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EXISTENCE RESULTS FOR NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

S. CHANDRASEKARAN* AND S. KARUNANITHI

Abstract. This paper is concerned with the existence of mild solutions for partial neutral functional integrodifferential equations with infinite delay in Banach spaces. The results are obtained by using resolvent operators and Krasnoselski-Schaefer type fixed point theorem. An example is provided to illustrate the results.

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1. Introduction

The aim of this paper is to establish the existence results for the following neutral functional integrodifferential equations with infinite delays:

$$\frac{d}{dt}\left[x(t) - g\left(t, x_t, \int_0^t a(t,s,x_s)ds\right)\right]$$

$$= Ax(t) + \int_0^t B(t-s)x(s)ds + f\left(t, x_t, \int_0^t k(t,s,x_s)ds\right),\ t \in J = [0,b] \quad (1)$$

$$x_0 = \phi \in \mathcal{B}_h \quad (2)$$

where A is the infinitesimal generator of a compact, analytic resolvent operator $R(t), t \geq 0$ in a Banach space X, $a : D \times \mathcal{B}_h \to X$, $g : J \times \mathcal{B}_h \times X \to X$, $k : D \times \mathcal{B}_h \to X$ and $f : J \times \mathcal{B}_h \times X \to X$ are given functions, where $\mathcal{B}_h$ is a phase space defined later and $D = \{(t,s) \in J \times J : s \leq t\}.0 < t_1 < t_2 < ... < t_m < b$ are bounded functions. $B(t), t \in J$ is a bounded linear operator.

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The histories $x_t : (-\infty, 0] \to X, x_t(s) = x(t+s), s \leq 0$, belong to an abstract phase space $\mathcal{B}_h$.

Neutral differential and integrodifferential equations arise in many areas of applied mathematics and for this reason these equations have been investigated extensively in the last decades. There are many contributions relative to this topic and we refer the reader to [1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 15].

The theory of nonlinear functional differential or integrodifferential equations with resolvent operators is an important branch of differential equations, which has an extensive physical background, see for instance [16, 17, 18].

Since many control systems arsing from realistic models depend heavily on histories (that is, the effect of infinite delay on the state equations [23]), there is real need to discuss the existence results for partial neutral functional integrodifferential equations with infinite delay. The development of the theory of functional differential equations with infinite delays depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space has required a separate development of the theory [20]. The common space is the phase space $\mathcal{B}$ proposed by Hale and Kato in [19].

The main purpose of this paper is to deal with the existence of mild solutions for the problem (1)-(2). Here, we use an abstract phase space adopted in [6, 24]. Sufficient conditions for the existence results are derived by means of the Krasnoselski-Schaefer type fixed point theorem combined with theories of analytic resolvent operators. The results generalise the results of [4, 6, 7, 21].

2. Main results

Throughout this paper, we assume that $(X, \|\cdot\|)$ is a Banach space, the notaion $L(X,Y)$ stands for the Banach space of all linear bounded operators from X into Y, and we abbreviate this notation to $L(X)$ when $X = Y$. $R(t),\ t > 0$ is compact, analytic resolvent operator generated by A.

Assume that

(A1) A is a densely defined, closed linear operator in a Banach space $(X, \|\cdot\|)$ and generates a C_0-semigroup $T(t)$. Hence $D(A)$ endowed with the graph norm $|x| = \|x\| + \|Ax\|$ is a Banach space which will be denoted by $(Y, |\cdot|)$.

(A2) $\{B(t) : t \in J\}$ is a family of continuous linear operator from $(Y, |\cdot|)$ into $(X, \|\cdot\|)$. Moreover, there is an integrable function $c : [0, b] \to \mathcal{R}^+$ such that for any $y \in Y$, the map $t \to B(t)y$ belongs to $W^{1,1}(J, X)$ and

$$\left\| \frac{d}{dt} B(t)y \right\| \leq c(t)|y|, \quad y \in Y,\ t \in J.$$

Definition 2.1. A family $\{R(t) : t \geq 0\}$ of continuous linear operators on X is called a resolvent operator for

$$\frac{dx}{dt} = Ax(t) + \int_0^t B(t-s)x(s)ds,$$

if:

(R1) $R(0) = I$ (the identity operator on X),

(R2) For all $x \in X$, the map $t \to R(t)x$ is continuous from J to X,

(R3) For all $t \in J, R(t)$ is a continuous linear operator on Y, and for all $y \in Y$, the map $t \to R(t)y$ belongs to $C(J,Y) \cap C'(J,X)$ and satisfies

$$\begin{aligned}\frac{d}{dt}R(t)y &= AR(t)y + \int_0^t B(t-s)R(s)yds\\ &= R(t)Ay + \int_0^t R(t-s)B(s)yds.\end{aligned}$$

Theorem 2.1 ([21]). *Let the assumptions (A1) and (A2) be satisfied. Then there exists a constant $H = H(b)$ such that*

$$\|R(t+h) - R(h)R(t)\|_{L(X)} \leq Hh, \quad \textit{for} \;\; 0 \leq h \leq t \leq b,$$

where $L(X)$ denotes the Banach space of continuous linear operators on X.

Next, if the C_0-semigroup $T(\cdot)$ generated by A is compact (that is, $T(t)$ is a compact operator for all $t > 0$), then the corresponding resolvent operator $R(\cdot)$ is also compact (that is, $R(t)$ is a compact operator for all $t > 0$) and is operator norm continuous (or continuous in the uniform operator topology) for $t > 0$.

Proof. Now, we define the abstract phase space $\mathcal{B}_h$ as given in [24, 7].

Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(t)dt < +\infty$. For any $a > 0$, we define

$$\mathcal{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \;\text{ is bounded and measurable}\},$$

and equip the space $\mathcal{B}$ with the norm

$$\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} |\psi(s)|, \quad \forall\; \psi \in \mathcal{B}.$$

Let us define

$$\begin{aligned}\mathcal{B}_h = \{\psi : (-\infty, 0] \to X \text{ such that for any } \; c > 0,\; \psi|_{[-c,0]} \in \mathcal{B}\\ \text{and } \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds < +\infty\}.\end{aligned}$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds,\; \forall\; \psi \in \mathcal{B}_h,$$

then it is clear that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space

$$\mathcal{B}'_h = \{x : (-\infty, b] \to X \;\text{ such that } x|_J \in C(J,X),\; x_0 = \phi \in \mathcal{B}_h\}.$$

Set $\|\cdot\|_b$ be a semi norm in $\mathcal{B}'_h$ defined by

$$\|x\|_b = \|\phi\|_{\mathcal{B}_h} + \sup\{|x(s)| : s \in [0,b]\},\; x \in \mathcal{B}'_h.$$

Next, we introduce the basic definitions and lemmas which are used throughout this paper.

Let $A : D(A) \to X$ be the infinitesimal generator of a compact, analytic resolvent operator $R(t),\ t \geq 0$. Let $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Further more, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha$$

defines a norm on $D(-A)^\alpha$.

Furthermore, we have the following properties appeared in [22]. □

Lemma 2.2. *The following properties hold:*
(i) If $0 < \beta < \alpha \leq 1$, then $X_\alpha \subset X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.
(ii) For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha R(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq b.$$

Lemma 2.3 ([5])**.** *Let Φ_1, Φ_2 be two operators satisfying Φ_1 is contraction and Φ_2 is completely continuous. Then either*
(i) the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution, or
(ii) the set $G = \{x \in X : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0,1)$.

Lemma 2.4 ([12])**.** *Let $v(\cdot), w(\cdot) : [0,b] \to [0,\infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0, 0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t \in J,$$

then

$$v(t) \leq e^{\theta^n \Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0,b]$ and every $n \in N$ such that $n\alpha > 1$, and $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.5 ([6])**.** *Assume $x \in \mathcal{B}_h'$, then for $t \in J, x_t \in \mathcal{B}_h$. Moreover,*

$$l|x(t)| \leq \|x_t\|_{\mathcal{B}_h} \leq \|x_0\|_{\mathcal{B}_h} + l \sup_{s\in[0,t]} |x(s)|,$$

where $l = \int_{-\infty}^0 h(t)dt < +\infty$.

Definition 2.2. A function $x : (-\infty, b] \to X$ is called a mild solution of problem (1)-(2) if the following holds: $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$; the restriction of $x(\cdot)$ to the interval J is continuous, and for each $s \in [0,t)$, the function $s \to AR(t-s)g(s, x_s, \int_0^s a(s,\tau,x_\tau)d\tau)$, is integrable and the integral equation

$$\begin{aligned}x(t) &= R(t)[\phi(0) - g(0,\phi,0)] + g\left(t, x_t, \int_0^t a(t,s,x_s)ds\right)\\&+ \int_0^t AR(t-s)g\left(s, x_s, \int_0^t a(t,\tau,x_\tau)d\tau\right)ds\\&+ \int_0^t R(t-s)\int_0^s B(s-\tau)g\left(\tau, x_\tau, \int_0^s a(s,\tau,x_\tau)d\tau\right)ds\\&+ \int_0^t R(t-s)f\left(s, x_s, \int_0^s k(t,\tau,x_\tau)d\tau\right)ds, \quad t \in J \qquad (3)\end{aligned}$$

is satisfied.

Definition 2.3. A map $f : J \times \mathcal{B}_h \times X \to X$ is said to be an L^1 -Caratheodory if

(i) For each $t \in J$, the function $f(t,\cdot,\cdot) : \mathcal{B}_h \times X \to X$ is continuous.

(ii) For each $(\phi, x) \in \mathcal{B}_h \times X$; the function $f(\cdot,\phi,x) : J \to X$ is strongly measurable.

(iii) For every positive integer $q > 0$, there exists $\alpha_q \in L^1(J, R_+)$ such that

$$\|f(t,\phi,x)\| \le \alpha_q(t) \text{ for all } \|\phi\|_{\mathcal{B}_h} \le q,\ \|x\| \le q \quad \text{and for almost all } t \in J.$$

3. Existence Results

In this section, we shall present and prove our main result. For the proof of the main result, we will use the following hypotheses:

(H1) ([see Lemma 2.2]) A is the infinitesimal generator of a compact analytic resolvent operator $R(t),\ t > 0$ and $0 \in \rho(A)$ such that

$$\|R(t)\| \le M_1, \qquad \|B(t)\| \le M_2 \text{ for all } t \ge 0 \quad \text{and}$$
$$\|(-A)^{1-\beta}R(t-s)\| \le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}, \quad 0 < t \le b.$$

(H2) There exist a constant $N_1 > 0$ such that

$$\left\|\int_0^t [a(t,s,x) - a(t,s,y)]ds\right\| \le N_1\|x-y\|_{\mathcal{B}_h}$$

(H3) There exist constants $0 < \beta < 1, C_0, c_1, c_2, N_2$ such that g is X_β -valued, $(-A)^\beta g$ is continuous, and

(i) $\|(-A)^\beta g(t, x_t, \int_0^t a(t,s,x_s)ds)\| \le c_1\|\ x_t\|_{\mathcal{B}_h} + c_2, \quad (t,x) \in J \times \mathcal{B}_h,$

