Journal for History of Mathematics http://dx.doi.org/10.14477/jhm.2015.28.6.311
Vol. 28 No. 6 (Dec. 2015), 311–320

Tongwen Suanzhi (同文算指) and transmission of bisuan (筆算 written calculation) in China: from an HPM (History and Pedagogy of Mathematics) viewpoint

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In 1613 the official-scholar LI Zhi-zao (李之藻) of the Ming Dynasty, in collaboration with the Italian Jesuit Matteo RICCI (利瑪竇), compiled the treatise *Tongwen Suanzhi* (同文算指). This is the first book which transmitted into China in a systematic and comprehensive way the art of written calculation that had been in common practice in Europe since the sixteenth century. This paper tries to see what pedagogical lessons can be gleaned from the book, in particular on the basic operations in arithmetic and related applications in various types of problems which form the content of modern day mathematics in elementary school education.

Keywords: basic operations in arithmetic, problems in arithmetic, *Tongwen Suanzhi*, written calculation, history and pedagogy of mathematics.

MSC:

1 Introduction

In the early part of the seventeenth century the official-scholar LI Zhi-zao (李之藻1565–1630) of the Ming Dynasty, in collaboration with the Italian Jesuit Matteo RICCI (利瑪竇1552–1610), compiled the treatise *Tongwen Suanzhi* (同文算指, literally meaning "rules of arithmetic common to cultures") [9], which first transmitted into China in a systematic and comprehensive way the art of *bisuan* (筆算 written calculation) that had been in common practice in Europe since the sixteenth century. This treatise, accomplished in 1613, was a compilation based on the 1583 European text *Epitome Arithmeticae Practicae* (literally meaning "abridgement of arithmetic in practice") of Christopher CLAVIUS (1538–1612) and the 1592 Chinese mathematical classic *Suanfa Tongzong* (算法統宗, literally meaning "unified source of computational methods") of CHENG Da-wei (程大位1533–1606) [2]. This work is also an

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Received on Dec. 2, 2015, revised on Dec. 22, 2015, accepted on Dec. 28, 2015.

attempt of LI Zhi-zao to integrate European mathematics with traditional Chinese mathematics, which was a prevalent intellectual trend of the time known as *zhongxi huitong* (中西會通, literally meaning "integration of Chinese and Western [learning]), started by the dedicated work of another official-scholar XU Guang-qi (徐光啟 1562–1633) who translated the first six books of Euclid's *Elements* (from a fifteen-book version compiled by Clavius in the latter part of the sixteenth century) also in collaboration with Ricci and published it as *Jihe Yuanben* (幾何原本, literally meaning "source of quantity") in 1607 [10].

Together with YANG Ting-jun (楊廷筠 1557–1627) the three noted high-ranking officials — Xu, Li and Yang — in the Ming Court, who were colleagues and friends and among the early Chinese converts to Catholicism, are hailed as the "three pillars of the Catholic Church in China". Li first got acquainted with Ricci in 1601 in Nanjing. He was deeply impressed by the map of the world that Ricci prepared, the *Kunyu Wanguo Quantu* [坤輿萬國全圖 Complete Map of the Myriad Countries of the World], which was later printed in Peking in 1602. Li himself had prepared a map of the fifteen provinces of China at the age of twenty and thought at the time he had well mastered the knowledge of cartography so that he was all the more amazed by this work of Ricci, whom he much admired and respected for Ricci's erudition.

The aim of the present paper is not to present a historical study of the book — its content, its historical context and its influence on Chinese mathematics in the eighteenth-century Qing Dynasty. For that historical aspect interested readers are invited to consult some more scholarly works [1, 13]. Rather, we try to adopt a viewpoint of HPM (History and Pedagogy of Mathematics) [11] and see what pedagogical lessons can be gleaned from the book, in particular on the basic operations in arithmetic and related applications in various types of problems which form the content of modern day mathematics in elementary school education.1)

2 Basic operations in arithmetic

The first book in the first part (Preliminary Part) of *Tongwen Suanzhi* explains the notation in positional system and the four basic operations in arithmetic. Apart from division the other three operations — addition, subtraction, multiplication — are done in the way a schoolboy of today is familiar with. Division is done by the so-called *galley* method, which will be illustrated below. The second book deals with the arithmetic of fractions ending with a collection of miscellaneous problems to consolidate the skill in written calculation that has just been learnt.

