac{\phi(0,0,0,0,s^n z)}{|s|^{4n}} \tag{3.11}$$

for all $z \in A$. As $n \to \infty$ in the inequality (3.11), we have

$$L(z^*) = L(z)^*$$

for all $z \in A$. This means that L is a quartic Lie $*$-derivation. Now, assume $L' : A \to A$ is another quartic $*$-derivation satisfying the inequality (3.5). Then

$$\begin{aligned} ||L(b) - L'(b)|| &= \frac{1}{|s|^{4n}}||L(s^n b) - L'(s^n b)|| \\ &\leq \frac{1}{|s|^{4n}}\Big(||L(s^n b) - f(s^n b)|| + ||f(s^n b) - L'(s^n b)||\Big) \\ &\leq \frac{1}{|s|^{4n+1}}\sum_{j=0}^{\infty}\frac{1}{|s|^{4j}}\phi(0, s^{j+n}b, 0, 0, 0) \\ &= \frac{1}{|s|^4}\sum_{j=n}^{\infty}\frac{1}{|s|^{4j}}\phi(0, s^j b, 0, 0, 0)\,, \end{aligned}$$

which tends to zero as $k \to \infty$, for all $b \in A$. Thus $L(b) = L'(b)$ for all $b \in A$. This proves the uniqueness of L. □

Corollary 3.3. *Let θ, r be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a, b)|| \leq \theta(||a||^r + ||b||^r)$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique quartic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(b) - L(b)|| \leq \frac{\theta||b||^r}{2(|s|^4 - |s|^r)}$$

for all $b \in A$.

Proof. The proof follows from Theorem 3.2 by taking $\phi(a, b, x, y, z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. □

In the following corollaries, we show the hyperstability for the quartic Lie $*$-derivations.

Corollary 3.4. *Let r be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a, b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq ||x||^r ||y||^r ||z||^r$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then f is a quartic Lie $*$-derivation on A.

Proof. By taking $\phi(a,b,x,y,z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r||z||^r)$ in Theorem 3.2 for all $a,b,x,y,z \in A$, we have $\widetilde{\phi}(0,b,0,0,0) = 0$. Hence the inequality (3.5) implies that $f = L$, that is, f is a quartic Lie $*$-derivation on A. □

Corollary 3.5. *Let r be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq ||x||^r(||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a,b,x,y,z \in A$. Then f is a quartic Lie $*$-derivation on A.

Proof. By taking $\phi(a,b,x,y,z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.2 for all $a,b,x,y,z \in A$, we have $\widetilde{\phi}(0,b,0,0,0) = 0$. Hence the inequality (3.5) implies that $f = L$, that is, f is a quartic Lie $*$-derivation on A. □

Now, we will investigate the stability of the given functional equation (3.1) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [11] and [15].

Definition 3.6. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 3.7 (The alternative of fixed point [11], [15])**.** *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant l. Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

(1) *$d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0} x, y) < \infty\};$$

(4) *$d(y, y^*) \leq \frac{1}{1-l} d(y, Ty)$ for all $y \in \triangle$.*

Theorem 3.8. *Let $f : A \to M$ be a continuous even mapping with $f(0) = 0$ and let $\phi : A^5 \to [0, \infty)$ be a continuous mapping such that*

$$||\Delta_\mu f(a, b)|| \le \phi(a, b, 0, 0, 0) \tag{3.12}$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \le \phi(0, 0, x, y, z) \tag{3.13}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. If there exists a constant $l \in (0, 1)$ such that

$$\phi(sa, sb, sx, sy, sz) \le |s|^4 l\phi(a, b, x, y, z) \tag{3.14}$$

for all $a, b, x, y, z \in A$, then there exists a quartic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(b) - L(b)|| \le \frac{1}{2|s|^4(1-l)}\phi(0, b, 0, 0, 0) \tag{3.15}$$

for all $b \in A$.