(ii) $\|(-A)^\beta g(t,x_1,y_1) - (-A)^\beta g(t,x_2,y_2)\| \le N_2\left[\|x_1 - x_2\|_{\mathcal{B}_h} + \|y_1 - y_2\|\right]$

for $t \in J,\ x_1, x_2 \in \mathcal{B}_h, y_1, y_2 \in X$, with

$$\|(-A)^{-\beta}\| = M_0 \quad \text{and} \quad C_0 = lN_2(1+N_1)\left[M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\right] < 1.$$

(H4) (i) For each $(t,s) \in D$, the function $k(t,s,\cdot) : \mathcal{B}_h \to X$ is continuous and for each $x \in \mathcal{B}_h$, the function $k(\cdot,\cdot,x) : D \to X$ is strongly measurable.
(ii) There exist an integrable function $m : J \to [0,\infty)$ and a constant $\gamma > 0$, such that

$$\|k(t,s,x)\| \leq \gamma m(s)\Omega(\|x\|_{\mathcal{B}_h}),$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function. Assume that the finite bound of $\int_0^t \gamma(s)ds$ is L_0.

(H5) The function $f : J \times \mathcal{B}_h \times X \to X$ satisfies the following caratheodory conditions:
(i) $t \to f(t,x,y)$ is measurable for each $(x,y) \in \mathcal{B}_h \times X$,
(ii) $(x,y) \to f(t,x,y)$ is continuous for almost all $t \in J$.

(H6) $\|f(t,x,y)\| \leq p(t)\Psi(\|x\|_{\mathcal{B}_h} + \|y\|)$ for almost all $t \in J$ and all $x \in \mathcal{B}_h, y \in X$, where
$p \in L^1(J,R_+)$ and $\Psi : R_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^b \widetilde{m}(s)ds < \int_{B_0K_1}^{\infty} \frac{ds}{s + \Psi(s) + \Omega(s)},$$

where

$$K_1 = \frac{\|\phi\|_{\mathcal{B}_h} + lF}{1 - lM_0N_2(1+N_1)}, \quad K_2 = \frac{lN_2(1+N_1)C_{1-\beta}}{1 - lM_0N_2(1+N_1)},$$
$$K_3 = \frac{M_0M_1M_2b}{1 - lM_0N_2(1+N_1)}, \quad \text{and} \quad K_4 = \frac{lM_1}{1 - lM_0N_2(1+N_1)}$$

with $\quad lM_0N_2(1+N_1) < 1, \quad B_0 = e^{K_2^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \left(\frac{K_2 b^\beta}{\beta}\right)^j,$

$\widetilde{m}(t) = \max\{B_0K_3, B_0K_4p(t), \gamma(t)\},$

$F = M_1[|\phi(0)| + M_0N_2\|\phi\|_{\mathcal{B}_h}] + c_2(1+M_1)M_0 + (N_2c_1 + c_2)\frac{C_{1-\beta}b^\beta}{\beta} + N_2C_1M_0,$
$c_1 = b \sup_{(t,s)\in D} a(t,s,0),\ c_2 = \|(-A)^\beta\| \sup_{t\in J} \|g(t,0,0)\|.$

We consider the operator $\Phi : \mathcal{B}_h{}' \to \mathcal{B}_h{}'$ defined by

$$\Phi x(t) = \begin{cases} \phi(t), & t \in (-\infty,0] \\ R(t)[\phi(0) - g(0,\phi,0)] + g\left(t, x_t, \int_0^t a(t,s,x_s)ds\right) \\ + \int_0^t AR(t-s)g\left(s, x_s, \int_0^t a(t,\tau,x_\tau)d\tau\right)ds \\ + \int_0^t R(t-s)\int_0^s B(s-\tau)g\left(\tau, x_\tau, \int_0^s a(s,\tau,x_\tau)d\tau\right)ds \\ + \int_0^t R(t-s)f\left(s, x_s, \int_0^s k(t,\tau,x_\tau)d\tau\right)ds, \quad t \in J. \end{cases} \tag{4}$$

From hypothesis $(H1)$, $(H2)$ and Lemma 2.3, the following inequality holds:

$$\left\|AR(t-s)g\left(s,x_s,\int_0^t a(t,\tau,x_\tau)d\tau\right)\right\| \leq \|(-A)^{1-\beta}R(t-s)\|\left\|(-A)^{\beta}g\left(s,x_s,\int_0^t a(t,\tau,x_\tau)d\tau\right)\right\|$$
$$\leq \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\Big[N_2(1+N_1)\|x_s\|_{\mathcal{B}_h}+N_2c_1+c_2\Big].$$

Then from Bochner theorem [25], it follows that $AR(t-s)g\left(s,x_s,\int_0^t a(t,\tau,x_\tau)d\tau\right)$ is integrable on $[0,t)$.

For $\phi\in\mathcal{B}_h$, we defined by $\tilde{\phi}$ by

$$\tilde{\phi}(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ R(t)\phi(0), & t\in J,\end{cases}$$

and then $\tilde{\phi}\in\mathcal{B}'_h$. Let $x(t)=y(t)+\tilde{\phi}(t), -\infty<t\leq b$. It is easy to see that x satisfies (3) if and only if y satisfies $y_0=0$ and

$$\begin{aligned} y(t) = & -R(t)g(0,\phi,0)+g\Big(t,y_t+\tilde{\phi}_t,\int_0^t a(t,s,y_s+\tilde{\phi}_s)ds\Big)\\ &+\int_0^t AR(t-s)g\Big(s,y_s+\tilde{\phi}_s,\int_0^s a(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds\\ &+\int_0^t R(t-s)\int_0^s B(s-\tau)g\Big(s,y_s+\tilde{\phi}_s,\int_0^s a(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds\\ &+\int_0^t R(t-s)f\Big(s,y_s+\tilde{\phi}_s,\int_0^s k(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds.\end{aligned}$$