1) This paper is an expanded version of a short presentation given at the study conference of the 23rd ICMI Study on Primary Mathematics Study on Whole Numbers held in University of Macau on June 3–7, 2015.

It would be instructive to compare the transmitted algorithms with the methods in traditional Chinese mathematics. We will look at how multiplication and division were done in ancient China as explained in *Sunzi Suanjing* (孫子算經, literally meaning "Master Sun's mathematical manual") of the fourth/fifth century [6, 7, 8]. The following two examples are taken from [8] (see Figure 1 and Figure 2, with the last item in modern notation inserted for comparison and with rod numerals replaced by what we are familiar with today).

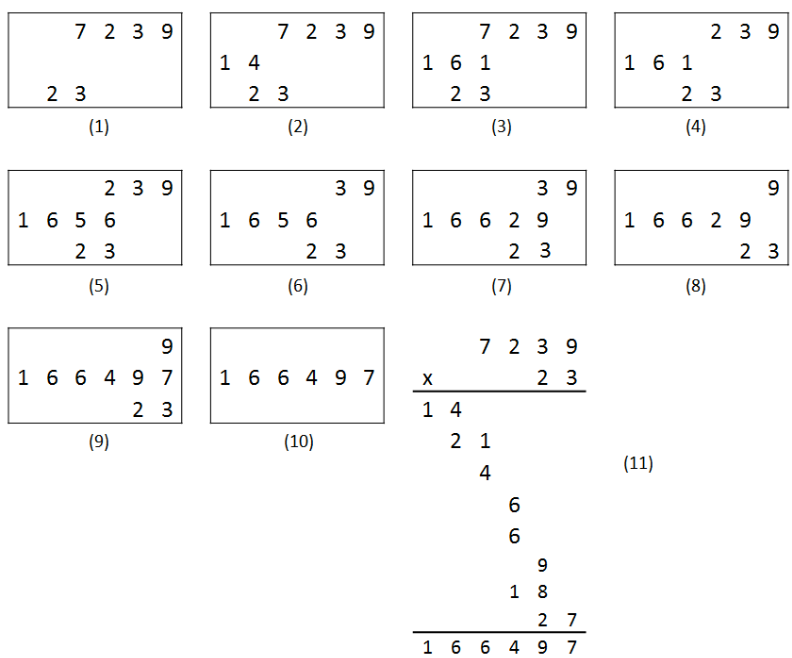

Figure 1. Multiplication in ancient China

In the Western world there was a movement of contest in efficiency of reckoning between the so-called "abacists" and "algorist" towards the latter part of the Middle Ages [12]. In particular, a method known as the *gelosia* method, coming from the Islamic world, was commonly used at the time. Written calculation did appear in Chinese texts even before *Tongwen Suanzhi*, for instance, in *Jiuzhang Suanfa Bilei Daquan* (九章算法比類大全, literally meaning "comprehensive collection of computational methods in *Nine Chapters* devised by analogy [with ancient problems and rules]") of 1450 by WU Jing (吳敬 15th century) [14] and in *Suanfa Tongzong* of 1592 by CHENG Da-wei [2], but not in a way as systematic and as comprehesive as in *Tongwen Suanzhi*. In both of these texts the *gelosia* method was introduced into China, called by CHENG Da-wei by the picturesque name of *pudijin* (鋪地錦, literally meaning "covering the floor with a glamorous carpet"). LI Zhi-zao seemed to prefer the

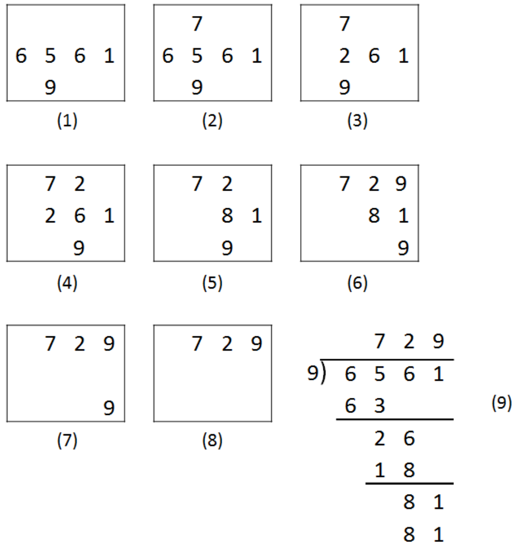

Figure 2. Division in ancient China

more modern method to this pictureqe *pudijin*. However, the latter may provide interesting exercise for a modern classroom (see Figure 3, with the last item in modern notation inserted for comparison).