Proof. Consider the set

$$\Omega = \{g \,|\, g : A \to A\,, g(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(g,\, h) = \inf\{c \in (0, \infty) \,|\, \parallel g(b) - h(b) \parallel \le c\phi(0, b, 0, 0, 0)\,, \text{for all } b \in A\}\,.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$T(g)(b) = \frac{1}{s^4} g(sb) \tag{3.16}$$

for all $b \in A$. Note that for all $g, h \in \Omega$, let $c \in (0,\, \infty)$ be an arbitrary constant with $d(g, h) \le c$. Then

$$||g(b) - h(b)|| \le c\phi(0, b, 0, 0, 0) \tag{3.17}$$

for all $b \in A$. Letting $b = sb$ in the inequality (3.17) and using (3.14) and (3.16), we have

$$\begin{aligned} ||T(g)(b) - T(h)(b)|| &= \frac{1}{|s|^4}||g(sb) - h(sb)|| \\ &\le \frac{1}{|s|^4}\, c\,\phi(0, sb, 0, 0, 0) \le c\, l\, \phi(0, b, 0, 0, 0)\,, \end{aligned}$$

that is,

$$d(Tg,\, Th) \le c\, l\,.$$

Hence we have that

$$d(Tg,\, Th) \le l\, d(g,\, h)\,,$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant l. Letting $\mu = 1$, $a = 0$ in the inequality (3.12), we get

$$||\frac{1}{s^4}f(sb) - f(b)|| \leq \frac{1}{2|s|^4}\phi(0, b, 0, 0, 0)$$

for all $b \in A$. This means that

$$d(Tf, f) \leq \frac{1}{2|s|^4}\,.$$

We can apply the alternative of fixed point and since $\lim_{n\to\infty} d(T^n f, L) = 0$, there exists a fixed point L of T in Ω such that

$$L(b) = \lim_{n\to\infty} \frac{f(s^n b)}{s^{4n}}\,, \tag{3.18}$$

for all $b \in A$. Hence

$$d(f, L) \leq \frac{1}{1-l}d(Tf, f) \leq \frac{1}{2|s|^4}\frac{1}{1-l}\,.$$

This implies that the inequality (3.15) holds for all $b \in A$. Since $l \in (0, 1)$, the inequality (3.14) shows that

$$\lim_{n\to\infty} \frac{\phi(s^n a, s^n b, s^n x, s^n y, s^n z)}{|s|^{4n}} = 0\,. \tag{3.19}$$

Replacing a, b by $s^n a$, $s^n b$, respectively, in the inequality (3.12), we have

$$\frac{1}{|s|^{4n}}||\Delta_\mu f(s^n a, s^n b)|| \leq \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{4n}}\,.$$

Taking the limit as k tend to infinity, we have $\Delta_\mu f(a, b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. The remains are similar to the proof of Theorem 3.2. □

Corollary 3.9. *Let θ, r be positive real numbers with $r < 4$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a, b)|| \leq \theta(||a||^r + ||b||^r)$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique quartic Lie ∗-derivation $L : A \to M$ satisfying

$$||f(b) - L(b)|| \leq \frac{\theta||b||^r}{2|s|^4(1-l)}$$

for all $b \in A$.

Proof. The proof follows from Theorem 3.8 by taking $\phi(a, b, x, y, z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. □

In the following corollaries, we show the hyperstability for the quartic Lie $*$-derivations.

Corollary 3.10. *Let r be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq ||x||^r ||y||^r ||z||^r$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then f is a quartic Lie $*$-derivation on A.

Proof. By taking $\phi(a,b,x,y,z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r ||z||^r)$ in Theorem 3.8 for all $a, b, x, y, z \in A$, we have $\widetilde{\phi}(0,b,0,0,0) = 0$. Hence the inequality (3.15) implies that $f = L$, that is, f is a quartic Lie $*$-derivation on A. □

Corollary 3.11. *Let r be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq ||x||^r (||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then f is a quartic Lie $*$-derivation on A.

Proof. By taking $\phi(a,b,x,y,z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.8 for all $a, b, x, y, z \in A$, we have $\widetilde{\phi}(0,b,0,0,0) = 0$. Hence the inequality (3.15) implies that $f = L$, that is, f is a quartic Lie $*$-derivation on A. □

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