Let $\mathcal{B}''_h=\{y\in\mathcal{B}'_h : y_0=0\in\mathcal{B}_h\}$. For any $y\in\mathcal{B}''_h$,

$$\begin{aligned}\|y\|_b &= \|y_0\|_{\mathcal{B}_h}+\sup\{|y(s)|:0\leq s\leq b\}\\ &=\sup\{|y(s)|:0\leq s\leq b\},\end{aligned}$$

thus $(\mathcal{B}''_h,\|\cdot\|_b)$ is a Banach space. Set $B_q=\{y\in\mathcal{B}''_h:\|y\|_b\leq q\}$ for some $q\geq 0$, then $B_q\subseteq\mathcal{B}''_h$ is uniformly bounded, and for $y\in B_q$, from Lemma 2.5, we have

$$\begin{aligned}\|y_t+\tilde{\phi}_t\|_{\mathcal{B}_h} &\leq \|y_t\|_{\mathcal{B}_h}+\|\tilde{\phi}_t\|_{\mathcal{B}_h}\\ &\leq l\sup_{s\in[0,t]}|y(s)|+\|y_0\|_{\mathcal{B}_h}+l\sup_{s\in[0,t]}|\tilde{\phi}(s)|+\|\tilde{\phi}_0\|_{\mathcal{B}_h}\\ &\leq l(q+M_1|\phi(0)|)+\|\phi\|_{\mathcal{B}_h}=q'. \end{aligned}\tag{5}$$

Define the operator $\bar{\Phi}:\mathcal{B}''_h\to\mathcal{B}''_h$ by

$$\bar{\Phi}y(t)=\begin{cases} 0, & t\in(-\infty,0]\\ -R(t)g(0,\phi,0)+g\Big(t,y_t+\tilde{\phi}_t,\int_0^t a(t,s,y_s+\tilde{\phi}_s)ds\Big) & \\ \quad+\int_0^t AR(t-s)g\Big(s,y_s+\tilde{\phi}_s,\int_0^s a(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds & \\ \quad+\int_0^t R(t-s)\int_0^s B(s-\tau)g\Big(s,x_s,\int_0^s a(s,\tau,x_\tau)d\tau\Big)ds & \\ \quad+\int_0^t R(t-s)f\Big(s,y_s+\tilde{\phi}_s,\int_0^s k(s,\tau,y_\tau+\tilde{\phi}_\tau)d\tau\Big)ds, & t\in J.\end{cases}$$

Now we decompose $\bar{\Phi}$ as $\bar{\Phi}_1 + \bar{\Phi}_2$ where

$$\begin{aligned}
\bar{\Phi}_1 y(t) &= -R(t)g(0,\phi,0) + g\Big(t, y_t + \tilde{\phi}_t, \int_0^t a(t,s,y_s + \tilde{\phi}_s)ds\Big) \\
&\quad + \int_0^t AR(t-s)g\Big(s, y_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds, \quad t \in J, \\
\bar{\Phi}_2 y(t) &= \int_0^t R(t-s)\int_0^s B(s-\tau)g\Big(s, y_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds \\
&\quad + \int_0^t R(t-s)f\Big(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds, \quad t \in J.
\end{aligned}$$

Obviously the operator Φ has a fixed point is equivalent to $\bar{\Phi}$ has one. Now, we shall show that the operators $\bar{\Phi}_1, \bar{\Phi}_2$ satisfy all the conditions of Lemma 2.3.

Lemma 3.1. *If assumptions (H1)-(H6) hold, then $\bar{\Phi}_1$ is a contraction and $\bar{\Phi}_2$ is completely continuous.*

Proof. First we show that $\bar{\Phi}_1$ is a contraction on $\mathcal{B}_h''$. Let $u, v \in \mathcal{B}_h''$. From (H1)-(H3) and Lemma 2.5, we have

$$\begin{aligned}
&\|\bar{\Phi}_1 u(t) - \bar{\Phi}_1 v(t)\| \\
&\leq \Big\|g\Big(t, u_t + \tilde{\phi}_t, \int_0^t a(t,s,u_s + \tilde{\phi}_s)ds\Big) - g\Big(t, v_t + \tilde{\phi}_t, \int_0^t a(t,s,v_s + \tilde{\phi}_s)ds\Big)\Big\| \\
&\quad + \int_0^t \Big\|AR(t-s)\Big[g\Big(s, u_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big) - g\Big(s, v_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)\Big]\Big\|ds \\
&\leq M_0 N_2\Big[\|u_t - v_t\|_{\mathcal{B}_h} + N_1\|u_t - v_t\|_{\mathcal{B}_h}\Big] + \int_0^t \frac{C_{1-\beta}N_2}{(t-s)^{1-\beta}}\Big[\|u_s - v_s\|_{\mathcal{B}_h} + N_1\|u_s - v_s\|_{\mathcal{B}_h}\Big]ds \\
&\leq M_0 N_2(1+N_1)\|u_t - v_t\|_{\mathcal{B}_h} + \frac{N_2(1+N_1)C_{1-\beta}b^\beta}{\beta}\|u_t - v_t\|_{\mathcal{B}_h} \\
&\leq N_2(1+N_1)\Big[M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\Big]\Big[l \sup_{s\in[0,t]}|u(s) - v(s)| + \|u_0\|_{\mathcal{B}_h} + \|v_0\|_{\mathcal{B}_h}\Big] \\
&\leq N_2(1+N_1)\Big[M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\Big] \sup_{s\in[0,b]}|u(s) - v(s)|.
\end{aligned}$$

Since $\|u_0\|_{\mathcal{B}_h} = 0, \|v_0\|_{\mathcal{B}_h} = 0$. Taking supremum over t,

$$\|\bar{\Phi}_1 u - \bar{\Phi}_1 v\| \leq C_0\|u - v\|,$$

where $C_0 = M_0 N_2(1+N_1)\Big[M_0 + \frac{C_{1-\beta}b^\beta}{\beta}\Big] < 1$. Thus $\bar{\Phi}_1$ is a contraction on $\mathcal{B}_h''$. □

Next we show that the operator $\bar{\Phi}_2$ is completely continuous. First we prove that $\bar{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{B}_h''$.