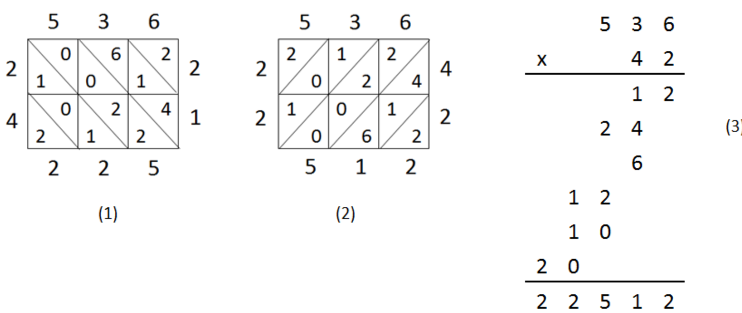

Figure 3. *Gelosia* method of multiplication

In *Tongwen Suanzhi* division is performed by the *galley* method. Let us illustrate this method by an example taken from [7], which in turn appeared in the *Treviso Arithmetic* of 1478 [12] (see Figure 4, with the last item in modern notation inserted for comparison) .

All the examples given above can be suitably utilized to prepare worksheets to enhance the understanding of the reasoning underlying the basic operations in arithmetic. Not to make this paper too lengthy we would not discuss the pros and cons of various methods, but its importance to teachers and mathematics educators is cer-

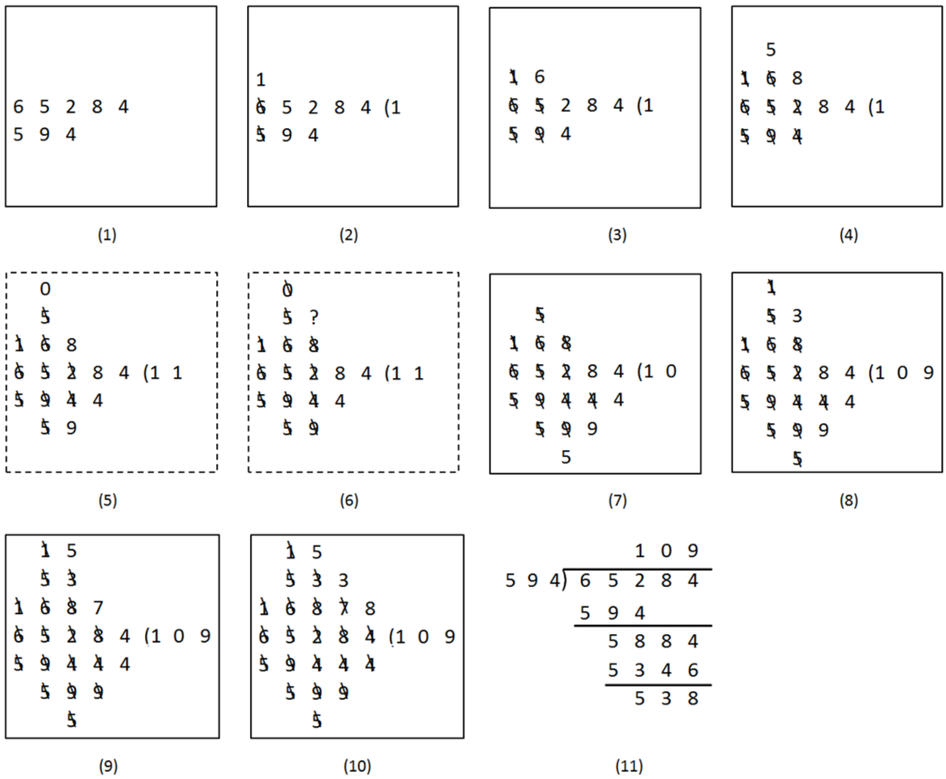

Figure 4. *Galley* method of division

tainly recognized [4].