Indeed, it is enough to show that there exists a positive constant Λ such that for each $y \in B_q = \{y \in \mathcal{B}_h'' : \|y\|_b \leq q\}$ one has $\|\bar{\Phi}_2 y\|_b \leq \Lambda$. Now for each $t \in J$,

$$\begin{aligned}
\bar{\Phi}_2 y(t) &= \int_0^t R(t-s)\int_0^s B(s-\tau)g\Big(s, y_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds \\
&\quad + \int_0^t R(t-s)f\Big(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\Big)ds.
\end{aligned}$$

By (H1)-(H6) and (5), we have for $t \in J$,

$$
\begin{aligned}
|\bar{\Phi}y(t)| &\le M_0M_1M_2b\int_0^t [N_2(1+N_1)\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}+N_2c_1+c_2]ds \\
&\quad + M_1\int_0^t p(s)\Psi\Big(\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}+\int_0^s \gamma_m(\tau)\Omega(\|y_\tau+\tilde{\phi}_\tau\|_{\mathcal{B}_h})d\tau\Big)ds \\
&\le M_0M_1M_2b^2[N_2(1+N_1)q'+N_2c_1+c_2]+M_1\Psi(q'+L_0\Omega(q'))\int_0^b p(s)ds=\Lambda.
\end{aligned}
$$

Then for each $y \in B_q$, we have $\|\bar{\Phi}_2 y\|_b \le \Lambda$.

Next we show that $\bar{\Phi}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}''_h$.

Let $0 < r_1 < r_2 \le b$, for each $y \in B_q = \{y \in \mathcal{B}''_h : \|y\|_b \le q\}$. Let $r_1, r_2 \in J - \{t_1, t_2, ..., t_m\}$. Then we have

$$
\begin{aligned}
&\|\bar{\Phi}_2y(r_1)-\bar{\Phi}_2y(r_2)\| \\
&\le M_0M_2b\int_0^{r_1-\epsilon}\|R(r_1-s)-R(r_2-s)\|_{L(X)}[N_2(1+N_1)q'+N_2c_1+c_2]ds \\
&\quad + M_0M_2b\int_{r_1-\epsilon}^{r_1}\|R(r_1-s)-R(r_2-s)\|_{L(X)}[N_2(1+N_1)q'+N_2c_1+c_2]ds \\
&\quad + M_0M_1M_2b\int_{r_1}^{r_2}[N_2(1+N_1)q'+N_2c_1+c_2]ds \\
&\quad + \int_0^{r_1-\epsilon}\|R(r_1-s)-R(r_2-s)\|_{L(X)}\alpha_{q'}(s)ds \\
&\quad + \int_{r_1-\epsilon}^{r_1}\|R(r_1-s)-R(r_2-s)\|_{L(X)}\alpha_{q'}(s)ds+M_1\int_{r_1}^{r_2}\alpha_{q'}(s)ds.
\end{aligned}
$$

The right-hand side from Theorem 2.1 of the above inequality tends to zero as $r_2 \to r_1$ and for ϵ sufficiently small. Thus the set $\{\bar{\Phi}_2 y : y \in B_q\}$ is equicontinuous. Here we consider only the case $0 < r_1 < r_2 \le b$, since the other cases $r_1 < r_2 \le 0$ or $r_1 \le 0 \le r_2 \le b$ are very simple.

Next, we show that $\bar{\Phi}_2 : \mathcal{B}''_h \to \mathcal{B}''_h$ is continuous.

Let $\{y^{(n)}(t)\}_{n=0}^{+\infty} \subseteq \mathcal{B}''_h$, with $y^{(n)} \to y$ in $\mathcal{B}''_h$. Then there is a number $q > 0$ such that $|y^{(n)}(t)| \le q$ for all n and a.e. $t \in J$, so $y^{(n)} \in B_q$ and $y \in B_q$. In view of (5), we have

$$\|y_t^{(n)}+\tilde{\phi}_t\|_{\mathcal{B}_h} \le q',\ t \in J.$$

By $(H3)$, $(H5)$ and Definition 2.2,

$f\left(t, y_t^{(n)}+\tilde{\phi}_t, \int_0^t k(t,s,y_s^{(n)}+\tilde{\phi}_s)ds\right) \to f\left(t, y_t+\tilde{\phi}_t, \int_0^t k(t,s,y_s+\tilde{\phi}_s)ds\right)$ for each $t \in J$ and since

$$\Big|f\left(t, y_t^{(n)}+\tilde{\phi}_t, \int_0^t k(t,s,y_s^{(n)}+\tilde{\phi}_s)ds\right) \to f\left(t, y_t+\tilde{\phi}_t, \int_0^t k(t,s,y_s+\tilde{\phi}_s)ds\right)\Big| \le 2\alpha_{q'}(t).$$

We have by the dominated convergence theorem that

$$\|\bar{\Phi}_2y^{(n)}-\bar{\Phi}_2y\|_b=\sup_{t\in J}\Big|\int_0^t R(t-s)\int_0^s B(s-\tau)\Big[g\left(t, y_\tau^{(n)}+\tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau^{(n)}+\tilde{\phi}_\tau)d\tau\right)$$

$$
\begin{aligned}
&\quad - g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau)d\tau\right)\Big]ds \\
&\quad + \int_0^t R(t-s)\Big[f\left(s, y_s^{(n)} + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau^{(n)} + \tilde{\phi}_\tau)d\tau\right) \\
&\quad - f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau^{(n)} + \tilde{\phi}_\tau)d\tau\right)\Big]ds\Big| \\
&\leq \int_0^b R(t-s)\int_0^s B(s-\tau)\Big[g\left(t, y_\tau^{(n)} + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau^{(n)} + \tilde{\phi}_\tau)\right) \\
&\quad - g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau)\right)\Big]d\tau ds \\
&\quad + \int_0^t R(t-s)\Big[f\left(s, y_s^{(n)} + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau^{(n)} + \tilde{\phi}_\tau)d\tau\right) \\
&\quad - f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau^{(n)} + \tilde{\phi}_\tau)d\tau\right)\Big]ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thus $\bar{\Phi}_2$ is continuous.

Next we show that $\bar{\Phi}_2$ maps B_q into a precompact in X. Let $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$, we define the operators

$$
\begin{aligned}
(\bar{\Phi}_2^\epsilon y)(t) &= R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)\int_0^s B(s-\tau)g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds \\
&\quad + R(\epsilon)\int_0^{t-\epsilon} R(t-s-\epsilon)f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds
\end{aligned}
$$

and

$$
\begin{aligned}
(\widetilde{\Phi}_2^\epsilon y)(t) &= \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds \\
&\quad + \int_0^{t-\epsilon} R(t-s)f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds
\end{aligned}
$$

From Theorem 2.1 and the compactness of the operator $R(\epsilon)$, the set $\widetilde{V}_\epsilon(t) = \{(\widetilde{\Phi}_2^\epsilon y)(t) : y \in B_q\}$is precompact in X, for every $\epsilon,\ 0 < \epsilon < t$. Moreover, by Theorem 2.1 and for each $y \in B_q$, we have

$$
\begin{aligned}
&\|(\bar{\Phi}_2^\epsilon y)(t) - (\widetilde{\Phi}_2^\epsilon y)(t)\| \\
&\leq \Big\|\int_0^{t-\epsilon} R(\epsilon)R(t-s-\epsilon)\int_0^s B(s-\tau)g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds \\
&\quad - \int_0^{t-\epsilon} R(t-s)\int_0^s B(s-\tau)g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds\Big\| \\
&\quad + \Big\|\int_0^{t-\epsilon} R(\epsilon)R(t-s-\epsilon)f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds \\
&\quad + \int_0^{t-\epsilon} R(t-s)f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)ds\Big\|
\end{aligned}
$$

$$\leq \int_0^{t-\epsilon} \|R(\epsilon)R(t-s-\epsilon) - R(t-s)\|_{L(X)}$$
$$(\times) \int_0^s \|B(s-\tau)\|\|g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)\|ds$$
$$+ \int_0^{t-\epsilon} \|R(\epsilon)R(t-s-\epsilon) - R(t-s)\|_{L(X)}$$
$$(\times)\|f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)\|ds$$

$$\leq \epsilon H \int_0^{t-\epsilon} \int_0^s \|B(s-\tau)\|\|g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)\|d\tau ds$$
$$+ \epsilon H \int_0^{t-\epsilon} \|f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)\|ds.$$

So the set $\widetilde{V}_\epsilon(t) = \{(\widetilde{\Phi}_2^\epsilon y)(t) : y \in B_q\}$ is precompact in X by using the total boundedness. Applying this idea again and observing that

$$|(\bar{\Phi}_2 y)(t) - (\widetilde{\Phi}_2^\epsilon y)(t)|$$
$$\leq \int_{t-\epsilon}^t |R(t-s)| \int_0^s |B(s-\tau)||g\left(t, y_\tau + \tilde{\phi}_\tau, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)\|d\tau ds$$
$$+ \int_{t-\epsilon}^t |R(t-s)||f\left(s, y_s + \tilde{\phi}_s, \int_0^s k(s,\tau,y_\tau + \tilde{\phi}_\tau)d\tau\right)|ds$$
$$\leq M_0 M_1 M_2 b \int_{t-\epsilon}^t (N_2(1+N_1)q' + N_2 c_1 + c_2)ds + M_1 \int_{t-\epsilon}^t \alpha_{q'}(s)ds.$$

Therefore,

$$|(\bar{\Phi}_2 y)(t) - (\widetilde{\Phi}_2^\epsilon y)(t)| \to 0, \text{ as } \quad \epsilon \to 0^+,$$

and there are precompact sets arbitrarily close to the set $\{(\bar{\Phi}_2 y)(t) : y \in B_q\}$. Thus the set $\{(\bar{\Phi}_2 y)(t) : y \in B_q\}$ is precompact in X.
Therefore from Arzela-Ascoli theorem, we can conclude that the operator $\bar{\Phi}_2$ is completely continuous. In order to study the existence results for the problem (1)-(2), we introduce a parameter $\lambda \in (0,1)$ and consider the following nonlinear operator equation

$$x(t) = \lambda \Phi x(t), \tag{6}$$

where Φ is already defined. The following lemma proves that an a priori bound exists for the solution of the above equation.