Another interesting feature in the first book of *Tongwen Suanzhi* is the discussion on methods of checking the answer. For each basic operation several methods are given. One is applying the reversed operation to the answer, which is logically questionable (as far as the presentation in a textbook is concerned) because before subtraction (respectively division) is explained one should not make use of it to check the answer obtained by addition (respectively multiplication)! On the other hand it indicates the awareness of the mutual reversibility of the operations involved. A second method is to carry out the operation with the given numbers in a different order, which indicates the awareness of the commutative law of the operation. The most interesting method is casting out nines (or sevens), which indicates an awareness of modulo arithmetic. As expected, it was assumed that the checking works without bothering about the mathematical fact that the method using modulo arithmetic tells when the working is wrong but cannot guarantee that the working is correct. Compared to the practical usefulness of these methods, such logical slip is a minor blemish.

3 More problems in arithmetic

The second part (General Part) of *Tongwen Suanzhi* contains a large collection of various problems, which appeared in mathematical texts in traditional Chinese mathematics such as *Suanfa Tongzong* of CHENG Da-wei, which in turn were handed down from the famous ancient Chinese mathematical classic *Jiuzhang Suanshu* (九章算術, literally meaning "nine chapters on the mathematical art") compiled between the second century B.C.E. and the first century C.E. [6]. These problems are treated in the newly introduced method of written calculation, thus amplifying the attempt of LI Zhi-zao in integrating Chinese mathematics with European mathematics.

Let us look at just two examples out of the many. The first example is on proportion treated by the method known in the Western world as "Rule of Three", or the so-called "Golden Rule". The second example is on "extraction of square root with an accompanying term", which is solving a quadratic equation in the Western term.

Example 1 (in Section 1) . "Suppose 4 *guan* (貫) of money can purchase 12 *jin* (斤) of goods, how many *jin* of goods can 20 *guan* purchase? "

Example 2 (in Section 14) . "Suppose a rectangular field has area 864 [square] *bu* (步) and its width is less than its length by 12 *bu*, what is its width? "

It is to be noted that the technique of "extraction of square root with an accompanying term" was not in *Epitome Arithmeticae Practicae* of Christopher CLAVIUS, but was a traditional Chinese method to handle pertinent problems. Moreover, LI Zhi-zao covered the topic in a more elaborate manner than how it was treated in traditional Chinese text, making use of the newly transmitted written calculation [1]. This indicates that in compiling *Tongwen Suanzhi* Li followed the footsteps of XU Guang-qi in attempting to integrate Chinese and Western learning.

4 Some pedagogical lessons

We now come to some pedagogical lessons that can be gleaned from the book.

(1) The techniques developed at the time fit in with the historical development of the time. One example is how the high cost of paper at the time explains why the *galley* method of multiplication was preferred before long division familiar to us today was developed and adopted later in history. Another example is the very detailed explanation in the calculation using fractions that arose as a necessity in commercial activities involving a diversity of European currencies. Indeed, the rise of written calculation has a lot to do with the upsurge of commercial activities since the sixteenth century in Europe [12]. The lesson for us is that the design of

curriculum has to take into account contemporary need (or diminishing need) so as to ride with time.

(2) Old techniques may provide instructive exercises for the benefit of teaching and learning. One example is multiplication by the *gelosia* method, also known as the *grid* method. Another example is division by the *galley* method. Still another example is the method of casting out nines. These techniques are no longer necessary skills to be learnt nowadays but may offer good ways to understand and to consolidate understanding if employed in a thoughtful manner. It has been pointed out by experts that "historical analysis is a source of inspiration as well as a means of control" and that the role of history of mathematics is to "reconstruct an epistemologically controlled genesis taking into account the specific constraints of the teaching content" [3]. When forced to work in an unfamiliar setting the learner may stand a better chance of going deeper into the underlying reasoning of a procedure. Besides, it may also add a humanistic touch to mathematical lessons by letting students see how people did things in the past.

(3) The first part (Preliminary Part) of *Tongwen Suanzhi* explains the four basic operations in arithmetic and the calculation with fractions. The second part (General Part) is a comprehensive account on various problems treating proportion, extraction of square and cube roots, method of double false position, and solving linear and quadratic equations. The third part (Special Part), which was undated and short, introduces basic knowledge in trigonometry. The second part plays a central role in which the authors tried to make use of these problems to consolidate the skill in written calculation, at the same time indicating the prowess of written calculation. This textbook design based on such a pedagogical objective is far superior to that of a heavy load of straightforward but boring drilling exercises in some modern day textbooks!