Lemma 3.2. *If hypotheses (H1)-(H6) are satisfied, then there exists an a priori bound $K > 0$ such that $\|x_t\|_{\mathcal{B}_h} \leq K,\ t \in J$, where K depends only on b and on the functions $\Psi, \tilde{m}$ and Ω*

Proof. From the equation (6), we have

$$
\begin{aligned}
|x(t)| &\leq M_1\Big(|\phi(0)| + M_0N_2\|\phi\|_{\mathcal{B}_h} + M_0c_2\Big) + (N_2c_1 + c_2)\frac{C_{1-\beta}b^{\beta}}{\beta} + (1+N_1)N_2c_{1-\beta} \\
&\quad (\times)\int_0^t \frac{\|x_s\|_{\mathcal{B}_h}}{(t-s)^{1-\beta}}ds + (1+M_1M_2b^2)M_0\Big[(1+N_1)N_2\|x_t\|_{\mathcal{B}_h} + N_2c_1\Big] \\
&\quad (\times)M_1\int_0^t p(s)\Psi\Big(\|x_s\|_{\mathcal{B}_h} + \int_0^s \gamma m(\tau)\Omega(\|x_\tau\|)d\tau\Big)ds \\
&\leq F + M_0(1+N_1)N_2\|x_t\|_{\mathcal{B}_h} + (1+N_1)N_2C_{1-\beta}\int_0^t \frac{\|x_s\|_{\mathcal{B}_h}}{(t-s)^{1-\beta}}ds + M_0M_1M_2bc_1 \\
&\quad (\times)\int_0^t \|x_s\|_{\mathcal{B}_h}ds + M_1\int_0^t p(s)\Psi\Big(\|x_s\|_{\mathcal{B}_h} + \int_0^s \gamma m(\tau)\Omega(\|x_\tau\|)d\tau\Big)ds.
\end{aligned}
$$

Thus from this proof and Lemma 2.4 it follows that

$$
\begin{aligned}
\|x_t\|_{\mathcal{B}_h} &\leq l\sup\{|x(s)| : 0 \leq s \leq t\} + \|\phi\|_{\mathcal{B}_h} \\
&\leq \|\phi\|_{\mathcal{B}_h} + lF + lM_0(1+N_1)N_2\sup_{0\leq s\leq t}\|x_s\|_{\mathcal{B}_h} + l(1+N_1)N_2C_{1-\beta} \\
&\quad (\times)\int_0^t \frac{\|x_s\|_{\mathcal{B}_h}}{(t-s)^{1-\beta}}ds + lM_0M_1M_2bc_1\int_0^t \|x_s\|_{\mathcal{B}_h}ds + M_1\int_0^t p(s) \\
&\quad (\times)\Psi\Big(\|x_s\|_{\mathcal{B}_h} + \int_0^s \gamma m(\tau)\Omega(\|x_\tau\|)d\tau\Big)ds
\end{aligned}
$$

Let $\mu(t) = \sup\{\|x_s\|_{B_h} : 0 \leq s \leq t\}$, then the function $\mu(t)$ is nondecreasing in J, and we have

$$
\begin{aligned}
\mu(t) &\leq \|\phi\|_{\mathcal{B}_h} + lF + lM_0(1+N_1)\mu(t) \\
&\quad + l(1+N_1)N_2C_{1-\beta}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds + lM_0M_1M_2b \\
&\quad (\times)\int_0^t \mu(s)ds + lM_1\int_0^t p(s)\Psi\Big(\mu(s) + \int_0^s \gamma m(\tau)\Omega(\mu(\tau))d\tau\Big)ds \\
&\leq \frac{\|\phi\|_{\mathcal{B}_h} + lF}{[1 - lM_0(1+N_1)N_2]} + \frac{l(1+N_1)N_2C_{1-\beta}}{[1 - lM_0(1+N_1)N_2]}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds \\
&\quad + \frac{M_0M_1M_2b}{[1 - lM_0(1+N_1)N_2]}\int_0^t \mu(s)ds + \frac{lM_1}{[1 - lM_0(1+N_1)N_2]} \\
&\quad (\times)\int_0^t p(s)\Psi\Big(\mu(s) + \int_0^s \gamma m(\tau)\Omega(\mu(\tau))d\tau\Big)ds \\
&\leq K_1 + K_2\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}ds + K_3\int_0^t \mu(s)ds \\
&\quad + K_4\int_0^t p(s)\Psi\Big(\mu(s) + \int_0^s \gamma m(\tau)\Omega(\mu(\tau))d\tau\Big)ds.
\end{aligned}
$$

By using lemma 2.5, we have

$$\mu(t) \leq B_0 \left(K_1 + K_3 \int_0^t \mu(s)ds + K_4 \int_0^t p(s)\Omega_0 \left(\mu(s) + \int_0^s \gamma m(\tau)\Omega(\mu(\tau))d\tau \right) ds \right)$$

where

$$B_0 = e^{K_2^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} (\frac{K_2 b^\beta}{\beta})^j.$$

Let us take the right hand side of the above inequality as $v(t)$. Then $v(0) = B_0K_1, \mu(t) \leq v(t), 0 \leq t \leq b$ and

$$v'(t) \leq B_0K_3\mu(t) + B_0K_4p(t)\Psi \left(\mu(t) + \int_0^t \gamma m(s)\Omega(\mu(s))ds \right)$$

Since Ψ and Ω are nondecreasing.

$$v'(t) \leq B_0K_3v(t) + B_0K_4p(t)\Psi \left(v(t) + \int_0^t \gamma m(s)\Omega(v(s))ds \right), \quad t \in J.$$

Let $w(t) = v(t) + \int_0^t \gamma m(s)\Omega(v(s))ds$. Then $w(0) = v(0)$ and $v(t) \leq w(t)$.

$$\begin{aligned} w'(t) &= v'(t) + \gamma m(t)\Omega(v(t)) \\ &\leq B_0K_3w(t) + B_0K_4p(t)\Psi(w(t)) + \gamma m(t)\Omega(w(t)) \\ &\leq \tilde{m}(t)\Big[w(t) + \Psi(w(t)) + \Omega(w(t)) \Big] \end{aligned}$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Psi(s) + \Omega(s)} \leq \int_0^b \tilde{m}(s)ds \leq \int_{B_0K_1}^{\infty} \frac{ds}{s + \Psi(s) + \Omega(s)}$$

This implies that $v(t) < \infty$. So there is a constant K such that $v(t) \leq K, t \in J$. So $\|x_t\|_{\mathcal{B}_h} \leq \mu(t) \leq v(t) \leq K, t \in J$, where K depends only on b and on the functions $\Psi, \tilde{m}$ and Ω. □

Theorem 3.3. *Assume that the hypotheses (H1)-(H6) hold. Then the problem (1)-(2) has at least one mild solution on J.*

Proof. Let us take the set

$$G(\bar{\Phi}) = \{y \in \mathcal{B}_h'' : y = \lambda\bar{\Phi}_1(\frac{y}{\lambda}) + \lambda\bar{\Phi}_2 y \quad \text{for some } \lambda \in (0,1)\}.$$

Then for any $y \in G(\bar{\Phi})$, we have by Lemma 3.2 that $\|x_t\|_{\mathcal{B}_h} \leq K, t \in J$, and we have

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathcal{B}_h} + \sup\{|y(t)| : 0 \leq t \leq b\} \\ &= \sup\{|y(t)| : 0 \leq t \leq b\} \\ &\leq \sup\{|x(t)| : 0 \leq t \leq b\} + \sup\{|\tilde{\phi}(t)| : 0 \leq t \leq b\} \\ &\leq \sup\{l^{-1}\|x_t\|_{\mathcal{B}_h} : 0 \leq t \leq b\} + \sup\{|R(t)\phi(0)| : 0 \leq t \leq b\} \\ &\leq l^{-1}K + M_1|\phi(0)|. \end{aligned}$$

which implies that the set G is bounded on J.

Consequently, by Krasnoselski-Schaefer type fixed point theorem and Lemma 3.2 the operator $\bar{\Phi}$ has a fixed point $y^* \in \mathcal{B}''_h$. Let $x(t) = y^*(t)+\tilde{\phi}(t),\ t \in (-\infty, b]$. Then x is a fixed point of the operator Φ which is a mild solution of the problem (1)-(2). $\square$

4. Example

Consider the following partial neutral integrodifferential equation of the form

$$\frac{\partial}{\partial t}\Big[z(t,x) + \int_{-\infty}^{t} a_1(t,x,s-t)P_1(z(s,x))ds + \int_0^t \int_{-\infty}^{t} k(s-\tau)P_2(z(\tau,x))d\tau ds\Big]$$
$$= \frac{\partial^2}{\partial x^2} z(t,x) + \int_0^t b(t-s)z(s,x)ds$$
$$+ \int_{-\infty}^{t} a_2(t,x,s-t)Q_1(z(s,x))ds + \int_0^t \int_{-\infty}^{t} k(s-\tau)Q_2(z(\tau,x))d\tau ds,$$
$$x \in [0,\pi], \quad t \in [0,b], \tag{7}$$
$$z(t,0) = z(t,\pi) = 0, \quad t \geq 0, \tag{8}$$
$$z(t,x) = \phi(t,x), \quad t \in (-\infty, 0], \quad x \in [0,\pi], \tag{9}$$

where $\phi \in \mathcal{B}_h$. We take $X = L^2[0,\pi]$ with the norm $|\cdot|_{L^2}$ and define $A : X \to X$ by $Aw = w^{''}$ with the domain

$D(A) = \{w \in X : w, w'$ are absolutely continuous, $w^{''} \in X, w(0) = w(\pi) = 0\}$.Then

$$Aw = \sum_{n=1}^{\infty} n^2 < w, w_n > w_n, \quad w \in D(A),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns),\ n = 1, 2, \ldots.$. is the orthogonal set of eigen vectors of A. It is well known that A generates a strongly continuous semigroup that is analytic, and resolvent operator $R(t)$ can be extracted this analytic semigroup and given by

$$R(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, w_n > w_n, \quad w \in X.$$

Since the analytic semigroup $R(t)$ is compact, there exists a constant $M_1 > 0$ such that $\|R(t)\| \leq M_1$. Especially, the operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n < w, w_n > w_n$$

with the domain $D((-A)^{\frac{1}{2}}) = \{w \in X : \sum_{n=1}^{\infty} n < w, w_n > w_n \in X\}$.

Let $h(s) = e^{2s}, s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2}$ and define

$$\|\phi\|_h = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} |\phi(\theta)|_{L^2} ds.$$

Hence for $(t, \phi) \in [0, b] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$z(t)(x) = z(t, x), \qquad g(t, \phi, B_1\phi)(x) = \int_{-\infty}^{0} a_1(t, x, \theta) P_1(\phi(\theta)(x)) d\theta + B_1\phi(x)$$

and

$$f(t, \phi, B_2\phi)(x) = \int_{-\infty}^{0} a_2(t, x, \theta) Q_1(\phi(\theta)(x)) d\theta + B_2\phi(x).$$

where

$$\begin{aligned} B_1\phi(x) &= \int_0^t \int_{-\infty}^{0} k(t-\theta) P_2(\phi(\theta)(x)) d\theta ds; \ B_2\phi(x) \\ &= \int_0^t \int_{-\infty}^{0} k(s-\theta) Q_2(\phi(\theta)(x)) d\theta ds; \end{aligned}$$

Then, the system (7)-(9) is the abstract formulation of the system (1)-(2). Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.3. We can conclude that system (7)-(9) has at least one mild solution on J.

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S. Chandrasekaran

Department of Mathematics, SNS College of Technology, Coimbatore - 641 035, Tamil Nadu, India.
e-mail: chandrusavc@gmail.com

S. Karunanithi

Department of Mathematics, Kongunadu Arts and Science College(Autonomous), Coimbatore - 641 029, Tamil Nadu, India.
e-mail: sknithi1957@yahoo.co.in