The emphasis *Tongwen Suanzhi* placed on the learning and teaching of arithmetic exerted influence on the subsequent writing of textbooks in China. Instead of paying attention to teaching algorithms through the aid of mnemonic poems the underlying reasoning was brought into the study. The use of counting rods and the abacus was gradually replaced by the use of written calculation.

(4) In ancient times people calculated by using manipulatives such as pebbles, sticks, counting rods, abacus, etc. By today's standard one may see these as clumsy and inefficient. However, with sufficient practice this needs not present an obstacle to efficient calculation. For an expert who had acquired the skill it can even mean a quick and convenient method. Likewise, the adoption of an ancient recording system of numerals by the grouping method may seem cumbersome to a modern

day schoolboy but not so for an ancient Egyptian scribe well versed in the art of calculation. Hence, what is so good about positional system in numeration and what is so good about written calculation?

The main advantage of written calculation lies in keeping intact a record of the intermediate steps which affords easy checking afterwards. It also allows one to see the procedure and to gain understanding of the underlying reasoning without having to memorize what is going on during the calculation. This is difficult to attain in calculation using manipulatives (although nowadays calculation by using manipulatives can gain its own pedagogical advantage in learning). Along with this benefit the advantage of positional system is revealed. Without the invention of a positional system written calculation as we know it cannot be invented.

But then this leads us to the next question in this age of computers. Ironically we are turning back the wheel of history in some sense in that we erase the intermediate steps when we calculate by punching a few keys on an electronic calculator! For all practical purposes it is definitely much more efficient to calculate by using an electronic calculator than to calculate by hand, just like one would not like to cook by setting up a fire instead of using a kitchen stove. It is true that because of that the emphasis in learning calculation would be shifted to skill and knowledge in estimation in order to guard against careless manipulation or errors in the calculating machine. But do we still need to pay so much attention to written calculation in schools? For instance, in some places there is a suggestion for de-emphasis of long division.

The rationale for learning written calculation, at least once in a person's lifetime, seems to be the acquirement of understanding of the underlying principle of the basic operations in arithmetic. For some learners this kind of understanding is essential in future endeavor. Let us just cite one example about a commonplace operation as multiplication. For the computation with very large numbers various algorithms had been developed to speed up the time by reducing the number of steps of simple multiplication of one-digit numbers, for instance the Karatsuba algorithm in the early 1960s, the Schönhage-Strassen algorithm in the early 1970s and the Fürer algorithm in the late 2000s [5]. In order to devise such algorithms one has to understand the underlying principle of multiplication. Admittedly, only a fairly small percentage of the population of all school pupils will need to have that kind of understanding in their future career. But it does not seem advisable to teach it only to these selected few after they reach a more advanced and specialized level. If it is going to be taught at all it would be advisable to teach it to all at the elementary school level. To include this topic in the elementary school level we can regard the art of calculation through the basic operations in arithmetic as a cultural heritage handed down to us by our ancestors and had undergone improvement with time, and is therefore something worth knowing even though tedious drilling in the past practice is no longer needed nor desirable in this computer age. Viewed in this light, written calculation still has its value in modern day education, but with a different emphasis. In this respect, looking at it through a historical perspective, supplemented with exercises suitably designed and based on historical material (as mentioned in (2)), maybe a good alternative.

(5) In the preface as well as in two forewords to *Tongwen Suanzhi* LI Zhi-zao and his friends and fellow official-scholars XU Guang-qi and YANG Ting-jun stressed the meaning of *tongwen* (literally meaning "common cultures") adopted as part of the title of the book [9], which exhibits their open mind and receptive attitude to foreign learning, at the same time indicating a deep appreciation of the common cultural root of mathematics despite different mathematical traditions. Let us look at some of their sayings to further illustrate this point.

"The origin of numbers, could it not be at the beginning of human history? Starting with one, ending with ten, the ten fingers symbolize them and are bent to calculate them, [numbers] are of unsurpassed utility! Across the five directions and myriad countries, changes in customs are multitudinous. When it comes to calculating numbers, there are none that are not the same; that all possess ten fingers, there are none that are not the same." [XU Guang-qi, Preface at the printing of *Tongwen Suanzhi*, 1613.]

"Across the seas of the East and the West the mind and reasoning are the same [同 *tong*]. The difference lies only in the language and the writing." [LI Zhi-zao, Preface to the reprinting of *Tianzhu Shiyi* (天主實義 The True Meaning of the Lord of Heaven, written by Matteo RICCI and printed in 1603 in Peking)].

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